A GENERALIZATION OF HOPF CROSSED PRODUCTS

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1. Introduction

The present paper deals both with the now classical theory of Hopf crossed products, as reviewed in Chapter 7 of Montgomery's book [8], and with a generalization of that theory.

Hopf crossed products were introduced by Doi and Takeuchi [5] and Blattner, Cohen and Montgomery [2] as a Hopf algebraic generalization of group crossed products.

Given a certain (weak, i. e. not fulfilling the axioms of a module) action $\rightharpoonup$ of a Hopf algebra H on an algebra B, and a 2-cocycle, that is, a map $\sigma\colon H \otimes H \to B$, the crossed product $B\#_\sigma H$ is $B \otimes H$ as a vector space with multiplication given by $(b\#g)(c\#h) = b(g_{(1)} \rightharpoonup c)\sigma(g_{(2)} \otimes h_{(1)})\#g_{(3)}h_{(2)}$. Certain 'cocycle identities' have to be required of the pair $(\rightharpoonup, \sigma)$ to ensure that $B\#_\sigma H$ becomes an associative algebra. With the comodule structure induced by the right tensorand, $B\#_\sigma H$ is then an H-comodule algebra, constructed from B and H and the data $\rightharpoonup, \sigma$ used for twisting the multiplication.

Doi and Takeuchi [5] showed that every cleft H-extension $B \subset A$ can be built up as such a Hopf crossed product, $A \cong B\#_\sigma H$ for a suitable weak action and cocycle. Here, a cleft H-extension A of $B \subset A$ is an H-comodule algebra A that admits a convolution invertible H-comodule map $\gamma\colon H \to A$, and such that $B = A^{\mathrm{co}\,H}$ is the subalgebra of A of all elements on which the coaction of H is trivial.

Moreover, cleft H-extensions were characterized in [5] as those H-Galois extensions that have a normal basis. Here, an H-Galois extension is by definition an H-comodule algebra A such that, for $B = A^{\mathrm{co}\,H}$, a certain canonical map $A \otimes_B A \to A \otimes H$ is a bijection. Hopf-Galois extensions are analogs of principal

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fiber bundles with a quantum structure group in the sense of noncommutative algebraic geometry. By definition, A has a normal basis if there is an isomorphism $A \cong B \otimes H$ of both left B-modules and right H-comodules.

The basic theory of cleft extensions and Hopf crossed products was made complete by Blattner and Montgomery [3], who showed that a Hopf crossed product $B\#_\sigma H$ is in fact always a cleft H-extension, provided the cocycle σ is convolution invertible. To do this one has to show that the obvious candidate for a cleft map, $\gamma\colon H \ni h \mapsto 1\#h \in B\#_\sigma H$, is convolution invertible, which was done in [3] using the cocycle identities.

In section 3 we will give a new short proof for most of the basic equivalence between Hopf-Galois extensions with a normal basis, cleft extensions, and Hopf crossed products. The only part that we will not be dealing with are the cocycle identities; this is possible because our new proof that crossed products with invertible cocycles are cleft does not need those identities. The only tool used in Section 3 is a general ring isomorphism $\mathrm{Hom}(C, R) \cong \mathrm{End}^C_R(R \otimes C)$ for an algebra R and coalgebra C, where the left hand side is an algebra by convolution, and the right hand side is an algebra by composition. The isomorphim is rather trivial to prove and appears as a technical tool in [6]. The comfortable switch between convolution and composition provided by the isomorphism allows us to reduce the key calculations in the characterization of cleft extensions to a couple of easy lines.

In Section 4 we discuss a generalization of the notion of cleft comodule algebras due to Masuoka and Doi [7]. The basic situation here is that we are given a Hopf algebra H together with a quotient coalgebra and right H-module $\overline{H}$ of H. (The situation studied in [7] is actually a little more general, but nonzero cleft comodule algebras only exist in the situation just described.) Now an H-comodule algebra A is called $\overline{H}$-cleft if an $\overline{H}$-colinear convolution invertible map $\gamma\colon \overline{H} \to A$ exists. Masuoka and Doi show that $\overline{H}$-cleft extensions are precisely those H-comodule algebras that are $\overline{H}$-Galois with a normal basis (we shall review the definitions below). We give a different proof using again the isomorphism $\mathrm{Hom}(C, R) \cong \mathrm{End}^C_R(R \otimes C)$.

In Section 5 we start covering new material. In the general case of a quotient coalgebra and right module $\overline{H}$ of a Hopf algebra H, there does not seem to be a reasonable generalization of crossed products that would describe how to build up $\overline{H}$-cleft extensions of B from B and $\overline{H}$. We study a special type of quotients: If $H' \subset H$ is a Hopf subalgebra, then one can form the coideal right ideal H'^+H and study the quotient $\overline{H} = H/H'^+H$. We are interested in the case that H

admits a 'weak projection' onto H' (so called because it generalizes the notion of a projection as studied by Radford [9].) A weak projection is by definition a coalgebra projection which is also left H'-linear. Given a Hopf algebra with a weak projection onto the Hopf subalgebra $H' \subset H$ and the corresponding quotient $\overline{H}$, we study a special type of $\overline{H}$-cleft extensions, in which the cleft map is required to be compatible in some sense with the chosen weak projection. We can show that these cleft extensions can be equivalently characterized as crossed products of a certain type in which the cocycle is again convolution invertible as a map from a twisted tensor product coalgebra $\overline{H}\tilde{\otimes}\overline{H}$ to B. The proof is a generalization of the new technique of proof given for the theory of usual cleft extensions in section 3. It should be noted that both Hopf algebras with a weak projection and cleft extensions of the type studied in section 5 over them exist in abundance. By an easy argument, Hopf algebras that have a triangular decomposition admit a weak projection; in particular, quantized enveloping algebras have weak projections onto their Cartan-like Hopf subalgebras. Moreover, whenever H has a weak projection onto H' and H' contains the coradical H_0 of H, then every comodule algebra which is a coflat H-comodule (in particular, every faithfully flat H-Galois extension) is $\overline{H}$-cleft.

We put off until section 6 the calculation of the identities for weak actions and cocycles relevant for our special $\overline{H}$-cleft extensions.

In the last section we describe briefly, but without giving proofs, the structure of Hopf algebras with a weak projection.

2. Preliminaries

Throughout the paper k is a fixed base field, and all algebras, unadorned tensor products and Hom's, algebras, coalgebras etc. are over k. We make free use of Sweedler's notation for comultiplications in the form $\Delta(c) = c_{(1)} \otimes c_{(2)}$ and for comodule structures in the form $\rho(m) = m_{(0)} \otimes m_{(1)}$. If H is a bialgebra and M is a right H-comodule, we write $M^{\mathrm{co}\,H} := \{m \in M | \rho(m) = m \otimes 1\}$. For a subset U of a coalgebra C we write $U^+ := \mathrm{Ker}(\varepsilon) \cap U$.

The following two Lemmas are very easy to prove, and at least the second is well known (see [8, p.91]). The reason why we still write them out in considerable detail is that they are the key ingredient in many proofs throughout the paper, in particular the new approach to cleft extensions and crossed products, and we would like to fix notations.

Lemma 2.1. *Let R be an algebra, M a right R-module, C a coalgebra and N a left C-comodule. Then there is an algebra anti-homomorphism*

$$\widetilde{T} := \widetilde{T}^{C,N}_{R,M} \colon \operatorname{Hom}(C,R) \to \operatorname{End}(M \otimes N);$$

here the left hand side is a k-algebra by convolution and the right hand side is a k-algebra by composition. The homomorphism is given by

$$\widetilde{T}(\gamma)(m \otimes n) = m\gamma(n_{(-1)}) \otimes n_{(0)}$$

for $m \in M$ and $n \in N$.

In particular, $\widetilde{T}(\gamma)$ is an isomorphism whenever γ is convolution invertible.

Lemma 2.2. *Let R be an algebra and C a coalgebra. Then we have an anti-isomorphism of algebras*

$$T := T^C_R \colon \operatorname{Hom}(C,R) \to \operatorname{End}^C_R(C \otimes R);$$

here the left hand side is a k-algebra by convolution. The right hand side denotes the set of all right C-colinear and left R-linear endomorphism, which is an algebra by composition. The isomorphism is given by

$$T(\gamma)(r \otimes c) = r\gamma(c_{(1)}) \otimes c_{(2)}$$

for $r \in R$ and $c \in C$, and its inverse by $T^{-1}(f)(c) = (\mathrm{id}_R \otimes \varepsilon) f(1 \otimes c)$.

In particular $\gamma \colon C \to R$ is convolution invertible if and only if $T^C_R(\gamma)$ is an isomorphism.

3. A quick tour of cleft comodule algebras and crossed products

Let H be a k-Hopf algebra and A a right H-comodule algebra with $B := A^{\mathrm{co}\,H}$. By definition A is H-cleft if there is a colinear, convolution invertible map $\gamma \colon H \to A$; it is H-Galois if the Galois map $\beta_A \colon A \otimes_B A \to A \otimes H$ with $\beta_A(x \otimes y) = xy_{(0)} \otimes y_{(1)}$ is a bijection; it has a normal basis if there is a left B-linear right H-colinear isomorphism $B \otimes H \to A$. It was proved by Doi and Takeuchi [5] that A is H-Galois with a normal basis iff it is cleft, and in this case A is a crossed product $A \cong B \#_\sigma H$ with an invertible cocycle $\sigma \colon H \otimes H \to B$. Blattner and Montgomery [3] have shown that crossed products with an invertible cocycle are cleft.

The purpose of this section is to give a new proof for these results; this new proof will also be the model for generalizations in subsequent sections.

Novelties in the proof will only appear after the statement of Theorem 3.2 below, but to set the stage and keep the treatment self-contained, we will repeat large parts of the theory.

For our purposes the following definition of crossed product will be suitable:

Definition 3.1. A crossed product $B\#_\sigma H$ is the structure of an associative algebra on the space $B\otimes H$ with unit $1\#1$ and multiplication of the form

$$(b\#g)(c\#h)=b(g_{(1)}\rightharpoonup c)\sigma(g_{(2)}|h_{(1)})\#g_{(3)}h_{(2)}$$

for certain maps $\rightharpoonup\colon H\otimes B\to B$ and $\sigma\colon H\otimes H\to B$.

We do not (need to) discuss the equations that have to be imposed on σ and $\rightharpoonup$ to make the above multiplication associative with the above unit. Clearly a crossed product is always an H-comodule algebra with an obvious normal basis.

Assume that the H-comodule algebra A has a normal basis $\psi\colon B\otimes H\to A$ satisfying $\psi(1\otimes 1)=1$. Then A is isomorphic (via ψ) to a crossed product. In fact we may as well assume $B\otimes H=A$ as B-modules and H-comodules. Define $h\rightharpoonup b=(B\otimes\varepsilon)((1\otimes h)(b\otimes 1))$ and $\sigma(g|h)=(B\otimes\varepsilon)((1\otimes g)(1\otimes h))$. Since multiplication is H-colinear, we find

$$\begin{aligned}(1\otimes g)(c\otimes 1)&=(B\otimes\varepsilon\otimes H)(B\otimes\Delta)((1\otimes g)(c\otimes 1))\\&=(B\otimes\varepsilon\otimes H)((1\otimes g_{(1)})(b\otimes 1)\otimes g_{(2)})=g_{(1)}\rightharpoonup b\otimes g_{(2)},\end{aligned}$$

$$\begin{aligned}(1\otimes g)(1\otimes h)&=(B\otimes\varepsilon\otimes H)(B\otimes\Delta)((1\otimes g)(1\otimes h))\\&=(B\otimes\varepsilon\otimes H)((1\otimes g_{(1)})(1\otimes h_{(1)})\otimes g_{(2)}h_{(2)})=\sigma(g_{(1)}|h_{(1)})\otimes g_{(2)}h_{(2)},\end{aligned}$$

and finally

$$\begin{aligned}(b\otimes g)(c\otimes h)&=(b\otimes 1)(1\otimes g)(c\otimes 1)(1\otimes h)=(b\otimes 1)(g_{(1)}\rightharpoonup c\otimes g_{(2)})(1\otimes h)\\&=b(g_{(1)}\rightharpoonup c)\sigma(g_{(2)}|h_{(1)})\otimes g_{(3)}h_{(2)}.\end{aligned}$$

Assume $\gamma\colon H\to A$ is a cleft map. Then $\rho\gamma=(\gamma\otimes H)\Delta$ is convolution invertible since ρ is an algebra map, and for $a\in A$

$$\begin{aligned}&T^H_{A\otimes H}(\rho\gamma)(\rho(a_{(0)}\gamma^{-1}(a_{(1)}))\otimes a_{(2)})=T^H_{A\otimes H}(\rho\gamma)T^H_{A\otimes H}(\rho\gamma^{-1})(\rho(a_{(0)})\otimes a_{(1)})\\&=a_{(0)}\gamma^{-1}(a_{(1)})\gamma(a_{(2)})\otimes a_{(3)}\otimes a_{(4)}=T^H_{A\otimes H}((\gamma\otimes H)\Delta)(a_{(0)}\gamma^{-1}(a_{(1)})\otimes 1\otimes a_{(2)})\end{aligned}$$

from which $\rho(a_{(0)}\gamma^{-1}(a_{(1)}))=a_{(0)}\gamma^{-1}(a_{(1)})\otimes 1$ follows by applying $(A\otimes\varepsilon)T(\rho\gamma)^{-1}$. Thus $\psi\colon B\otimes H\ni b\otimes h\mapsto b\gamma(h)\in A$ is a normal basis, since $A\ni a\mapsto a_{(0)}\gamma^{-1}(a_{(1)})\otimes a_{(2)}\in B\otimes H$ is well defined and easily checked to be an inverse for ψ. Note that if γ satisfies $\gamma(1) = 1$, then also $\psi(1 \otimes 1) = 1$. If $\tilde{\gamma}$ is any cleft map, then $\tilde{\gamma}(1)$ is a unit in B, and thus $\gamma(h) = \tilde{\gamma}(1)^{-1}\tilde{\gamma}(h)$ defines another cleft map which satisfies $\gamma(1) = 1$.

Theorem 3.2. *Let H be a Hopf algebra and A a right H-comodule algebra with $B := A^{\mathrm{co}H}$. The following are equivalent:*

1. *A is H-cleft.*
2. *A is H-Galois with a normal basis.*
3. *A is isomorphic to a crossed prodcut $B\#_\sigma H$ such that the cocycle $\sigma\colon H \otimes H \to B$ is convolution invertible.*

Proof. We have already shown that under any of the three hypotheses we can assume that $A \cong B\#_\sigma H = B \otimes H$ is a crossed product. In particular, $\alpha\colon B \otimes H \otimes H \to A \otimes_B A$ with $\alpha(b \otimes g \otimes h) = b \otimes g \otimes 1 \otimes h$ is an isomorphism. We set $\gamma(h) = 1 \otimes h$ and can assume γ is the cleft map if we started with (1).

Now, for $b \in B$, $g, h \in H$ we have

$$\begin{aligned}\beta_A\alpha(b \otimes g \otimes h) = \beta_A(b \otimes g \otimes \gamma(h)) = (b \otimes g)\gamma(h_{(1)}) \otimes h_{(2)} = T_A^H(\gamma)(b \otimes g \otimes h)\\ = b\gamma(g)\gamma(h_{(1)}) \otimes h_{(2)} = b\sigma(g_{(1)}|h_{(1)}) \otimes g_{(2)}h_{(2)} \otimes h_{(3)}\\ = (B \otimes \beta_H)(b\sigma(g_{(1)}|h_{(1)}) \otimes g_{(2)} \otimes h_{(2)}) = (B \otimes \beta_H)T_B^{H\otimes H}(\sigma)(b \otimes g \otimes h)\end{aligned}$$

that is, $\beta_A\alpha = T_A^H(\gamma) = (B \otimes \beta_H)T_B^{H\otimes H}(\sigma)$, so that, if any one of β_A, $T_A^H(\gamma)$ and $T_B^{H\otimes H}(\sigma)$ is bijective, so are the other two. $\square$

The original proof of the implication (3)⇒(1) in [3] works by giving explicitly the convolution inverse for $\gamma\colon H \ni h \mapsto 1\#h \in B\#_\sigma H$. Note that the inverse for γ can be calculated from our proof by the formula

$$\gamma^{-1} = (T_A^H)^{-1}\left(T_B^{H\otimes H}(\sigma^{-1})(B \otimes \beta_H^{-1})\right).$$

We shall do this explicitly in a more general case below.

4. A generalization of cleft comodule algebras

In this section we will review the main results from Section 1 of [7] with a different proof which is largely based on Lemmas 2.1 and 2.2.

Throughout the section, we shall assume the following situation: H is a bialgebra, $\overline{H}$ is a right H-module coalgebra, and $\nu\colon H \to \overline{H}$ is a right H-module coalgebra map; we write $\nu(h) =: \overline{h}$. We will usually denote H-comodule structure maps by ρ and $\overline{H}$-comodule structure maps by $\overline{\rho}$.

We will be studying the following generalizations of cleft comodule algebras, introduced in [7]:

Definition 4.1. Let A be a right H-comodule algebra with comodule structure ρ and let $\overline{\rho} := (A \otimes \nu)\rho$ be the induced $\overline{H}$-comodule structure. Put $B := A^{\mathrm{co}\,\overline{H}} := \{a \in a | \overline{\rho}(a) = a \otimes \overline{1}\}$.

1. A is $\overline{H}$-cleft if there is a convolution invertible $\overline{H}$-colinear map $\gamma\colon \overline{H} \to A$.
2. A $(B, \overline{H})$-normal basis for A is an isomorphism $\psi\colon B \otimes \overline{H} \to A$ of left B-modules and right $\overline{H}$-comodules.
3. A is an $\overline{H}$-Galois extension of B if the Galois map $\overline{\beta}_A\colon A \otimes_B A \to A \otimes \overline{H}$ given by $\overline{\beta}_A(x \otimes y) = xy_{(0)} \otimes \overline{y_{(1)}}$ is a bijection.

Remark 4.2. The notation we have used for $\nu\colon H \to \overline{H}$ suggests that $\overline{H}$ is a quotient $\overline{H} = H/I$ of H by some coideal right ideal I. This is not too misleading for the situations we are interested in, since if $0 \neq A$ is $\overline{H}$-Galois, then it follows easily from the definition that ν is onto.

Lemma 4.3. *Let A be a right H-comodule algebra and $B := A^{\mathrm{co}\,\overline{H}}$. Assume that $\gamma\colon \overline{H} \to A$ is $\overline{H}$-colinear and convolution invertible. Then*

1. *A left B-linear projection $\pi\colon A \to B$ is defined by $\pi(a) = a_{(0)}\gamma^{-1}(\overline{a_{(1)}})$.*
2. *The map $\psi\colon B \otimes \overline{H} \ni b \otimes \overline{h} \mapsto b\gamma(\overline{h}) \in A$ is an isomorphism with $\psi^{-1}(a) = \pi(a_{(0)}) \otimes \overline{a_{(1)}}$.*

Proof. To show that π is well defined, we have to show that $\overline{\rho}(a_{(0)}\gamma^{-1}(\overline{a_{(1)}})) = a_{(0)}\gamma^{-1}(\overline{a_{(1)}}) \otimes \overline{1}$ holds in $A \otimes \overline{H}$. Clearly it is sufficient to show

$$\overline{\rho}(a_{(0)}\gamma^{-1}(\overline{a_{(1)}})) \otimes \overline{a_{(2)}} = a_{(0)}\gamma^{-1}(\overline{a_{(1)}}) \otimes \overline{1} \otimes \overline{a_{(2)}} \tag{4.1}$$

as an equality in $A \otimes \overline{H} \otimes \overline{H}$. Consider $A \otimes \overline{H}$ as a right A-module via $(x \otimes \overline{h})y = xy_{(0)} \otimes \overline{hy_{(1)}}$. Note that with this module strucure, $\overline{\rho}\colon A \to A \otimes \overline{H}$ is an A-module map. Consider

$$\widetilde{T} := \widetilde{T}^{\overline{H},\overline{H}}_{A,A\otimes\overline{H}}\colon \operatorname{Hom}(\overline{H}, A) \to \operatorname{End}(A \otimes \overline{H} \otimes \overline{H})$$

as in 2.1; this yields mutually inverse automorphisms $\widetilde{T}(\gamma)$ and $\widetilde{T}(\gamma^{-1})$ of $A \otimes \overline{H} \otimes \overline{H}$. While $\overline{\rho}(a_{(0)}\gamma^{-1}(\overline{a_{(1)}})) \otimes \overline{a_{(2)}} = \widetilde{T}(\gamma^{-1})(\overline{\rho}(a_{(0)}) \otimes \overline{a_{(1)}})$, we have

$$\begin{aligned}\widetilde{T}(\gamma)(a_{(0)}\gamma^{-1}(\overline{a_{(1)}}) \otimes \overline{1} \otimes \overline{a_{(2)}}) &= a_{(0)}\gamma^{-1}(\overline{a_{(1)}})\gamma(\overline{a_{(2)}})_{(0)} \otimes \overline{\gamma(\overline{a_{(2)}})_{(1)}} \otimes \overline{a_{(3)}} \\ &= a_{(0)}\gamma^{-1}(\overline{a_{(1)}})\gamma(\overline{a_{(2)}}) \otimes \overline{a_{(3)}} \otimes \overline{a_{(4)}} = a_{(0)} \otimes \overline{a_{(1)}} \otimes \overline{a_{(2)}};\end{aligned}$$

this proves (4.1). One now proves that ψ is an isomorphism with the claimed inverse by direct calculation: The equality $\psi\psi^{-1} = \mathrm{id}$ is obvious, and using

$$\pi\gamma(\overline{h}) = \gamma(\overline{h})_{(0)}\gamma^{-1}(\overline{\gamma(\overline{h})_{(1)}}) = \gamma(\overline{h}_{(1)})\gamma^{-1}(\overline{h}_{(2)}) = \varepsilon(h)$$

we obtain

$$\begin{aligned}\psi^{-1}\psi(b\otimes\overline{h}) = \psi^{-1}(b\gamma(\overline{h})) &= \pi(b_{(0)}\gamma(\overline{h})_{(0)})\otimes\overline{b_{(1)}\gamma(\overline{h})_{(1)}}\\ &= \pi(b\gamma(\overline{h})_{(0)})\otimes\overline{\gamma(\overline{h})_{(1)}} = b\pi(\gamma(\overline{h}_{(1)}))\otimes\overline{h}_{(2)} = b\otimes\overline{h} \quad\square\end{aligned}$$

Proposition 4.4. *Let H be a bialgebra, $I\subset H$ a coideal right ideal and $\overline{H} := H/I$. Let A be a right H-comodule algebra and $B := A^{\mathrm{co}\,\overline{H}}$.*

Then A is $\overline{H}$-cleft if and only if A is $\overline{H}$-Galois with a normal basis.

Proof. If we are given a normal basis ψ, we can define an $\overline{H}$-colinear map $\gamma\colon\overline{H}\to A$ by the equation

$$\psi(1\otimes\overline{h}) = \gamma(\overline{h}); \tag{4.2}$$

if we are given an $\overline{H}$-cleft map γ, we have a normal basis ψ satisfying (4.2) by Lemma 4.3. It remains to show that γ is convolution invertible if and only if the Galois map $\overline{\beta}_A$ is an isomorphism. Let $\alpha\colon A\otimes\overline{H}\to A\otimes_B B\otimes\overline{H}$ denote the canonical isomorphism; then it will suffice to show

$$T_A^{\overline{H}}(\gamma) = \overline{\beta}_A(A\otimes_B\psi)\alpha\colon A\otimes\overline{H}\to A\otimes\overline{H}$$

by Lemma 2.2. But indeed

$$\begin{aligned}\overline{\beta}_A(A\otimes_B\psi)\alpha(x\otimes\overline{h}) = \overline{\beta}_A(A\otimes_B\psi)(x\otimes 1\otimes\overline{h}) &= \overline{\beta}_A(x\otimes\gamma(\overline{h}))\\ &= x\gamma(\overline{h})_{(0)}\otimes\overline{\gamma(\overline{h})_{(1)}} = x\gamma(\overline{h}_{(1)})\otimes\overline{h}_{(2)} = T(\gamma)(x\otimes\overline{h}) \quad\square\end{aligned}$$

Corollary 4.5 (to the proof). *If A is $\overline{H}$-Galois over B, then a bijection between normal bases ψ and $\overline{H}$-cleft maps γ is given by $\gamma(\overline{h}) = \psi(1\otimes\overline{h})$ and $\psi(b\otimes\overline{h}) = b\gamma(\overline{h})$. We have $\psi^{-1}(a) = \pi(a_{(0)})\otimes\gamma(\overline{a_{(1)}}) = a_{(0)}\gamma^{-1}(\overline{a_{(1)}})\otimes\overline{a_{(2)}}$ and*

$$\gamma^{-1} = \left(T_A^{\overline{H}}\right)^{-1}(\alpha^{-1}(A\otimes_B\psi^{-1})\overline{\beta}_A^{-1}).$$

If γ is an $\overline{H}$-cleft map, then the inverse of the Galois map is given by $\overline{\beta}_A^{-1} = (A\otimes_B\psi)\alpha T(\gamma^{-1})$, that is, $\overline{\beta}_A^{-1}(b\otimes\overline{h}) = b\gamma^{-1}(\overline{h}_{(1)})\otimes\gamma(\overline{h}_{(1)})$.

Remark 4.6. If $\gamma\colon \overline{H} \to A$ is an $\overline{H}$-cleft map, then $\gamma^{-1}(1) = \pi(1) \in B$. Thus

$$\begin{aligned}\overline{\rho}(\gamma^{-1}(1)\gamma(\overline{h})) &= \pi(1)_{(0)}\gamma(\overline{h})_{(0)} \otimes \overline{\pi(1)_{(1)}\gamma(\overline{h})_{(1)}} \\ &= \pi(1)\gamma(\overline{h})_{(0)} \otimes \overline{\gamma(\overline{h})_{(1)}} = \gamma^{-1}(1)\gamma(\overline{h}_{(1)}) \otimes \overline{h}_{(2)}\end{aligned}$$

so that

$$\tilde{\gamma}\colon \overline{H} \ni \overline{h} \mapsto \gamma^{-1}(1)\gamma(\overline{h}) \in A$$

is an $\overline{H}$-cleft map satisfying $\tilde{\gamma}(1) = 1$. The corresponding normal basis ψ satisfies $\psi(1 \otimes 1) = 1$.

5. A generalization of crossed products

The characterization of $\overline{H}$-cleft extensions following [7] in the preceding section is a generalization of (1)⇔(2) in Theorem 3.2, due to [5]. In the general situation discussed above, there does not seem to be a reasonable generalization of the notion of crossed products, nor consequently of the equivalence of (1) and (2) with (3) in Theorem 3.2.

In this section we shall discuss a generalization of crossed products and their equivalence with cleft extensions suited to the case that the quotient $\overline{H}$ of the bialgebra H we are considering is of a special type.

Radford [9] has studied Hopf algebras H with the special property that there is a Hopf subalgebra H' and a projection $\pi\colon H \to H'$, that is, a Hopf algebra map satisfying $\pi|_{H'} = \mathrm{id}_{H'}$. In this case, the Hopf algebra H can be written as a semidirect product algebra and cosemidirect product coalgebra $H = H' \otimes \overline{H}$, where, as a coalgebra, $\overline{H} = H/H'^{+}H$. In [9] it was calculated (before the advent of that terminology) that to build up a Hopf algebra H with a projection as a semidirect cosemidirect product $H = H' \otimes \overline{H}$ the structure needed for the tensorand $\overline{H}$ is that of a Hopf algebra in the braided category of Yetter-Drinfeld H'-modules.

For our purposes, a substantially weaker setup is sufficient, although the case of Hopf algebras with a projection in the sense of Radford should be kept in mind as a special case.

Definition 5.1. Let H be a bialgebra and $H' \subset H$ a subbialgebra that is a Hopf algebra. A weak projection $\pi\colon H \to H'$ is a left H'-linear coalgebra map satisfying $\pi|_{H'} = \mathrm{id}$.

The following easy observation shows in particular that quantized enveloping algebras admit a weak projection onto their Cartan subalgebras — due to the well-known triangular decomposition.

Proposition 5.2. *Let H be a bialgebra, and $H', P, Q \subset H$ subbialgebras such that H' is a Hopf algebra, $H' \subset P$ and $H' \subset Q$, and such that multiplication in H induces an isomorphism $P \otimes_{H'} Q \cong H$.*

Assume that both P and Q admit weak projections onto H'. Then so does H.

Proof. Let $\pi_P \colon P \to H'$ and $\pi_Q \colon Q \to H'$ be weak projections. Since π_Q is left H'-linear, the map $\pi' \colon P \otimes_{H'} Q \ni p \otimes q \mapsto \pi_P(p\pi_Q(q)) \in H'$ is well defined. Because π_P is left H'-linear, so is π'. Via the assumed isomorphism $P \otimes_{H'} Q \cong H$, π' induces a map $\pi \colon H \to H'$ with $\pi(pq) = \pi_P(p\pi_Q(q))$ for $p \in P$ and $q \in Q$. Clearly $\pi|_{H'} = \mathrm{id}_{H'}$, and π is left H'-linear. Moreover

$$\begin{aligned}\Delta\pi(pq) = \Delta(\pi_P(p\pi_Q(q))) &= \pi_P(p_{(1)}\pi_Q(q)_{(1)}) \otimes \pi_P(p_{(2)}\pi_Q(q)_{(2)}) \\ &= \pi_P(p_{(1)}\pi_Q(q_{(1)})) \otimes \pi_P(p_{(2)}\pi_Q(q_{(2)})) = \pi(p_{(1)}q_{(1)}) \otimes \pi(p_{(2)}q_{(2)})\end{aligned}$$

shows that π is a coalgebra map. □

Notation 5.3. For the rest of this section assume we are given a Hopf algebra H with a weak projection $\pi \colon H \to H' \subset H$. We let $\pi^{-1} = S\pi$ denote the convolution inverse of π. We consider the coideal right ideal $H'^+H \subset H$ and the quotient coalgebra and right module $\overline{H} := H/H'^+H$. $\overline{H}$ is a right H'-module coalgebra with the right action $\leftharpoonup$ induced by the regular right action of H on itself. We will denote H-comodule structures by ρ, $\overline{H}$-comodule structures by $\overline{\rho}$ and H'-comodule structures by ρ'.

We make $\overline{H}$ a right H-comodule by

$$\rho(\overline{h}) = \overline{h}_{(2)} \otimes \pi^{-1}(h_{(1)})h_{(3)}$$

(which is easily checked to be well defined and a comodule structure). Then $\overline{H}$ also becomes an H'-comodule via π; we denote this comodule structure by

$$\rho' \colon \overline{H} \ni \overline{h} \mapsto \overline{h}_{[0]} \otimes \overline{h}_{[1]} = \overline{h}_{(2)} \otimes \pi^{-1}(h_{(1)})\pi(h_{(3)}) \in \overline{H} \otimes H'.$$

One checks that $\overline{H}$ is in this way an H'-comodule coalgebra.

Define a map $\varphi \colon \overline{H} \to H$ by $\varphi(\overline{h}) = \pi^{-1}(h_{(1)})h_{(2)}$. One checks easily that φ is well defined; if H is a Hopf algebra, then φ is convolution invertible with inverse $\varphi^{-1}(\overline{h}) = S(h_{(1)})\pi(h_{(2)})$.

Let M be an H-comodule. Then we have, for $m \in M$, the equality

$$\rho(m) = m_{(0)} \otimes m_{(1)} = m_{(0)} \otimes \pi(m_{(1)})\gamma(\overline{m_{(2)}}) = m_{(0)(0)} \otimes \pi(m_{(0)(1)})\gamma(\overline{m_{(1)}}).$$

We deduce easily that a k-linear map $f\colon M \to N$ between two H-comodules M and N is H-colinear if and only if it is both H'-colinear (via π) and $\overline{H}$-colinear. In fact the only if part is clear and for the converse we calculate

$$\begin{aligned} f(m)_{(0)} \otimes f(m)_{(1)} &= f(m)_{(0)(0)} \otimes \pi(f(m)_{(0)(1)})\gamma(\overline{f(m)_{(1)}}) \\ &= f(m_{(0)})_{(0)} \otimes \pi(f(m_{(0)})_{(1)})\gamma(\overline{m_{(1)}}) \\ &= f(m_{(0)(0)}) \otimes \pi(m_{(0)(1)})\gamma(\overline{m_{(1)}}) \\ &= f(m_{(0)}) \otimes m_{(1)} \end{aligned}$$

Definition 5.4. Let H be a Hopf algebra with a weak projection $\pi\colon H \to H' \subset H$. Let A be an H-comodule algebra and $B := A^{\mathrm{co}\,\overline{H}}$

1. A π-cleft map for A is an H-colinear convolution invertible map $\phi\colon \overline{H} \to A$. If a π-cleft map exists we call A a π-cleft extension of B.
2. A π-normal basis for A is an isomorphism $\psi\colon B \otimes \overline{H} \to A$ of right H-comodules and left B-modules.

Proposition 5.5. *Let A be an H-comodule algebra. Assume that A is coflat as an H-comodule and the Hopf subalgebra H' contains the coradical of H. Then A is π-cleft.*

Proof. By [4] there exists an H-colinear map $\tilde{\gamma}\colon H \to A$ satisfying $\tilde{\gamma}(1) = 1$. Define $\gamma = \tilde{\gamma}\varphi$. Then γ is H-colinear and satisfies $\gamma(\overline{1}) = 1$. Since $\overline{1}$ spans the coradical of $\overline{H}$ (see for example [8, Cor.5.3.5.]) it follows that γ is convolution invertible by [8, Lem.5.2.10.]. □

Example 5.6. The H-comodule algebra H itself has a π-normal basis $\Psi\colon H' \otimes \overline{H} \to H$ defined by $\Psi(x \otimes p) = x\varphi(p)$ with inverse $\Psi^{-1}(h) = \pi(h_{(1)}) \otimes \overline{h_{(2)}}$.

Proposition 5.7. *The following are equivalent for an H-comodule algebra A:*

1. *A is $\overline{H}$-Galois with a π-normal basis.*
2. *A is π-cleft.*

Proof. The proof consists in the observation that the bijection described in Corollary 4.5 restricts to the subsets of π-normal bases and π-cleft maps. □

Remark 5.8. Let $\gamma\colon \overline{H} \to A$ be a π-cleft map. Then in particular $\rho\gamma(1) = \gamma(1)\otimes 1$, hence also $\rho\gamma^{-1}(1) = \gamma^{-1}(1) \otimes 1$ (for the left hand sides of these equations are inverse to each other, as well as the right hand sides). Thus,

$$\tilde{\gamma}\colon \overline{H} \ni \overline{h} \mapsto \gamma^{-1}(1)\gamma(\overline{h}) \in A$$

is a π-cleft map satisfying $\tilde{\gamma}(1) = 1$.

Definition 5.9. Let B be an H'-comodule algebra, $\sigma\colon \overline{H} \otimes \overline{H} \to B$ and $\rightharpoonup\colon \overline{H} \otimes B \to B$ two k-linear maps. Assume that $1 \rightharpoonup b = b$ and $\overline{h} \rightharpoonup 1 = \varepsilon(h)1$ hold for all $h \in H$ and $b \in B$. The pair $(\rightharpoonup, \sigma)$ will be called π-crossed data if $B\#_\sigma\overline{H} := B \otimes \overline{H}$ is an H-comodule algebra with unit $1 \otimes 1$ when equipped with the algebra structure

$$(b\#\overline{g})(c\#\overline{h}) = b(\overline{g}_{(1)} \rightharpoonup c_{(0)})\sigma(\overline{g_{(2)}c_{(1)}}|\overline{h}_{(2)}) \otimes \overline{g_{(3)}c_{(2)}\pi^{-1}(h_{(1)})h_{(3)}} \tag{5.1}$$

We then call $B\#_\sigma\overline{H}$ a π-crossed product with action $\rightharpoonup$ and cocycle σ.

Proposition 5.10. *The following are equivalent for an H-comodule algebra A and $B = A^{\mathrm{co}\,\overline{H}}$:*

1. *A has a π-normal basis $\psi\colon B \otimes \overline{H} \to A$ with $\psi(1 \otimes 1) = 1$.*
2. *A is isomorphic to a π-crossed product $B\#_\sigma\overline{H}$.*

Proof. The implication (2)$\Rightarrow$(1) is obvious. For the reverse implication we may as well assume that $A = B \otimes \overline{H}$ with the given right H-coaction and a suitable algebra structure with unit $1 \otimes 1$. Define

$$\begin{aligned} \overline{h} \rightharpoonup b &:= (\mathrm{id} \otimes \varepsilon)\left((1 \otimes \overline{h})(b \otimes 1)\right) \\ \sigma(\overline{g}|\overline{h}) &:= (\mathrm{id} \otimes \varepsilon)\left((1 \otimes \overline{g})(1 \otimes \overline{h})\right). \end{aligned}$$

It remains to show that with this pair $(\rightharpoonup, \sigma)$ the multiplication in $B \otimes \overline{H}$ takes the form described in the definition of a crossed product. Note first that

$$(\mathrm{id}_B \otimes \varepsilon \otimes \mathrm{id}_{\overline{H}})\overline{\rho} = \mathrm{id}_{B\otimes\overline{H}},$$

where $\overline{\rho}$ denotes the $\overline{H}$-comodule structure on $B \otimes \overline{H}$. Thus, for $b, c \in B$ and $g, h \in H$ we have

$$\begin{aligned} (1 \otimes \overline{g})(c \otimes 1) &= (\mathrm{id}_B \otimes \varepsilon \otimes \mathrm{id}_{\overline{H}})\overline{\rho}\left((1 \otimes \overline{g})(c \otimes 1)\right) \\ &= (\mathrm{id}_B \otimes \varepsilon \otimes \mathrm{id}_{\overline{H}})\left((1 \otimes \overline{g_{(2)}})(c_{(0)} \otimes 1) \otimes \overline{\pi^{-1}(g_{(1)})g_{(3)}c_{(1)}}\right) \\ &= \overline{g_{(1)}} \rightharpoonup c_{(0)} \otimes \overline{g_{(2)}c_{(1)}} \end{aligned}$$

and

$$\begin{aligned}
&(1\otimes\overline{g})(1\otimes\overline{h})\\
&\qquad=(\mathrm{id}_B\otimes\varepsilon\otimes\mathrm{id}_{\overline{H}})\overline{\rho}\left((1\otimes\overline{g})(1\otimes\overline{h})\right)\\
&\qquad=(\mathrm{id}_B\otimes\varepsilon\otimes\mathrm{id}_{\overline{H}})\left((1\otimes\overline{g_{(2)}})(1\otimes\overline{h_{(2)}})\otimes\overline{\pi^{-1}(g_{(1)})g_{(3)}\pi^{-1}(h_{(1)})h_{(3)}}\right)\\
&\qquad=\sigma(\overline{g_{(1)}}|\overline{h_{(2)}})\otimes\overline{g_{(2)}\pi^{-1}(h_{(1)})h_{(3)}},
\end{aligned}$$

and, combining these,

$$\begin{aligned}
(b\otimes\overline{g})(c\otimes\overline{h})&=(b\otimes 1)(1\otimes\overline{g})(c\otimes 1)(1\otimes\overline{h})\\
&=(b\otimes 1)(\overline{g_{(1)}}\rightharpoonup c_{(0)}\otimes\overline{g_{(2)}c_{(1)}})(1\otimes\overline{h})\\
&=(b(\overline{g_{(1)}}\rightharpoonup c_{(0)})\otimes 1)(1\otimes\overline{g_{(2)}c_{(1)}})(1\otimes\overline{h})\\
&=(b(\overline{g_{(1)}}\rightharpoonup c_{(0)})\otimes 1)(\sigma(\overline{g_{(2)}c_{(1)}}|\overline{h_{(2)}})\otimes\overline{g_{(3)}c_{(2)}\pi^{-1}(h_{(1)})h_{(3)}}\\
&=b(\overline{g_{(1)}}\rightharpoonup c_{(0)})\sigma(\overline{g_{(2)}c_{(1)}}|\overline{h_{(2)}})\otimes\overline{g_{(3)}c_{(2)}\pi^{-1}(h_{(1)})h_{(3)}}\quad\square
\end{aligned}$$

Example 5.11. Recall that H itself has a π-normal basis. The corresponding action $\rightharpoonup$ and cocycle τ are given by $\overline{h}\rightharpoonup x=\pi^{-1}(h_{(1)})\pi(h_{(2)}x)$ and $\tau(\overline{g}|\overline{h})=\pi^{-1}(g_{(1)})\pi(g_{(2)}\pi^{-1}(h_{(1)})h_{(2)})$.

In this example, the formula (5.1) for multiplication in $H'\otimes\overline{H}\cong H$ does not really break down the multiplicative structure of H into two pieces, like one would expect from a construction like a crossed product. In fact, (5.1) *uses* multiplication in H, and not only structures pertaining to $\overline{H}$, H' and B to define a multiplication on $B\#_\sigma\overline{H}$. To amend this, we need one more structure on $\overline{H}$: We define a (nonassociative) multiplication

$$\nabla\colon\overline{H}\otimes\overline{H}\ni p\otimes q\mapsto p\bullet q:=p\varphi(q)\in\overline{H}\tag{5.2}$$

on $\overline{H}$, that is, $\overline{g}\bullet\overline{h}=\overline{g\pi^{-1}(h_{(1)})h_{(2)}}$.

Then multiplication in a crossed product $B\#_\sigma\overline{H}$ takes the form

$$(b\#p)(c\#q)=b(p_{(1)}\rightharpoonup c_{(0)})\sigma(p_{(2)}\leftharpoonup c_{(1)}|q_{(1)[0]})\#(p_{(3)}\leftharpoonup c_{(2)}q_{(1)[1]})\bullet q_{(2)}\tag{5.3}$$

for $b,c\in B$ and $p,q\in\overline{H}$. In fact

$$\begin{aligned}b(\overline{g}_{(1)}\rightharpoonup c_{(0)})\sigma(\overline{g}_{(2)}\leftharpoonup c_{(1)}|\overline{h}_{(1)[0]})\#(\overline{g}_{(3)}\leftharpoonup c_{(2)}\overline{g}_{(1)[1]})\bullet\overline{h}_{(2)}\\ =b(\overline{g_{(1)}}\rightharpoonup c_{(0)})\sigma(\overline{g_{(2)}c_{(1)}}|\overline{h_{(2)}})\#\overline{g_{(3)}c_{(2)}\pi^{-1}(g_{(1)})\pi(g_{(3)})\varphi(\overline{g_{(4)}})}\\ =b(\overline{g_{(1)}}\rightharpoonup c_{(0)})\sigma(\overline{g_{(2)}c_{(1)}}|\overline{h_{(2)}})\#\overline{g_{(3)}c_{(2)}\pi^{-1}(g_{(1)})g_{(3)}}.\end{aligned}$$

In the special case of H as a π-crossed product

$$(x\#p)(y\#q)=x(p_{(1)}\rightharpoonup y_{(1)})\tau(p_{(2)}\leftharpoonup y_{(2)}|q_{(1)[0]})\#(p_{(3)}\leftharpoonup y_{(2)}q_{(1)[1]})\bullet q_{(2)} \tag{5.4}$$

for $x,y\in H'$ and $p,q\in\overline{H}$.

It is easy to check that the comultiplication induced in $H'\#_\tau\overline{H}$ via the isomorphism with H takes the form of a cosemidirect product

$$\Delta(x\otimes p)=x_{(1)}\otimes p_{(1)[0]}\otimes x_{(2)}p_{(1)[1]}\otimes p_{(2)}. \tag{5.5}$$

An important special case is that π is a projection in the sense of [9]. Then both $p\rightharpoonup x=\varepsilon(p)x$ and $\tau(p|q)=\varepsilon(p)\varepsilon(q)$ for $p,q\in\overline{H}$ and $x\in H'$ and hence

$$(x\#p)(y\#q)=xy_{(1)}\#(p\leftharpoonup y_{(2)})\bullet q$$

holds for multiplication in $H'\#_\tau\overline{H}\cong H$. From [9] we recall that this implies that $\overline{H}$ with multiplication $\bullet$ is a bialgebra in the category of Yetter-Drinfeld modules over H'.

Lemma 5.12. *Let R be a bialgebra, C a right R-module coalgebra with module structure denoted by $\leftharpoonup$ and D a right R-comodule coalgebra with comodule structure denoted by $d\mapsto d_{[0]}\otimes d_{[1]}$.*

Then $C\tilde{\otimes}D:=C\otimes D$ is a coalgebra with comultiplication

$$\Delta(c\otimes d)=c_{(1)}\otimes d_{(1)[0]}\otimes c_{(2)}\leftharpoonup d_{(1)[1]}\otimes d_{(2)}$$

and counit $\varepsilon_C\otimes\varepsilon_D$.

Proof. Formally dual to the construction of a generalized smash product algebra from a module algebra and a comodule algebra in [1] □

Theorem 5.13. *Let H be a Hopf algebra with a weak projection $\pi\colon H\to H'\subset H$ and $\overline{H}:=H/(H'^+H)$.*

The following are equivalent for an H-comodule algebra A and $B:=A^{\mathrm{co}\,\overline{H}}$:

(1) *A/B is $\overline{H}$-Galois with a π-normal basis.*

(2) *A is a π-cleft extension of B.*
(3) *A is isomorphic to a π-crossed product $B\#_\sigma\overline{H}$ such that the cocycle σ is invertible in* $\mathrm{Hom}(\overline{H}\tilde{\otimes}\overline{H}, B)$.

Remark 5.14. The proof of the Theorem will not use directly the assumption that H is a Hopf algebra, but the following consequence of it: φ is convolution invertible, hence $\beta_{\overline{H}} = \tilde{T}^{\overline{H},\overline{H}}_{H,\overline{H}}(\varphi)$ is a bijection. Explicitly $\beta_{\overline{H}}(p\otimes q) = p\varphi(q_{(1)})\otimes q_{(2)} = p\bullet q_{(1)}\otimes q_{(2)}$ or $\beta_{\overline{H}}(\overline{g}\otimes\overline{h}) = \overline{g\pi^{-1}(h_{(1)})h_{(2)}}\otimes\overline{h_{(3)}}$.

Proof. We have already proved the equivalence of (1) and (2). For the equivalence with (3) we may, by Proposition 5.7, Remark 5.8 and Proposition 5.10, already assume that $A = B\#_\sigma\overline{H}$ with some crossed data $(\rightharpoonup, \sigma)$; we have to show that the relative Galois map $\overline{\beta}_A$ is a bijection if and only if σ is convolution invertible.

Let $\alpha\colon B\otimes\overline{H}\otimes\overline{H}\to A\otimes_B A$ denote the canonical isomorphism.

By Lemma 2.2 it is sufficient to prove the equation $\overline{\beta}_A\alpha = (B\otimes\beta_{\overline{H}})T_B^{\overline{H}\tilde{\otimes}\overline{H}}(\sigma)$. We calculate

$$\begin{aligned}
\overline{\beta}_A\alpha(b\otimes p\otimes q) &= \overline{\beta}_A(b\otimes p\otimes 1\otimes q)\\
&= (b\otimes p)(1\otimes q_{(1)})\otimes q_{(2)}\\
&= b\sigma(p_{(1)}|q_{(1)[0]})\otimes(p_{(2)}\leftharpoonup q_{(1)[1]})\bullet q_{(2)}\otimes q_{(3)}\\
&= b\sigma(p_{(1)}|q_{(1)[0]})\otimes\beta_{\overline{H}}(p_{(2)}\leftharpoonup q_{(1)[1]}\otimes q_{(2)})\\
&= (B\otimes\beta_{\overline{H}})T_B^{\overline{H}\tilde{\otimes}\overline{H}}(\sigma)(b\otimes p\otimes q)\quad\square
\end{aligned}$$

Corollary 5.15 (of the proof). *Assume that $A = B\#_\sigma\overline{H}$ is a π-crossed product with invertible cocycle σ.*

Define $\overline{S}\colon\overline{H}\to\overline{H}$ by $\overline{S}(\overline{h}) = \overline{\varphi^{-1}(\overline{h})} = \overline{S(h_{(1)})\pi(h_{(2)})}$.

Then the convolution inverse for the π-cleft map

$$\gamma\colon\overline{H}\ni p\mapsto 1\#p\in B\#_\sigma\overline{H}$$

is given by

$$\gamma^{-1}(p) = \sigma^{-1}(\overline{S}(p_{(1)})_{(1)}|p_{(2)[0]})\#\overline{S}(p_{(1)})_{(2)}\leftharpoonup p_{(2)[1]}.$$

Proof. Note first that $\beta^{-1}_{\overline{H}}(1\otimes p) = \overline{S}(p_{(1)})\otimes p_{(2)}$. Combining $T_A^{\overline{H}}(\gamma) = \overline{\beta}_A\alpha$ from the proof of Proposition 4.4 with $\overline{\beta}_A\alpha = (B\otimes\beta_{\overline{H}})T_B^{\overline{H}\tilde{\otimes}\overline{H}}(\sigma)$ from the proof of

Theorem 5.13 we obtain $T_A^{\overline{H}}(\gamma) = (B \otimes \beta_{\overline{H}})T_B^{\overline{H}\tilde{\otimes}\overline{H}}(\sigma)$, hence

$$\begin{aligned}
\gamma^{-1}(p) &= \left(T_A^{\overline{H}}\right)^{-1}\left(T_B^{\overline{H}\tilde{\otimes}\overline{H}}(\sigma^{-1})(B \otimes \beta_{\overline{H}}^{-1})\right)(p)\\
&= (\mathrm{id}_A \otimes \varepsilon)T_B^{\overline{H}\tilde{\otimes}\overline{H}}(\sigma^{-1})(B \otimes \beta_{\overline{H}}^{-1})(1 \otimes 1 \otimes p) = (\mathrm{id} \otimes \varepsilon)T_B^{\overline{H}\tilde{\otimes}\overline{H}}(\sigma^{-1})(1 \otimes \overline{S}(p_{(1)}) \otimes p_{(2)})\\
&= (\mathrm{id} \otimes \varepsilon)(\sigma^{-1}(\overline{S}(p_{(1)})_{(1)} \otimes p_{(2)[0]}) \otimes \overline{S}(p_{(1)})_{(2)} \leftharpoonup h_{(2)[1]} \otimes h_{(3)}\\
&= \sigma^{-1}(\overline{S}(p_{(1)})_{(1)} \otimes p_{(2)[0]})\#\overline{S}(p_{(1)})_{(2)} \leftharpoonup p_{(2)[1]} \quad \square
\end{aligned}$$

Note that in the special case that π is a projection, $\overline{S}$ is the antipode of the Yetter-Drinfeld module Hopf algebra $\overline{H}$.

6. Cocycle identities

In our treatment of π-crossed products so far we have not discussed the conditions that one has to impose on $\rightharpoonup$ and σ to ensure that they are crossed data in the sense of Definition 5.9.

Lemma 6.1. *Let B be an H'-comodule algebra. Let $\sigma\colon \overline{H} \otimes \overline{H} \to B$ and $\rightharpoonup\colon \overline{H} \otimes B \to B$ be k-linear maps such that $1 \rightharpoonup b = 1$ and $\overline{h} \rightharpoonup 1 = \varepsilon(h)$ for all $h \in H$ and $b \in B$.*

Then $(\rightharpoonup, \sigma)$ are crossed data if and only if the following conditions are satisfied:

$$\rho'(\overline{h} \rightharpoonup b) = \overline{g}_{(2)} \rightharpoonup c_{(0)} \otimes \pi^{-1}(g_{(1)})\pi(g_{(3)}c_{(1)}) \tag{6.1}$$

$$\rho'\sigma(\overline{g}|\overline{h}) = \sigma(\overline{g}_{(2)}|\overline{h}_{(2)}) \otimes \pi^{-1}(g_{(1)})\pi(g_{(3)}\pi^{-1}(h_{(1)})h_{(3)}) \tag{6.2}$$

$$\overline{h} \rightharpoonup bc = (\overline{h}_{(1)} \rightharpoonup b_{(0)})(\overline{h_{(2)}b_{(1)}} \rightharpoonup c) \tag{6.3}$$

$$\sigma(\overline{h}|1) = \sigma(1|\overline{h}) = \varepsilon(h) \tag{6.4}$$

$$\begin{aligned}
&(\overline{f}_{(1)} \rightharpoonup \sigma(\overline{g}_{(2)}|\overline{h}_{(3)}))\sigma(\overline{f_{(2)}\pi^{-1}(g_{(1)})\pi(g_{(3)}\pi^{-1}(h_{(2)})h_{(4)})}|\overline{g_{(4)}\pi^{-1}(h_{(1)})h_{(5)}})\\
&\qquad = \sigma(\overline{f}_{(1)}|\overline{g}_{(2)})\sigma(\overline{f_{(2)}\pi^{-1}(g_{(1)})g_{(3)}}|h)
\end{aligned} \tag{6.5}$$

$$\begin{aligned}
&\sigma(\overline{g}_{(1)}|\overline{h}_{(2)})(\overline{g_{(2)}\pi^{-1}(h_{(1)})h_{(3)}} \rightharpoonup b)\\
&\qquad = (\overline{g}_{(1)} \rightharpoonup \overline{h}_{(2)} \rightharpoonup b_{(0)})\sigma(\overline{g_{(2)}\pi^{-1}(h_{(1)})\pi(h_{(3)}b_{(1)})}|\overline{h_{(4)}b_{(2)}})
\end{aligned} \tag{6.6}$$

Proof. We define a (not necessarily associative nor H-colinear) multiplication $\nabla\colon B\#\overline{H} \otimes B\#\overline{H} \to B\#\overline{H}$ by the formula in the definition of a crossed product and omit checking that (6.4) is satisfied if and only if $1\#1$ acts as a unit element for ∇.

We first examine the conditions for multiplication being H-colinear. This is equivalent to being $\overline{H}$-colinear as well as H'-colinear. We note first that multiplication is always $\overline{H}$-colinear:

$$
\begin{aligned}
&(b\#\overline{g})_{(0)}(c\#\overline{h})_{(0)} \otimes \overline{(b\#\overline{g})_{(1)}(c\#\overline{h})_{(1)}} \\
&\quad = (b_{(0)}\#\overline{g}_{(2)})(c_{(0)}\#\overline{h}_{(2)}) \otimes \overline{b_{(1)}\pi^{-1}(g_{(1)})g_{(3)}c_{(1)}\pi^{-1}(h_{(1)})h_{(3)}} \\
&\quad = (b\#\overline{g}_{(1)})(c_{(0)}\#\overline{h}_{(2)}) \otimes \overline{g_{(2)}c_{(1)}\pi^{-1}(h_{(1)})h_{(3)}} \\
&\quad = b(\overline{g}_{(1)} \rightharpoonup c_{(0)})\sigma(\overline{g_{(2)}c_{(1)}}|\overline{h}_{(3)})\#\overline{g_{(3)}c_{(2)}\pi^{-1}(h_{(2)})h_{(4)}} \otimes \overline{g_{(4)}c_{(3)}\pi^{-1}(h_{(1)})h_{(5)}} \\
&\quad = \overline{\rho}\left(b(\overline{g}_{(1)} \rightharpoonup c_{(0)})\sigma(\overline{g_{(2)}c_{(1)}}|\overline{h}_{(2)})\#\overline{g_{(3)}c_{(2)}\pi^{-1}(h_{(1)})h_{(3)}}\right) \\
&\quad = \overline{\rho}\left((b\#\overline{g})(c\#\overline{h})\right)
\end{aligned}
$$

Assuming (6.1) and (6.2), we can also show that ∇ is H'-colinear: We can rewrite (6.1) as

$$(\overline{g}_{(1)} \rightharpoonup c_{(0)})_{(0)} \otimes (\overline{g}_{(1)} \rightharpoonup c_{(0)})_{(1)}\pi^{-1}(g_{(2)}c_{(1)}) = \overline{g}_{(2)} \rightharpoonup c \otimes \pi^{-1}(g_{(1)})$$

and (6.2) as

$$\sigma(\overline{g}_{(1)}|\overline{h}_{(2)})_{(0)} \otimes \sigma(\overline{g}_{(1)}|\overline{h}_{(2)})_{(1)}\pi^{-1}(g_{(2)}\pi^{-1}(h_{(1)})h_{(3)}) = \sigma(\overline{g}_{(2)}|\overline{h}) \otimes \pi^{-1}(g_{(2)})$$

and then calculate

$$
\begin{aligned}
&\rho'\left((b\#\overline{g})(c\#\overline{h})\right) \\
&\quad = \rho'\left(b(\overline{g}_{(1)} \rightharpoonup c_{(0)})\sigma(\overline{g_{(2)}c_{(1)}}|\overline{h}_{(2)})\#\overline{g_{(3)}c_{(2)}\pi^{-1}(h_{(1)})h_{(3)}}\right) \\
&\quad = b_{(0)}(\overline{g}_{(1)} \rightharpoonup c_{(0)})_{(0)}\sigma(\overline{g_{(2)}c_{(1)}}|\overline{h}_{(4)})_{(0)}\#\overline{g_{(4)}c_{(3)}\pi^{-1}(h_{(2)})h_{(6)}} \\
&\qquad\quad \otimes b_{(1)}(\overline{g}_{(1)} \rightharpoonup c_{(0)})_{(1)}\sigma(\overline{g_{(2)}c_{(1)}}|\overline{h}_{(4)})_{(1)} \\
&\qquad\qquad \pi^{-1}(g_{(3)}c_{(2)}\pi^{-1}(h_{(3)})h_{(5)})\pi(g_{(5)}c_{(4)}\pi^{-1}(h_{(1)})h_{(7)}) \\
&\quad = b_{(0)}(\overline{g}_{(1)} \rightharpoonup c_{(0)})_{(0)}\sigma(\overline{g_{(3)}c_{(2)}}|\overline{h}_{(3)})\#\overline{g_{(4)}c_{(3)}\pi^{-1}(h_{(2)})h_{(4)}} \\
&\qquad\quad \otimes b_{(1)}(\overline{g}_{(1)} \rightharpoonup c_{(0)})_{(1)}\pi^{-1}(g_{(2)}c_{(1)})\pi(g_{(5)}c_{(4)}\pi^{-1}(h_{(1)})h_{(5)}) \\
&\quad = b_{(0)}(\overline{g}_{(2)} \rightharpoonup c_{(0)})\sigma(\overline{g_{(3)}c_{(1)}}|\overline{h}_{(3)})\#\overline{g_{(4)}c_{(2)}\pi^{-1}(h_{(2)})h_{(4)}} \\
&\qquad\quad \otimes b_{(1)}\pi^{-1}(g_{(1)})\pi(g_{(5)}c_{(3)}\pi^{-1}(h_{(1)})h_{(5)}) \\
&\quad = (b_{(0)}\#\overline{g}_{(2)})(c_{(0)}\#\overline{h}_{(2)}) \otimes \pi\left(b_{(1)}\pi^{-1}(g_{(1)})g_{(3)}c_{(1)}\pi^{-1}(h_{(1)})h_{(3)}\right) \\
&\quad = (b\#\overline{g})_{(0)}(c\#\overline{h})_{(0)} \otimes \pi((b\#\overline{g})_{(1)}(c\#\overline{h})_{(1)})
\end{aligned}
$$

On the other hand, the assumption that ∇ be H'-colinear implies (6.1) and (6.2). In fact note first that

$$\begin{aligned}(\mathrm{id}_B\otimes\varepsilon\otimes\pi)\rho(b\#\overline{g}) &= (\mathrm{id}_B\otimes\varepsilon\otimes\pi)(b_{(0)}\#\overline{g}_{(2)}\otimes b_{(1)}\pi^{-1}(g_{(1)})g_{(3)})\\ &= b_{(0)}\otimes b_{(1)}\varepsilon(g)=\varepsilon(g)\rho'(b)\end{aligned}$$

holds for $b\in B$ and $g\in H$. Now colinearity of ∇ implies (6.1) by the calculation

$$\begin{aligned}\rho'(\overline{g}\rightharpoonup c) &= (\mathrm{id}_B\otimes\varepsilon\otimes\pi)\rho\left(\overline{g}_{(1)}\rightharpoonup c_{(0)}\#\overline{g_{(2)}c_{(1)}}\right) = (\mathrm{id}_B\otimes\varepsilon\otimes\pi)\rho\left((1\#\overline{g})(c\#1)\right)\\ &= (\mathrm{id}_B\otimes\varepsilon)\left((1\#\overline{g})_{(0)}(c\#1)_{(0)}\right)\otimes\pi\left((1\#\overline{g})_{(1)}(c\#1)_{(1)}\right)\\ &= (\mathrm{id}_B\otimes\varepsilon)\left((1\#\overline{g}_{(2)})(c_{(0)}\#1)\right)\otimes\pi(\pi^{-1}(g_{(1)})g_{(3)}c_{(1)})\\ &= \overline{g}_{(2)}\rightharpoonup c_{(0)}\otimes\pi^{-1}(g_{(1)})\pi(g_{(3)}c_{(1)})\end{aligned}$$

and it also implies (6.2) by the calculation

$$\begin{aligned}\rho'(\sigma(\overline{g}|\overline{h})). &= (\mathrm{id}_B\otimes\varepsilon\otimes\pi)\rho\left(\sigma(\overline{g}_{(1)}|\overline{h}_{(2)})\#\overline{g_{(2)}\pi^{-1}(h_{(1)})h_{(3)}}\right)\\ &= (\mathrm{id}_B\otimes\varepsilon\otimes\pi)\rho\left((1\#\overline{g})(1\#\overline{h})\right)\\ &= (\mathrm{id}_B\otimes\varepsilon)\left((1\#\overline{g})_{(0)}(1\#\overline{h})_{(0)}\right)\otimes\pi\left((1\#\overline{g})_{(1)}(1\#\overline{h})_{(1)}\right)\\ &= (\mathrm{id}_B\otimes\varepsilon)\left((1\#\overline{g}_{(2)})(1\#\overline{h}_{(2)})\right)\otimes\pi\left(\pi^{-1}(g_{(1)})g_{(3)}\pi^{-1}(h_{(1)})h_{(3)}\right)\\ &= \sigma(g_{(2)}|h_{(2)})\otimes\pi^{-1}(g_{(1)})\pi(g_{(3)}\pi^{-1}(h_{(1)})h_{(3)}\end{aligned}$$

For the rest of the proof we shall assume that equations (6.1) and (6.2) are satisfied.

For $b,c,d\in B$ and $g,h\in H$ we have

$$\begin{aligned}((b\#\overline{g})(c\#1))\,(d\#\overline{h}) &= \left(b(\overline{g}_{(1)}\rightharpoonup c_{(0)})\#\overline{g_{(2)}c_{(1)}}\right)(d\#\overline{h})\\ &= b(\overline{g}_{(1)}\rightharpoonup c_{(0)})(\overline{g_{(2)}c_{(1)}}\rightharpoonup d_{(0)})\sigma(\overline{g_{(3)}c_{(2)}d_{(1)}}|\overline{h}_{(2)})\#\overline{g_{(4)}c_{(3)}d_{(2)}\pi^{-1}(h_{(1)})h_{(3)}}\end{aligned}$$

and

$$\begin{aligned}(b\#\overline{g})\left((c\#1)(d\#\overline{h})\right) &= (b\#\overline{g})(cd\#\overline{h})\\ &= b(\overline{g}_{(1)}\rightharpoonup c_{(0)}d_{(0)})\sigma(\overline{g_{(2)}c_{(1)}d_{(1)}}|\overline{h}_{(2)})\#\overline{g_{(3)}c_{(2)}d_{(2)}\pi^{-1}(h_{(1)})h_{(3)}}.\end{aligned}$$

We conclude that $(b\#\overline{g})((c\#1)(d\#\overline{h}))=((b\#\overline{g})(c\#1))(d\#\overline{h})$ holds for all $b,c,d\in B$ and $g,h\in H$ if and only if (6.3) holds; for the only if part, we just have to specialize $b=1$, $h=1$ and apply ε to the second tensorand.

For the rest of the proof we shall assume that (6.3) is satisfied. Then we know that $B\#_\sigma\overline{H}$ is a right B-module by $(b\#\overline{h})c = (b\#\overline{h})(c\#1)$, and that ∇ induces a left B-linear map $\tilde{\nabla}\colon (B\#\overline{H}) \otimes_B (B\#\overline{H}) \to B\#\overline{H}$.

If ∇ makes $B\#\overline{H}$ an associative algebra, then $\tilde{\nabla}$ is also a right B-module map. Now $\tilde{\nabla}$ is a left B-module map and we have an isomorphism

$$B \otimes \overline{H} \otimes \overline{H} \ni b \otimes \overline{g} \otimes \overline{h} \mapsto b\#\overline{g} \otimes 1\#\overline{h} \in (B\#\overline{H}) \otimes_B (B\#\overline{H}).$$

Thus $\tilde{\nabla}$ is right B-linear if and only if, for all $g, h \in H$ and $b \in B$, we have

$$((1\#\overline{g})(1\#\overline{h}))(b\#1) = (1\#\overline{g})((1\#\overline{h})(b\#1).$$

Now

$$\begin{aligned}
&\Big((1\#\overline{g})(1\#\overline{h})\Big)\,(b\#1)\\
&= \Big(\sigma(\overline{g}_{(1)}|\overline{h}_{(2)})\#\overline{g_{(2)}\pi^{-1}(h_{(1)})h_{(3)}}\Big)\,(b\#1)\\
&= \sigma(\overline{g}_{(1)}|\overline{h}_{(3)})(\overline{g_{(2)}\pi^{-1}(h_{(2)})h_{(4)}} \rightharpoonup b_{(0)})\#\overline{g_{(3)}\pi^{-1}(h_{(1)})h_{(5)}b_{(1)}}
\end{aligned}$$

and

$$\begin{aligned}
&(1\#\overline{g})\,\Big((1\#\overline{h})(b\#1)\Big)\\
&\quad= (1\#\overline{g})\,\Big(\overline{h}_{(1)} \rightharpoonup b_{(0)}\#\overline{h_{(2)}b_{(1)}}\Big)\\
&\quad= \Big(\overline{g}_{(1)} \rightharpoonup (\overline{h}_{(1)} \rightharpoonup b_{(0)})_{(0)}\Big)\,\sigma(\overline{g_{(2)}(\overline{h}_{(1)} \rightharpoonup b_{(0)})_{(1)}}|\overline{h_{(3)}b_{(2)}})\\
&\qquad\quad \#\overline{g_{(3)}(\overline{h}_{(1)} \rightharpoonup b_{(0)})_{(2)}\pi^{-1}(h_{(2)}b_{(1)})h_{(4)}b_{(3)}}\\
&\quad= (\overline{g}_{(1)} \rightharpoonup \overline{h}_{(3)} \rightharpoonup b_{(0)})\sigma(\overline{g_{(2)}\pi^{-1}(h_{(2)})\pi(h_{(4)}b_{(1)})}|\overline{h_{(7)}b_{(4)}})\\
&\qquad\quad \#\overline{g_{(3)}\pi^{-1}(h_{(1)})\pi(h_{(5)}b_{(2)})\pi^{-1}(h_{(6)}b_{(3)})h_{(8)}b_{(5)}}\\
&\quad= (\overline{g}_{(1)} \rightharpoonup \overline{h}_{(3)} \rightharpoonup b_{(0)})\sigma(\overline{g_{(2)}\pi^{-1}(h_{(2)})\pi(h_{(4)}b_{(1)})}|\overline{h_{(5)}b_{(2)}})\\
&\qquad\quad \#\overline{g_{(3)}\pi^{-1}(h_{(1)})h_{(6)}b_{(3)}}.
\end{aligned}$$

We see that $((1\#\overline{g})(1\#\overline{h}))(b\#1) = (1\#\overline{g})((1\#\overline{h})(b\#1)$ is implied by (6.6); the converse implication follows by applying ε to the second tensorand. We shall assume that (6.6) is satisfied for the rest of the proof.

Now $\tilde{\nabla}$ is left B-linear and we have an isomorphism

$$B\otimes\overline{H}\otimes\overline{H}\otimes\overline{H} \ni b\otimes\overline{f}\otimes\overline{g}\otimes\overline{h} \mapsto b\#\overline{f}\otimes 1\#\overline{g}\otimes 1\#\overline{h} \in (B\#\overline{H})\otimes_B(B\#\overline{H})\otimes_B(B\#\overline{H}).$$

Thus ∇ is associative if and only if we have, for all $f, g, h \in H$, the equality $((1\#\overline{f})(1\#\overline{g}))(1\#\overline{h}) = (1\#\overline{f})((1\#\overline{g})(1\#\overline{h}))$. Let $f, g, h \in H$. Then we have

$$\begin{aligned}\left((1\#\overline{f})(1\#\overline{g})\right)(1\#\overline{h}) &= \left(\sigma(\overline{f}_{(1)}|\overline{g}_{(1)})\#\overline{f_{(2)}\pi^{-1}(g_{(1)})g_{(3)}}\right)(1\#\overline{h})\\ &= \sigma(\overline{f}_{(1)}|\overline{g}_{(3)})\sigma(\overline{f_{(2)}\pi^{-1}(g_{(2)})g_{(4)}}|\overline{h}_{(2)})\#\overline{f_{(3)}\pi^{-1}(g_{(1)})g_{(5)}\pi^{-1}(h_{(1)})h_{(3)}}\end{aligned}$$

and

$$\begin{aligned}&(1\#\overline{f})\left((1\#\overline{g})(1\#\overline{h})\right)\\ &= (1\#\overline{f})\left(\sigma(\overline{g}_{(1)}|\overline{h}_{(2)})\#\overline{g_{(2)}\pi^{-1}(h_{(1)})h_{(3)}}\right)\\ &= (\overline{f}_{(1)} \rightharpoonup \sigma(\overline{g}_{(1)}|\overline{h}_{(4)})_{(0)})\sigma(\overline{f}_{(2)}\sigma(\overline{g}_{(1)}|\overline{h}_{(4)})_{(1)}|\overline{g_{(3)}\pi^{-1}(h_{(2)})h_{(6)}})\\ &\quad \#\overline{f_{(3)}\sigma(\overline{g}_{(1)}|\overline{h}_{(4)})_{(2)}\pi^{-1}(g_{(2)}\pi^{-1}(h_{(3)})h_{(5)})g_{(4)}\pi^{-1}(h_{(1)})h_{(7)}}\\ &= (\overline{f}_{(1)} \rightharpoonup \sigma(\overline{g}_{(3)}|\overline{h}_{(6)}))\sigma(\overline{f_{(2)}\pi^{-1}(g_{(2)})\pi(g_{(4)}\pi^{-1}(h_{(5)})h_{(7)})}|\overline{g_{(7)}\pi^{-1}(h_{(2)})h_{(10)}})\\ &\quad \#\overline{f_{(3)}\pi^{-1}(g_{(1)})\pi(g_{(5)}\pi^{-1}(h_{(4)})h_{(8)})\pi^{-1}(g_{(6)}\pi^{-1}(h_{(3)})h_{(9)})g_{(8)}\pi^{-1}(h_{(1)})h_{(11)}}\\ &= (\overline{f}_{(1)} \rightharpoonup \sigma(\overline{g}_{(3)}|\overline{h}_{(4)}))\sigma(\overline{f_{(2)}\pi^{-1}(g_{(2)})\pi(g_{(4)}\pi^{-1}(h_{(3)})h_{(5)})}|\overline{g_{(5)}\pi^{-1}(h_{(2)})h_{(6)}})\\ &\quad \#\overline{f_{(3)}\pi^{-1}(g_{(1)})g_{(6)}\pi^{-1}(h_{(1)})h_{(7)}}\end{aligned}$$

We see that $((1\#\overline{f})(1\#\overline{g}))(1\#\overline{h}) = (1\#\overline{f})((1\#\overline{g})(1\#\overline{h}))$ is implied by (6.5); the converse implication follows by applying ε to the second tensorand. $\square$

Remark 6.2. Under the assumption of (6.1) and (6.2), equations (6.6) and (6.5) are equivalent to

$$\begin{aligned}&\sigma(\overline{g_{(1)}}|\overline{h_{(2)}})(\overline{g_{(2)}\pi^{-1}(h_{(1)})h_{(3)}} \rightharpoonup b)\\ &\qquad (\overline{g_{(1)}} \rightharpoonup (\overline{h_{(1)}} \rightharpoonup b_{(0)})_{(0)})\sigma(\overline{g_{(2)}(\overline{h_{(1)}} \rightharpoonup b_{(0)})_{(1)}}|\overline{h_{(2)}b_{(1)}})\end{aligned} \tag{6.7}$$

and

$$\begin{aligned}&\sigma(\overline{f_{(1)}}|\overline{g_{(2)}})\sigma(\overline{f_{(2)}\pi^{-1}(g_{(1)})g_{(3)}}|\overline{h})\\ &\qquad = (\overline{f_{(1)}} \rightharpoonup \sigma(\overline{g_{(1)}}|\overline{h_{(2)}})_{(0)})\sigma(\overline{f_{(2)}\sigma(\overline{g_{(1)}}|\overline{h_{(2)}})_{(1)}}|\overline{g_{(2)}\pi^{-1}(h_{(1)})h_{(3)}}),\end{aligned} \tag{6.8}$$

respectively.

Finally we can rewrite the cocycle identities using only the operations $\rightharpoonup$, τ, $\bullet$ on $\overline{H}$ (and H') instead of operations in H as follows

Lemma 6.3. *Let B be an H'-comodule algebra, let $\rightharpoonup\colon \overline{H}\otimes B \to B$ and $:\overline{H}\otimes \overline{H} \to H'$ be linear maps. with $p \rightharpoonup 1 = \varepsilon(p)1$ and $1 \rightharpoonup b = b$ for all $p \in \overline{H}$ and $b \in B$. Then $(\rightharpoonup, \sigma)$ are π-crossed data if and only if*

$$\begin{aligned}
\rho'(p \rightharpoonup b) &= p_{(1)[0]} \rightharpoonup b_{(0)} \otimes p_{(1)[1]}(p_{(2)} \rightharpoonup b_{(1)})\\
\rho'\sigma(p|q) &= \sigma(p_{(1)[0]}|q_{(1)[0]}) \otimes p_{(1)[1]}(p_{(2)} \rightharpoonup q_{(1)[1]})\tau(p_{(3)} \leftharpoonup q_{(1)[2]}|q_{(2)})\\
\sigma(1|p) &= \sigma(p|1) = \varepsilon(p)1\\
p \rightharpoonup bc &= (p_{(1)} \rightharpoonup b_{(0)})((p_{(2)} \leftharpoonup b_{(1)}) \rightharpoonup c)
\end{aligned}$$

$$\begin{aligned}
&\sigma(p_{(1)}|q_{(1)[0]})\sigma((p_{(2)} \leftharpoonup q_{(1)[1]}) \bullet q_{(2)}|r)\\
&\qquad = (p_{(1)} \rightharpoonup \sigma(q_{(1)}|r_{(1)[0]})_{(0)})\sigma(p_{(2)} \leftharpoonup \sigma(q_{(1)}|r_{(1)[0]})_{(1)}|(q_{(2)} \leftharpoonup r_{(1)[1]}) \bullet r_{(2)})
\end{aligned}$$

$$\begin{aligned}
&(p_{(1)} \rightharpoonup (q_{(1)} \rightharpoonup b_{(0)})_{(0)})\sigma(p_{(2)} \leftharpoonup (q_{(1)} \rightharpoonup b_{(0)})_{(1)}|q_{(2)} \leftharpoonup b_{(1)})\\
&\qquad\qquad = \sigma(p_{(1)}|q_{(1)[0]})((p_{(2)} \leftharpoonup q_{(1)[1]}) \bullet q_{(2)} \rightharpoonup b)
\end{aligned}$$

hold for $b, c \in B$ and $p, q, r \in \overline{H}$.

This can be verified by just substituting the definitions of ω, τ and ∇.

Remark 6.4. If π is a Hopf algebra map, then the first two equations in Lemma 6.3 just say that both $\rightharpoonup\colon \overline{H}\otimes B \to B$ and $\sigma\colon \overline{H}\otimes\overline{H} \to B$ are H'-colinear maps.

In fact, in this case $\omega = \varepsilon$ and $\tau = \varepsilon$.

7. The structure of Hopf algebras with a weak projection

Assume that H is a bialgebra with a weak projection π onto the Hopf subalgebra H'. Put $Q := \overline{H}$. In the preceding section we have introduced maps

$$\omega\colon Q\otimes H' \ni q\otimes x \mapsto q \rightharpoonup x \in H' \tag{7.1}$$
$$\tau\colon Q\otimes Q \ni p\otimes q \mapsto \tau(p|q) \in H' \tag{7.2}$$
$$\nabla\colon Q\otimes Q \ni p\otimes q \mapsto p\bullet q \in Q \tag{7.3}$$

in addition to H'-module and H'-comodule coalgebra structures on Q. We have shown that $H \cong H'\#_\tau Q := H'\otimes Q$ with a certain multiplication and comultiplication expressed in terms of the above structures, namely

$$(x\#p)(y\#q) = x(p_{(1)} \rightharpoonup y_{(1)})\tau(p_{(2)} \leftharpoonup y_{(2)}|q_{(1)[0]})\#(p_{(3)} \leftharpoonup y_{(2)}q_{(1)[1]}) \bullet q_{(2)} \tag{7.4}$$

$$\Delta(x \otimes p) = x_{(1)} \otimes p_{(1)[0]} \otimes x_{(2)}p_{(1)[1]} \otimes p_{(2)}. \tag{7.5}$$

for $x, y \in H'$ and $p, q \in Q$.

It is a natural question what axioms for an H'-action and H'-coaction on a coalgebra Q and maps as in equations (7.1) through (7.3) are necessary and sufficient for the multiplication and comultiplication in (7.4) and (7.5) to define a bialgebra (or Hopf algebra) with unit $1\#1$ and counit $\varepsilon \otimes \varepsilon$. Note that when $H := H'\#_\tau Q$ is a bialgebra, then it has an (obvious) weak projection onto H', and $\overline{H} = Q$. Thus one can find necessary conditions by substituting the definitions of $\tau, \bullet, \rightharpoonup$ in the preceding section. In this way one finds the following list of axioms (7.6) through (7.27):

Q is an H'-module coalgebra, that is:

$$\Delta(q \leftharpoonup x) = q_{(1)} \leftharpoonup x_{(1)} \otimes q_{(2)} \leftharpoonup x_{(2)} \tag{7.6}$$

$$\varepsilon(q \leftharpoonup x) = \varepsilon(q)\varepsilon(x) \tag{7.7}$$

Q is an H'-comodule coalgebra, that is:

$$\Delta(q_{[0]}) \otimes q_{[1]} = q_{(1)[0]} \otimes q_{(2)[0]} \otimes q_{(1)[1]}q_{(2)[1]} \tag{7.8}$$

$$\varepsilon(q_{[0]})q_{[1]} = \varepsilon(q) \cdot 1 \tag{7.9}$$

$$(p_{(2)} \leftharpoonup x_{(2)})_{[0]} \otimes (p_{(1)} \rightharpoonup x_{(1)})(p_{(2)} \leftharpoonup x_{(2)})_{[1]} = p_{(1)[0]} \leftharpoonup x_{(1)} \otimes p_{(1)[1]}(p_{(2)} \rightharpoonup x_{(2)}) \tag{7.10}$$

$$\Delta(p \rightharpoonup x) = p_{(1)[0]} \rightharpoonup x_{(1)} \otimes p_{(1)[1]}(p_{(2)} \rightharpoonup x_{(2)}) \tag{7.11}$$

$$\varepsilon(p \rightharpoonup x) = \varepsilon(p)\varepsilon(x) \tag{7.12}$$

$$\Delta(\tau(p|q)) = \tau(p_{(1)[0]}|q_{(1)[0]}) \otimes p_{(1)[1]}(p_{(2)} \rightharpoonup q_{(1)[1]})\tau(p_{(3)} \leftharpoonup q_{(1)[2]}|q_{(2)}) \tag{7.13}$$

$$\varepsilon(\tau(p|q)) = \varepsilon(p)\varepsilon(q) \tag{7.14}$$

$$((p_{(2)} \leftharpoondown q_{(1)[1]}) \bullet q_{(2)})_{[0]} \otimes \tau(p_{(1)}|q_{(1)[0]})((p_{(2)} \leftharpoondown q_{(1)[1]}) \bullet q_{(2)})_{[1]}$$
$$= p_{(1)[0]} \bullet q_{(1)[0]} \otimes p_{(1)[1]}(p_{(2)} \rightharpoonup q_{(1)[1]})\tau(p_{(3)} \leftharpoondown q_{(1)[2]}|q_{(2)}) \quad (7.15)$$

$$(p \bullet q) \leftharpoondown x = (p \leftharpoondown (q_{(1)} \rightharpoonup x_{(1)})) \bullet (q_{(2)} \leftharpoondown x_{(2)}) \quad (7.16)$$
$$1 \leftharpoondown x = \varepsilon(x)1 \quad (7.17)$$

$$\Delta(p \bullet q) = p_{(1)} \bullet q_{(1)[0]} \otimes (p_{(2)} \leftharpoondown q_{(1)[1]}) \bullet q_{(2)} \quad (7.18)$$
$$\varepsilon(p \bullet q) = \varepsilon(p)\varepsilon(q) \quad (7.19)$$

$$(p \bullet q) \bullet r = (p \leftharpoondown \tau(q_{(1)}|r_{(1)[0]})) \bullet ((q_{(2)} \leftharpoondown r_{(1)[1]}) \bullet r_{(2)}) \quad (7.20)$$
$$1 \bullet q = q \bullet 1 = 1 \quad (7.21)$$

$$p \rightharpoonup xy = (p_{(1)} \rightharpoonup x_{(1)})((p_{(2)} \leftharpoondown x_{(2)}) \rightharpoonup y) \quad (7.22)$$
$$p \rightharpoonup 1 = \varepsilon(p)1 \quad (7.23)$$

$$\tau(p_{(1)}|q_{(1)[0]})\tau((p_{(2)} \leftharpoondown q_{(1)[1]}) \bullet q_{(2)}|r)$$
$$= (p_{(1)} \rightharpoonup \tau(q_{(1)}|r_{(1)[0]})_{(1)})\tau(p_{(2)} \leftharpoondown \tau(q_{(1)}|r_{(1)[0]})_{(2)}|(q_{(2)} \leftharpoondown r_{(1)[1]}) \bullet r_{(2)}) \quad (7.24)$$

$$\tau(1|p) = \tau(p|1) = \varepsilon(p)1 \quad (7.25)$$

$$(p_{(1)} \rightharpoonup (q_{(1)} \rightharpoonup x_{(1)})_{(1)})\tau(p_{(2)} \leftharpoondown (q_{(1)} \rightharpoonup x_{(1)})_{(2)}|q_{(2)} \leftharpoondown x_{(2)})$$
$$= \tau(p_{(1)}|q_{(1)[0]})((p_{(2)} \leftharpoondown q_{(1)[1]}) \bullet q_{(2)} \rightharpoonup x) \quad (7.26)$$

$$1 \rightharpoonup x = x \quad (7.27)$$

I hope to be able to prove in the near future that the list of axioms is complete, that is, that these axioms suffice to make $H'\#_\tau Q$ a bialgebra. (In principle, one can attempt to show this by directly checking the bialgebra axioms, but the calculations tend to be of an unwieldy size.)

If $H'\#_\tau Q$ is a bialgebra, then I expext that it is a Hopf algebra if and only if τ is convolution invertible in $\mathrm{Hom}(Q\tilde{\otimes}Q, H')$, and the map

$$\beta_Q\colon Q \otimes Q \ni p \otimes q \mapsto p \bullet q_{(1)} \otimes q_{(2)} \in Q \otimes Q$$

is a bijection.

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