GALOIS CORRESPONDENCES FOR HOPF BIGALOIS EXTENSIONS

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Abstract. Let H be a Hopf algebras with bijective antipode over a commutative ring k. A right H-Galois extension of k is a right H-comodule algebra A such that $k = A^{\mathrm{co}\,H}$ and a certain canonoical map $A \otimes A \to A \otimes H$ is a bijection. We investigate Galois connections for Hopf-Galois extensions that can be formulated with the help of an additional Hopf algebra L over which A is also a left L-Galois extension of k, and an L-H-bicomodule — such an additional Hopf algebra always exists and is unique up to isomorphism. The Galois connection between quotient coalgebras and left modules of L and H-subcomodule algebras of A induces a bijection between those quotients over which L is faithfully coflat, and those subalgebras over which A is faithfully flat. As a consequence, the lattices of Hopf subalgebras of L and H over which these are faithfully flat are isomorphic.

1. Introduction

We consider all algebras etc. over a fixed commutative ring k. Let H be a Hopf algebra. A right H-Galois extension of k is a right H-comodule algebra A with $k = A^{\mathrm{co}\,H}$ such that the map $A \otimes A \ni x \otimes y \mapsto xy_{(0)} \otimes y_{(1)} \in A \otimes H$ is a bijection.

The notion of a Hopf-Galois extension generalizes that of a Galois extension of commutative rings with finite Galois group G (in this case $H = k^G$; of course, classically Galois field extensions are a special case), that of purely inseparable Galois field extensions (in this case H is the dual of the restricted enveloping algebra of a p-Lie algebra) and that of a strongly G-graded algebra (in this case $H = kG$ is the group algebra).

Consider a classically Galois field extension A/k with Galois group G, and put $H = k^G$. The fundamental theorem of Galois theory translates to the language of Hopf algebras as follows:

> $H/I \mapsto A^{\mathrm{co}\,H/I}$ defines a bijective correspondence between quotient Hopf algebras H/I of H and k-subalgebras of A (all of which are necessarily fields). Under this correspondence, conormal quotients H/I (i.e. normal Hopf ideals $I \subset H$) correspond to normal extensions of k.

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For a field extension A/k which is H-Galois for a Hopf algebra H, there is no precise analog of the fundamental theorem: Although, at least if H is finite dimensional, the map $I \mapsto A^{\operatorname{co} H/I}$ is injective and inclusion-preserving, it is not clear which intermediate fields can be obtained in this way. In particular, a result of [?] states that any normal separable field extension A/k is H-Galois with a Hopf algebra for which the subfields of the form $A^{\operatorname{co} H/I}$ are precisely the *normal* intermediate fields between k and A.

Let H be any Hopf algebra. Then we have maps

$$\{\text{Coideal left ideals of } H\} \rightleftarrows \{\text{Right coideal subalgebras of } H\}$$
$$I \mapsto {}^{\operatorname{co} H/I}H$$
$$HB^+ \mapsfrom B$$

The map from left to right is the analog of the Galois correspondence, where we consider H as a left H-Galois extension of k. In many cases, the maps are known to be inverse bijections and/or to restrict to inverse bijections on certain classes of coideal left ideals, respectively right coideal subalgebras. Note, however, that the set on the right hand side, and consequently the map from right to left, only makes sense if we know the right H-comodule structure of H, although we are dealing with H as a left H-Galois extension to define the map from left to right. Thus, the Galois connection between coideal left ideals and right coideal subalgebras of a Hopf algebra does not generalize easily to the case of general Hopf Galois extensions.

The situation is amended by a construction that first appears in [?] in the commutative case, and generalized to the noncommutative case (with the coinvariant subalgebra equal to k, which is no loss of generality in the commutative case) in [?]. See also [?, ?]. We use the conventions of [?]. Given a faithfully flat right H-Galois extension A of k, there exists a unique Hopf algebra $L = L(A, H)$ and left L-comodule algebra structure on A such that A is a left L-Galois extension of k and an L-H-bicomodule. We call such a bicomodule algebra an L-H-bigalois extension. In the commutative case, L is constructed in [?] and called the OZ-transform in [?]. In the case that A is an H-Galois field extension of a base field k, the following version of a fundamental theorem of Hopf Galois theory is proved in [?]: Under the map $I \mapsto {}^{\operatorname{co} L/I}A$ Hopf ideals in L correspond bijectively to H-costable subfields of A.

At first sight, this does not answer the original question, namely which subfields of A correspond to Hopf ideals of H via $I \mapsto A^{\operatorname{co} H/I}$. However, it was observed in [?] that the OZ-transformation is involutive. The same holds mutatis mutandis in the noncommutative case. Thus, the roles of L and H are interchangeable throughout the theory. However, we will discuss below that L and H may have different lattices of quotients, so that it seems that we do need a second Hopf algebra L to describe which subobjects of A correspond to quotients of H.

The main results of the present paper concern a generalization of the fundamental theorem of Hopf Galois theory of [?]. For a faithfully flat L-H-bigalois extension of k we have maps

$$\{\text{Coideal left ideals in } L\} \underset{\mathcal{I}}{\overset{\mathcal{F}}{\rightleftarrows}} \{H\text{-subcomodule algebras of } A\}$$

with $\mathcal{F}(I) = {}^{\mathrm{co}\,L/I}A$. The map $\mathcal{I}$ is essentially due to [?], see [?]; we will review it in Proposition **??**. The maps $\mathcal{F}$ and $\mathcal{I}$ form a Galois connection if we order the set on the right hand side by reverse inclusion. Under favorable hypotheses they induce precise correspondences between subsets of suitable coideal left ideals and suitable subcomodule algebras. We have already mentioned the first correspondence [?, Thm.4.4] of this type. We have proved two variations on this result in [?]: One of them is a noncommutative analog of [?, Thm.4.4] with a skew field A. The other is valid in the case of a finite commutative Hopf algebra and a commutative algebra A; here we have restricted the Hopf ideals to those for which L/I is projective and the intermediate rings B to those for which A_B is faithfully projective.

In the present paper we prove a version of the correspondence theorem which holds whenever H has bijective antipode. Here, we consider k-flat quotient coalgebras and right modules $\overline{L}$ of L with L faithfully coflat over $\overline{L}$, and subcomodule algebras $B \subset A$ such that A is right faithfully flat over B. The condition on $\overline{L}$ should be considered as a weak technical condition which is known to be satisfied automatically in many interesting cases. The correspondence restricts to Hopf algebra quotients on the one hand and Yetter-Drinfeld module subalgebras (with respect to the Miyashita-Ulbrich action of H on A) on the other, and further to normal quotients of L and those Yetter-Drinfeld module subalgebras that are also L-subcomodules. As a corollary, we obtain a certain structural similarity between H and $L(A,H)$: The lattices of those of their Hopf subalgebras over which they are faithfully flat are isomorphic. One should note that the same is not true in general for the lattices of those of their k-flat quotient Hopf algebras over which they are faithfully coflat, even if H is the dual of a group algebra and $k \subset A$ is a field extension.

2. Preliminaries

Throughout the paper let k denote a commutative base ring. All modules, algebras, coalgebras etc. are over k if nothing else is indicated. We denote the multiplication of an algebra by ∇, the comultiplication of a coalgebra by Δ, the antipode of a Hopf algebra by S, a right (resp. left) comodule structure map by δ_r (resp. δ_ℓ). We use the following versions of Sweedler's notation: $\Delta(x) = x_{(1)} \otimes x_{(2)}, \delta_r(x) = x_{(0)} \otimes x_{(1)}$ and $\delta_\ell(x) = x_{(-1)} \otimes x_{(0)}$. We denote by ε the counit of a coalgebra C. For any k-submodule M of C we let $M^+ := \mathrm{Ker}(\varepsilon|_M)$.

Let H be a Hopf algebra. The right adjoint module H_{ad} is H with the right module structure $g \otimes h \mapsto S(h_{(1)})gh_{(2)}$. The left adjoint action is $g \otimes h \mapsto g_{(1)}hS(g_{(2)})$, we denote this module by ${}_{\mathrm{ad}}H$. A Hopf subalgebra $H' \subset H$ is said to be normal if it is stable under both adjoint actions. If H and H' have bijective antipodes, and H' is a submodule of H_{ad} or ${}_{\mathrm{ad}}H$, then H' is normal. The left coadjoint comodule ${}^{\mathrm{coad}}H$ is H with the left comodule structure $h \mapsto h_{(1)}S(h_{(3)}) \otimes h_{(2)}$. The right coadjoint comodule H^{coad} is H with the right comodule structure $h \mapsto h_{(2)} \otimes S(h_{(1)})h_{(3)}$. A quotient Hopf algebra $\overline{H}$ of H is said to be (co)normal, if $\overline{H}$ is a quotient comodule of H^{coad} and ${}^{\mathrm{coad}}H$. In case H and $\overline{H}$ have bijective antipodes, one of these conditions is sufficient. A Hopf ideal $I \subset H$ is (co)normal if H/I is.

Let H be a Hopf algebra. A right H-comodule algebra A is said to be an H-Galois extension of k if $k = A^{\mathrm{co}\,H} := \{a \in A | \delta_r(a) = a \otimes 1\}$ and the canonical map $\kappa_r : A \otimes A \ni x \otimes y \mapsto xy_{(0)} \otimes y_{(1)} \in A \otimes H$ is an isomorphism. The definition of a left H-Galois extension of k is analogous; we denote the corresponding canonical map by κ_ℓ. We will also use the map $\kappa'_\ell : A \otimes A \to H \otimes A$, defined, for a left H-comodule algebra A, by $\kappa'_\ell(x \otimes y) = y_{(-1)} \otimes xy_{(0)}$. Let A be a right H-Galois extension of k. We write $\kappa_r^{-1}(1 \otimes h) = \sum \ell_i(h) \otimes r_i(h)$. The Miyashita-Ulbrich action of H on A is defined [?, ?] by $a \leftharpoonup h := \sum \ell_i(h) a r_i(h)$. It makes A a (right-right) Yetter-Drinfeld H-module, which means that the comodule and module structures satisfy the compatibility condition $a_{(0)} \leftharpoonup h_{(0)} \otimes a_{(1)}h_{(1)} = (a \leftharpoonup h_{(2)})_{(0)} \otimes h_{(1)}(a \leftharpoonup h_{(2)})_{(1)}$. (Yetter-Drinfeld modules were introduced as crossed modules in [?].) Also, A is an H-module algebra with the Miyashita-Ulbrich action, and thus an algebra in the monoidal category of Yetter-Drinfeld modules. Moreover, this category is prebraided with prebraiding $V \otimes W \ni v \otimes w \mapsto w_{(0)} \otimes v \leftharpoonup w_{(1)} \in W \otimes V$ for two Yetter-Drinfeld modules V, W, and A is a commutative algebra with respect to this prebraiding, which means that $yx = x_{(0)}(y \leftharpoonup x_{(1)})$ holds for $x, y \in A$. In case the antipode of H is bijective, the prebraiding is a braiding with $\sigma^{-1}(w \otimes v) = v \leftharpoonup S^{-1}(w_{(1)}) \otimes w_{(0)}$.

Now let L and H be two Hopf algebras. An L-H-bigalois extension [?, Def.3.4] is an L-H-bicomodule A which is both a left L-Galois extension of k and a right H-Galois extension of k with respect to these comodule structures. Note that our definition is more restrictive than that of [?]: The algebra A in a faithfully flat bi-Galois extension (U, W, H, A) as defined in [?, Def.2.6] is a W-H-Bigalois extension of k in our sense if $U = W$ and, moreover, $A^{\mathrm{co}\,H} = {}^{\mathrm{co}\,W}A = k$. We have proved in [?, Sec.3] that for every right H-Galois extension A of k which is faithfully flat over k, there exists a unique up to isomorphism Hopf algebra L such that A is an L-H-Bigalois extension of k (and conversely, for any Hopf algebra L and left L-Galois extension of k, there is a unique H making it a Bigalois extension). The construction is due to [?] for the commutative case. We need some facts on the structure of L. Recall first, [?], that for a faithfully flat H-Galois extension A of k we have an equivalence of categories $\mathcal{M}_A^H \to \mathcal{M}_k$ given by $M \mapsto M^{\mathrm{co}\,H}$. Here $\mathcal{M}_A^H$ denotes the category of relative Hopf modules, that is, right H-comodules and A-modules M satisfying $\delta(ma) = m_{(0)}a_{(0)} \otimes m_{(1)}a_{(1)}$ for all $m \in M$ and $a \in A$. The inverse equivalence is given by $N \mapsto N \otimes A$ with the comodule and module structures induced by those of the right tensorand. Now if A is an L-H-Bigalois extension of k, then $\kappa_\ell : A \otimes A \to L \otimes A$ is an isomorphism in

$\mathcal{M}_A^H$ if we endow the source with the codiagonal comodule structure. It follows that $L \cong (A \otimes A)^{\mathrm{co}\,H}$. We also have an isomorphism

$$\hat{\kappa}_\ell : A \otimes A \otimes A \xrightarrow{\kappa_\ell \otimes A} L \otimes A \otimes A \xrightarrow{L \otimes \kappa_\ell} L \otimes L \otimes A$$

The comultiplication Δ_L of L makes the diagram

$$\begin{tikzcd} L \otimes A \arrow[r, "\Delta_L \otimes A"] & L \otimes L \otimes A \\ A \otimes A \arrow[u, "\kappa_\ell"] \arrow[r, "\tilde{\Delta}"'] & A \otimes A \otimes A \arrow[u, "\hat{\kappa}_\ell"'] \end{tikzcd} \tag{2.1}$$

commute, where $\tilde{\Delta}(x \otimes y) = x \otimes 1 \otimes y$. In fact $\hat{\kappa}_\ell \tilde{\Delta}(x \otimes y) = \hat{\kappa}_\ell(x \otimes 1 \otimes y) = x_{(-1)} \otimes \kappa_\ell(x_{(0)} \otimes y) = x_{(-2)} \otimes x_{(-1)} \otimes x_{(0)} y = (\Delta_L \otimes A)\kappa_\ell(x \otimes y)$. Explicitly, Δ_L is given (with the identification $L = (A \otimes A)^{\mathrm{co}\,H}$) by $\Delta(\sum x_i \otimes y_i) = \sum x_{i(0)} \otimes \ell_j(x_{i(1)}) \otimes r_j(x_{i(1)}) \otimes y_i$, and its left coaction on A by $\delta_\ell(x) = \sum x_{(0)} \otimes \ell_i(x_{(1)}) \otimes r_i(x_{(1)})$.

The multiplication of L is that for which $L \subset L \otimes A \cong A \otimes A^{\mathrm{op}}$ is an algebra map, that is to say, for which $L \cong (A \otimes A)^{\mathrm{co}\,H} \subset A \otimes A^{\mathrm{op}}$ as a subalgebra. Indeed if $\kappa_\ell(\sum w_i \otimes x_i) = f \otimes 1$ and $\kappa_\ell(\sum y_j \otimes z_j) = g \otimes 1$, then $\kappa_\ell(\sum w_i y_j \otimes z_j x_i) = \sum w_{i(-1)} y_{j(-1)} \otimes w_{i(0)} y_{j(0)} z_j x_i = \sum w_{i(-1)} g \otimes w_{i(0)} x_i = fg \otimes 1$. Finally, if we denote by $\sigma_{AA} : A \otimes A \to A \otimes A$ the prebraiding in the category of Yetter-Drinfel'd modules, $\sigma(x \otimes y) = y_{(0)} \otimes x \leftharpoonup y_{(1)}$, then

$$\begin{tikzcd} A \otimes A \arrow[r, "\kappa_\ell"] \arrow[d, "\sigma_{AA}"'] & L \otimes A \arrow[d, "\alpha"] \\ A \otimes A \arrow[r, "\kappa_\ell"'] & L \otimes A \end{tikzcd} \tag{2.2}$$

commutes, where $\alpha(f \otimes a) = S(f) a_{(-1)} \otimes a_{(0)}$. It follows that

$$\begin{tikzcd} (A \otimes A)^{\mathrm{co}\,H} \arrow[r, "\cong"] \arrow[d, "\sigma^{\mathrm{co}\,H}"'] & L \arrow[d, "S"] \\ (A \otimes A)^{\mathrm{co}\,H} \arrow[r, "\cong"'] & L \end{tikzcd} \tag{2.3}$$

also commutes. In particular, if the antipode of H is bijective, then so is the antipode of L, with $S_L^{-1} = (\sigma^{-1})^{\mathrm{co}\,H}$.

3. The Galois Correspondence

The additional structure of a left L-Galois extension on any right H-Galois extension of k allows one to formulate the Hopf algebraic analog of the Galois connection; this idea is due to [?], where the maps $\mathcal{F}$ and $\mathcal{I}$ below are used for the case of a Hopf Galois field extension, and a fundamental theorem of Galois Theory is proved in this case. The construction of the maps extends readily to the general case, see [?, Thm.6.4].

Definition 3.1. Let A be a right H-comodule algebra. We set

$$\mathrm{Sub}(A) := \{H\text{-subcomodule algebras of } A\}.$$

$\mathrm{Sub}(A)$ is partially ordered by inclusion. Let L be a Hopf algebra. We let

$$\mathrm{Quot}(L) := \{\text{Coideal left ideals in } L\}$$

which we identify with the set of quotient left modules and coalgebras of L. We define $\mathrm{Quot}(L)$ as a partially ordered set with the ordering defined by $I \leq I' \Leftrightarrow I' \subset I$.

Note that in the case that $L = (kG)^*$ is the dual of the group algebra of a finite group G with a field k, the quotient left modules and coalgebras of $(kG)^*$ correspond to the subgroups of G, and the chosen ordering of $\mathrm{Quot}(L)$ corresponds to the ordering of subgroups by inclusion

Proposition 3.2. *Let L and H be Hopf algebras and A a faithfully flat L-H-bigalois extension of k. Then we have a Galois connection*

$$\mathrm{Quot}(L) \underset{\mathcal{I}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathrm{Sub}(A)$$

with $\mathcal{F}(I) = {}^{\mathrm{co}\,L/I}A$ and $L/\mathcal{I}(B) := (A \underset{B}{\otimes} A)^{\mathrm{co}\,H}$ under the identification $L \cong (A \otimes A)^{\mathrm{co}\,H}$.

It is easy to see that $\mathcal{F}$ is well defined. That $\mathcal{I}(B)$ is a left ideal is easy since $L \cong (A \otimes A)^{\mathrm{co}\,H}$ as a subalgebra of $A \otimes A^{\mathrm{op}}$. That $\mathcal{I}(B)$ is a coideal follows from the characterization of the comultiplication of L in diagram (??). That $\mathcal{F}$ and $\mathcal{I}$ define a Galois connection means that we have

$$\begin{aligned} \mathcal{I}\mathcal{F}(I) &\subset I \\ \mathcal{F}\mathcal{I}(B) &\supset B \\ B \subset B' &\Rightarrow \mathcal{I}(B) \subset \mathcal{I}(B') \\ I \subset I' &\Rightarrow \mathcal{F}(I) \subset \mathcal{F}(I') \end{aligned}$$

which we have observed in [?, Thm.6.4]. The last two implications are clear. Let us briefly recall the first two inclusions: We have $\mathcal{I}\mathcal{F}(I) \subset I$ since the canonical map

$L/\mathcal{IF}(I) = (A \underset{\mathrm{co}\, L/I_A}{\otimes} A)^{\mathrm{co}\, H} \to L/I$ is well defined, which is a consequence of the natural map $A \underset{\mathrm{co}\, L/I_A}{\otimes} A \to L/I \otimes A$ being well defined. For $B \in \mathrm{Sub}(A)$ we have

$$\begin{aligned}
\mathcal{FI}(B) &= {}^{\mathrm{co}\, L/\mathcal{I}(B)}A \\
&= {}^{\mathrm{co}\, (A \otimes_B A)^{\mathrm{co}\, H}}A \\
&= \{x \in A | \sum x_{(0)} \underset{B}{\otimes} \ell_i(x_{(1)}) \otimes r_i(x_{(1)}) = 1 \underset{B}{\otimes} 1 \otimes x \in A \underset{B}{\otimes} A \otimes A\} \\
&\supset B
\end{aligned}$$

where the last inclusion holds since B is an H-subcomodule and $\sum x_{(0)}\ell_i(x_{(1)}) \otimes r_i(x_{(1)}) = 1 \otimes x$.

A well known easy consequence of the axioms of a Galois connection is that $\mathcal{IFI}(B) = \mathcal{I}(B)$ and $\mathcal{FIF}(I) = I$ for all B and I.

However, $\mathcal{F}$ and $\mathcal{I}$ are not in general inverse bijections; this can already be seen in the case $A = H$: Let B be a subbialgebra of H which is not a Hopf algebra, and let $\hat{B}$ be the subbialgebra generated by B and $S(B)$. Then $H \underset{\hat{B}}{\otimes} H \cong H \underset{B}{\otimes} H$, since, for $b \in B$ and $g, h \in H$ the following holds in $H \underset{B}{\otimes} H$:

$$gS(b) \otimes h = gS(b_{(1)}) \otimes b_{(2)}S(b_{(3)})h = gS(b_{(1)})b_{(2)} \otimes S(b_{(3)})h = g \otimes S(b)h.$$

Hence, $\mathcal{I}(B) = \mathcal{I}(\hat{B})$. Similarly, if I is a biideal, but not a Hopf ideal in L, $B = \mathcal{F}(I)$, and $J = \mathcal{I}(B)$, then by construction

$$A \underset{B}{\otimes} A \ni x \otimes y \mapsto \overline{x_{(-1)}} \otimes x_{(0)}y \in L/J \otimes A$$

is a bijection, in other words, $J = I$ would imply that A is a left L/I-Galois extension of B, and hence I is a Hopf ideal by [**?**]. Thus, to get a one-to-one Galois correspondence, we need to restrict ourselves to certain subsets of $\mathrm{Sub}(A)$ and $\mathrm{Quot}(L)$.

Definition 3.3. Let C be a coalgebra and $C \to \overline{C}$ a quotient coalgebra. We call $\overline{C}$ right (left) admissible if it is k-flat (hence faithfully k-flat) and C is right (left) faithfully coflat over $\overline{C}$. We call a coideal $I \subset C$ right (left) admissible if C/I is.

For an algebra A and a subalgebra $B \subset A$ we say B is right (left) admissible if A is faithfully flat as a right (left) B-module.

Admissible shall mean right and left admissible.

If H is a finitely generated projective (for short, a finite) Hopf algebra and $\overline{H}$ is a k-split quotient Hopf algebra, then $\overline{H}$ is admissible. This follows by localization from the dual of the Nichols-Zoeller theorem; cf. [**?**, Sec.2], where it is proved that H is faithfully projective over any k-split Hopf subalgebra. In particular any k-split Hopf subalgebra of a finite Hopf algebra is admissible.

If k is a field, then by [?, Cor.1] every Hopf subalgebra of a commutative Hopf algebra is admissible. A note added in proof to that paper also announces that any quotient Hopf algebra of a commutative Hopf algebra is also admissible. By [?, Thm.1.3] every Hopf subalgebra of H and every coideal left ideal of H is admissible if the coradical of H is cocommutative. Roughly speaking, admissibility of a coideal left ideal I in Thm. **??** below should be considered a technical condition which is automatic in many interesting cases, especially for Hopf ideals; the same type of condition is used in [?, ?]. On the other hand some restriction on intermediate algebras in an extension $k \subset A$ is already necessary in classical Galois theory of commutative ring extensions: If such an extension A/k with Galois group G has only trivial idempotents, then subgroups of G are in bijective correspondence to separable subalgebras of A. We will discuss after Cor. **??** that these are precisely the subalgebras B such that A_B is faithfully projective. Our weaker admissibility restriction on subalgebras corresponds to the fact that we admit quotient Hopf algebras of k^G that are not k-projective, hence do not correspond to subgroups of G; see Cor. **??**.

Proposition 3.4. *Let L, H be Hopf algebras and A a faithfully flat L-H-Bigalois extension of k.*

1. *If $B \in \mathrm{Sub}(A)$ is left admissible, then so is $\mathcal{I}(B)$, and $\mathcal{F}\mathcal{I}(B) = B$.*
2. *If $I \in \mathrm{Quot}(L)$ is right admissible, then so is $\mathcal{F}(I)$, and $\mathcal{I}\mathcal{F}(I) = I$.*

Proof. Assume first that $I \subset L$ is a right admissible coideal left ideal. Put $B = {}^{\mathrm{co}\, L/I}A$. Then by [?, Thm. 1.4] A_B is faithfully flat, and the map

$$A \underset{B}{\otimes} A \ni x \otimes y \mapsto \overline{x_{(-1)}} \otimes x_{(0)} y \in L/I \otimes A$$

is a bijection, thus $L/I \cong (A \underset{B}{\otimes} A)^{\mathrm{co}\, H}$.

Now let B be a left admissible H-subcomodule algebra of A, and put $I := \mathcal{I}(B)$. We have $B \subset \mathcal{F}\mathcal{I}(B) = {}^{\mathrm{co}\, L/I}A$ and by definition of $\mathcal{I}$ the canonical map $A \underset{B}{\otimes} A \cong L/I \otimes A$ is a bijection. We can refer to [?, Rem.1.2,(1)] to conclude $B = {}^{\mathrm{co}\, L/I}A = \mathcal{F}\mathcal{I}(B)$. Moreover, A left faithfully flat over B and faithfully flat over k implies L/I (faithfully) flat over k if $A \underset{B}{\otimes} A \cong L/I \otimes A$. This puts us in the situation of [?, Rem.1.2,(2)], so that A is faithfully coflat as a left L/I-comodule. This in turn implies that L is left faithfully coflat as an L/I-comodule, since for any left L/I-comodule V we have $(V \,\square_{L/I}\, L) \,\square_L\, A \cong V \,\square_{L/I}\, A$ and A is faithfully coflat over L as well. $\square$

To conclude that the Galois connection is a precise correspondence on subsets of (left or right) admissible subalgebras and quotients, we need a means to switch sides. We can do this if L (or, equivalently, H) has bijective antipode. Note first that in this case the opposite algebra A^{op} is a faithfully flat L^{op}-H^{op}-Bigalois extension of k. In the hope of making our formulas more understandable, we denote the Galois connection for this extension by $\mathcal{F}^{\mathrm{op}}$ and $\mathcal{I}^{\mathrm{op}}$.

Proposition 3.5. *Let L be a k-flat Hopf algebra with bijective antipode.*

The map $\mathrm{Quot}(L) \ni I \mapsto I^{\mathrm{op}} := S_L(I) \in \mathrm{Quot}(L^{\mathrm{op}})$ *is a bijection with inverse* $\mathrm{Quot}(L^{\mathrm{op}}) \ni J \mapsto J^{\mathrm{op}} := S_{L^{\mathrm{op}}}(J) \in \mathrm{Quot}(L)$. *The coideal $S_L(I)$ is right (left) admissible iff I is left (right) admissible.*

For any left L-comodule algebra A and $I \in \mathrm{Quot}(L)$ we have ${}^{\mathrm{co}\, L^{\mathrm{op}}/I^{\mathrm{op}}}(A^{\mathrm{op}}) = \left({}^{\mathrm{co}\, L/I}A\right)^{\mathrm{op}}$

Proof. Clearly $I \mapsto S_L I$ is a bijection with inverse $J \mapsto S_L^{-1}(J) = S_{L^{\mathrm{op}}}(J)$.

The antipode induces an isomorphism $L/I \mapsto L/SI$, so that L/SI is k-flat. Let V be a left L/SI-comodule. We make V a right L/I-comodule by $v \mapsto v_{(0)} \otimes S^{-1}(v_{(1)})$. Then we have

$$L \,\square_{L/SI}\, V \cong V \,\square_{L/I}\, L$$
$$\sum f_i \otimes v_i \mapsto \sum v_i \otimes f_i$$

which is, of course, natural in V, so that L is right faithfully coflat over L/SI, and thus SI is right admissible, if and only if I is left admissible. That I is right admissible iff SI is left admissible is proved in the same way.

Now let A be a left H-comodule algebra and $I \in \mathrm{Quot}(L)$. Let $a \in {}^{\mathrm{co}\, L/I}A$, that is $a_{(-1)} \otimes a_{(0)} - 1 \otimes a \in I \otimes A$. We have a map $\alpha : I \otimes A \ni f \otimes a \mapsto S(f)a_{(-1)} \otimes a_{(0)} \in S(I) \otimes A$ since $S(I)$ is a right ideal. It follows that $S(I) \otimes A \ni \alpha(a_{(-1)} \otimes a_{(0)} - 1 \otimes a) = S(a_{(-2)})a_{(-1)} \otimes a_{(0)} - a_{(-1)} \otimes a_{(0)} = 1 \otimes a - a_{(-1)} \otimes a_{(0)}$ whence $a \in {}^{\mathrm{co}\, L/SI}A$. We have found $\left({}^{\mathrm{co}\, L/I}A\right)^{\mathrm{op}} \subset \left({}^{\mathrm{co}\, L/SI}A\right)^{\mathrm{op}} = {}^{\mathrm{co}\, L^{\mathrm{op}}/I^{\mathrm{op}}}(A^{\mathrm{op}})$. The other inclusion follows when we apply the same argument to A^{op} instead of A. □

Theorem 3.6. *Let H and L be Hopf algebras with bijective antipodes and A a faithfully flat L-H-Bigalois extension of k. The maps $\mathcal{F}$ and $\mathcal{I}$ from Proposition* **??** *induce mutually inverse bijections between the (left, right) admissible coideal left ideals $I \subset L$, and those H-subcomodule algebras $B \subset A$ such that A is (left, right) faithfully flat over B.*

Proof. Let $B \subset A$ be right admissible. Then $B^{\mathrm{op}} \subset A^{\mathrm{op}}$ is left admissible, whence $\mathcal{I}^{\mathrm{op}}(B^{\mathrm{op}})$ is left admissible and $B^{\mathrm{op}} = \mathcal{F}^{\mathrm{op}}\mathcal{I}^{\mathrm{op}}(B^{\mathrm{op}})$, thus $B = (\mathcal{F}^{\mathrm{op}}\mathcal{I}^{\mathrm{op}}(B^{\mathrm{op}}))^{\mathrm{op}} = \mathcal{F}(\mathcal{I}^{\mathrm{op}}(B^{\mathrm{op}})^{\mathrm{op}})$. Moreover, $\mathcal{I}^{\mathrm{op}}(B^{\mathrm{op}})^{\mathrm{op}}$ is right admissible, and we conclude that $\mathcal{I}(B) = \mathcal{I}\mathcal{F}(\mathcal{I}^{\mathrm{op}}(B^{\mathrm{op}})^{\mathrm{op}}) = \mathcal{I}^{\mathrm{op}}(B^{\mathrm{op}})^{\mathrm{op}}$.

The last equality also holds, of course, for A^{op} as an L^{op}-H^{op}-bigalois extension, where it reads $\mathcal{I}^{\mathrm{op}}(B^{\mathrm{op}}) = \mathcal{I}(B)^{\mathrm{op}}$ whenever B^{op} is right admissible, that is, B is left admissible. Now let I be left admissible. Then I^{op} is right admissible, so is $\mathcal{F}^{\mathrm{op}}(I^{\mathrm{op}}) = \mathcal{F}(I)^{\mathrm{op}}$, whence $\mathcal{F}(I)$ is left admissible, and $I^{\mathrm{op}} = \mathcal{I}^{\mathrm{op}}\mathcal{F}^{\mathrm{op}}(I^{\mathrm{op}}) = \mathcal{I}^{\mathrm{op}}(\mathcal{F}(I)^{\mathrm{op}}) = (\mathcal{I}\mathcal{F}(I))^{\mathrm{op}}$ whence $I = \mathcal{I}\mathcal{F}(I)$. □

Note that the Theorem just proved contains [**?**, Thm. 4.4] as a special case in view of the fact that any Hopf ideal of a commutative Hopf algebra over a field is admissible.

The generalization [?, Thm. 6.7] is not directly a special case; comparing, we conclude that every coideal left ideal of a Hopf algebra H is admissible provided H admits an H-Galois skew field extension of k.

To include [?, Thm. 6.6] as a special case of Theorem **??** we need to add the following observation:

Corollary 3.7. *In the situation of* **??** *let* $B \in \mathrm{Sub}(A)$ *be admissible, and put* $I = \mathcal{I}(B)$. *Then* A *is left or right finitely generated projective over* B *iff* L/I *is finitely generated projective over* k.

Proof. Let A be right finitely generated projective over B. Then $L/I \otimes A \cong A \underset{B}{\otimes} A$ is finitely generated projective as a right A-module, and by faithful flatness of A over k it follows that L/I is finitely generated projective over k. Conversely, assume that L/I is finitely generated projective over k. Again, it follows that $A \underset{B}{\otimes} A \cong L/I \otimes A$ is finitely generated projective, hence finitely presented, as a right A-module. Thus, by faithful flatness of ${}_BA$, it follows that A is a finitely presented right B-module. Since A_B is faithfully flat, it is finitely generated projective. □

Theorem 3.8. *Let* H *be a Hopf algebra with bijective antipode and* A *a faithfully flat* L-H-*Bigalois extension of* k. *Let* $B \subset A$ *be a right or left admissible* H-*subcomodule algebra, and let* $I := \mathcal{I}(B) \subset L$ *be the corresponding coideal left ideal.*

1. *I is a Hopf ideal if and only if B is an H-submodule of A with respect to the Miyashita-Ulbrich action. In this case $B \subset A$ is admissible.*
2. *I is an L-subcomodule of L with respect to the left coadjoint coaction if and only if B is an L-subcomodule of A.*
3. *I is a conormal Hopf ideal if and only if B is both an H-submodule and an L-subcomodule of A.*

Proof. (1) Follows from [?, Prop.6.5], but we can give a slightly different proof in the present case. Assume that L/I is a Hopf algebra. Then $SI \subset I$. Since $A \underset{B}{\otimes} A \cong L/I \otimes A$ via κ_ℓ, it follows from diagram (**??**) that $\sigma' : A \underset{B}{\otimes} A \ni x \otimes y \mapsto y_{(0)} \otimes x \leftharpoonup y_{(1)} \in A \underset{B}{\otimes} A$ is well defined. Let $b \in B$ and $h \in H$. Then in $A \underset{B}{\otimes} A$ we have $1 \otimes b \leftharpoonup h = \sum \ell_i(h) r_i(h)_{(0)} \otimes b \leftharpoonup r_i(h)_{(1)} = \sum \ell_i(h)\sigma'(b \otimes r_i(h)) = \sum \ell_i(h)\sigma'(1 \otimes b r_i(h)) = \sum \ell_i(h) b r_i(h) \otimes 1 = 1 \otimes b \leftharpoonup h$, whence $b \leftharpoonup h \in B$ by faithful flatness. Conversely assume that B is an H-submodule. Note that $SI \subset L$ is a ideal. We need only show that $SI = I$, then L/I is a Hopf algebra (which, moreover, has bijective antipode). Looking at diagram (**??**), it suffices to show that $\sigma, \sigma^{-1} : A \otimes A \to A \otimes A$ induce well defined endomorphisms of $A \underset{B}{\otimes} A$. Denote by $\pi : A \otimes A \to A \underset{B}{\otimes} A$ the projection. Then $\pi\sigma(xb \otimes y) = y_{(0)} \otimes (xb) \leftharpoonup y_{(1)} = y_{(0)} \otimes (b_{(0)}(x \leftharpoonup b_{(1)}) \leftharpoonup y_{(1)} = y_{(0)} \otimes (b_{(0)} \leftharpoonup y_{(1)})(x \leftharpoonup b_{(1)} y_{(2)}) = y_{(0)}(b_{(0)} \leftharpoonup y_{(1)}) \otimes x \leftharpoonup b_{(1)} y_{(2)} = b_{(0)} y_{(0)} \otimes x \leftharpoonup$

$b_{(1)}y_{(1)} = \pi\sigma(x \otimes by)$ and $\pi\sigma^{-1}(x \otimes by) = (by) \leftharpoonup S^{-1}(x_{(1)}) \otimes x_{(0)} = ((y \leftharpoonup S^{-1}(b_{(1)}))b_{(0)}) \leftharpoonup S^{-1}(x_{(1)}) \otimes x_{(0)} = (y \leftharpoonup S^{-1}(x_{(2)}b_{(1)}) \otimes b_{(0)} \leftharpoonup S^{-1}(x_{(1)})x_{(0)} = (y \leftharpoonup S^{-1}(x_{(1)}b_{(1)}) \otimes x_{(0)}b_{(0)} = \pi\sigma^{-1}(xb \otimes y)$.
The statement on admissibility is clear since a Hopf ideal is left admissible iff it is right admissible.

(2) It is straightforward to check that $\kappa_\ell : A \otimes A \to {}^{\mathrm{coad}}L \otimes A$ is an isomorphism of left L-comodules, if we endow the codomain and domain with the codiagonal comodule structures. Consequently ${}^{\mathrm{coad}}L \cong (A \otimes A)^{\mathrm{co}H}$ as left comodules, where the right hand side has the structure of a subcomodule of the codiagonal comodule structure. Now if B is an L-subcomodule of A, then $A \underset{B}{\otimes} A$ is a quotient left L-comodule of $A \otimes A$, and thus L/I is a quotient comodule of ${}^{\mathrm{coad}}L$. Conversely, if I is a subcomodule of ${}^{\mathrm{coad}}L$, then $A \underset{B}{\otimes} A \cong L/I \otimes A$ is a quotient comodule of $A \otimes A$. We can conclude that B is an L-subcomodule: For $b \in B$ we have $\delta_\ell(1 \otimes b) = b_{(-1)} \otimes 1 \otimes b_{(0)}$ and $\delta_\ell(b \otimes 1) = b_{(-1)} \otimes 1 \otimes b_{(0)}$. For $A \underset{B}{\otimes} A$ to be a quotient L-comodule of $A \otimes A$, we have to have $b_{(-1)} \otimes 1 \otimes b_{(0)} = b_{(-1)} \otimes b_{(0)} \otimes 1$ in $L \otimes A \underset{B}{\otimes} A$, whence $b_{(-1)} \otimes b_{(0)} \in L \otimes A$ is in the equalizer of $L \otimes A \rightrightarrows L \otimes A \underset{B}{\otimes} A$, with the two maps defined by $f \otimes x \mapsto f \otimes 1 \otimes x$ and $f \otimes x \mapsto f \otimes x \otimes 1$; this equalizer equals $L \otimes B$ by faithful flatness of A_B and flatness of L.

(3) is clear from (1) and (2), taking into account that L/I has bijective antipode if I is a Hopf ideal, so that I is normal iff it is a subcomodule of ${}^{\mathrm{coad}}L$. □

If H is cocommutative and A is a faithfully flat H-Galois extension of k, then it is also an H-H-Bigalois extension with identical left and right H-comodule structures.

Corollary 3.9. *Let H be a cocommutative Hopf algebra and A a faithfully flat H-Galois extension of k. Then we have a bijection between admissible Hopf ideals $I \subset H$ and Yetter-Drinfeld submodule algebras of A over which A is faithfully flat.*

Our main theorem specializes for $A = H$ to a well known correspondence between admissible quotient coalgebras and left modules and right coideal subalgebras of H. The study of correspondences of this type goes back to [**?**]. At least the first part of the theorem seems to be folklore even if k is not a field. In case k is a field it can be found explicitly in [**?**, Thm.1.11], which is also a very nice survey of the literature. Part (3) of our theorem is contained in [**?**, Thm.1.4] if k is a field (and it is no surprise it also holds if k is not a field).

Theorem 3.10. *Let H be a k-flat Hopf algebra with bijective antipode. Then we have a Galois connection*

$$\{\textit{Coideal left ideals of } H\} \underset{\mathcal{I}'}{\overset{\mathcal{F}}{\rightleftarrows}} \{\textit{Right Coideal subalgebras of } H\}$$

given by $\mathcal{F}(I) = {}^{\mathrm{co}\,H/I}H$ *and* $\mathcal{I}'(B) = HB^+$. *The connection induces a bijection between the (right, left) admissible coideal left ideals and the (right, left) admissible right coideal subalgebras.*

Let I *be a (right, left) admissible coideal left ideal of* H*, and put* $H' := {}^{\mathrm{co}\,H/I}H$.

(1) I *is a subcomodule of* ${}^{\mathrm{coad}}H$ *iff* H' *is a Hopf subalgebra.*
(2) I *is a Hopf ideal iff* H' *is a submodule of* H_{ad}.
(3) I *is a normal Hopf ideal iff* H' *is a normal Hopf subalgebra.*

In any of these cases, I *and* H' *are admissible.*

Proof. We omit the proof that we have a Galois connection. To prove bijectivity on the indicated subsets, it then only remains to check that $\mathcal{I}'(B) = \mathcal{I}(B)$ whenever B is right or left admissible. But this follows easily from the properties of Galois connections and Theorem **??**: Put $I = \mathcal{I}(B)$, so that I is right or left admissible and $\mathcal{F}(I) = B$. It follows that $\mathcal{F}\mathcal{I}'(B) = \mathcal{F}\mathcal{I}'\mathcal{F}(I) = \mathcal{F}(I) = B$, hence $\mathcal{I}(B) = \mathcal{I}\mathcal{F}\mathcal{I}'(B) \subset \mathcal{I}'(B)$ and also $\mathcal{I}'(B) = \mathcal{I}'\mathcal{F}\mathcal{I}(B) \subset \mathcal{I}(B)$.

Now (1) follows from part (2) of **??** and **??**, since a faithfully flat k-submodule of H is an H-H-sub-bicomodule iff it is a subbialgebra. Statement (2) follows from part (1) of **??**, since the Miyashita-Ulbrich action of H on itself is the right adjoint action, and (3) is a consequence of (1) and (2) since a Hopf subalgebra with bijective antipode is normal if it is a subcomodule of the right adjoint action. $\square$

If A is a general L-H-Bigalois extension, then we can describe the relation between ${}^{\mathrm{co}\,L/I}L$ and ${}^{\mathrm{co}\,L/I}A$ for an admissible coideal left ideal of L.

Theorem 3.11. *Let* L *and* H *be Hopf algebras with bijective antipodes and* A *a faithfully flat* L-H-*Bigalois extension of* k. *There is a bijection between right admissible right coideal subalgebras* $L' \subset L$ *and right admissible right* H-*comodule subalgebras of* A *given by* $L' \mapsto L' \,\square_H\, A$.

Assume that L' *is an admissible Hopf subalgebra of* L*, and put* $B = L' \,\square_L\, A$. *Then the left comodule structure of* A *restricts to a map* $\delta : B \to L' \otimes B$*, which makes* B *a left* L'-*Galois extension of* k.

Proof. Right admissible coideal subalgebras are in bijection with right admissible coideal left ideals by $I \mapsto {}^{\mathrm{co}\,L/I}L$, as well as right admissible H-subcomodule algebras are, via $I \mapsto {}^{\mathrm{co}\,L/I}A$ Let I be a right admissible coideal left ideal, put $\overline{L} := L/I$, $B := {}^{\mathrm{co}\,L/I}A$ and $L' := {}^{\mathrm{co}\,L/I}L$. For any left L-comodule V we have ${}^{\mathrm{co}\,L/I}V = k \,\square_{L/I}\, V$ where k has the comodule structure defined by $\overline{1} \in L/I$. Thus $B = k \,\square_{L/I}\, A \cong k \,\square_{L/I}\, (L \,\square_L\, A) \cong (k \,\square_{L/I}\, L) \,\square_L\, A \cong L' \,\square_L\, A$, where the second isomorphism uses faithful coflatness of A over L and flatness of L over k.

Now assume that L' is a Hopf subalgebra of L, which implies that L is a subcomodule of ${}^{\mathrm{coad}}L$, which implies that B is an L-subcomodule of A, that is, $\delta_\ell(B) \subset L \otimes B$. Since $\delta_\ell : A \to L \otimes A$ is left L-colinear, we also have $\delta_\ell(B) = \delta_\ell({}^{\mathrm{co}\,L/I}A) \subset {}^{\mathrm{co}\,L/I}(L \otimes A) \cong L' \otimes A$. By faithful flatness of A, B, L, L' clearly $\delta_\ell(B) \subset L' \otimes B$, so that B is a left L'-comodule algebra. It follows that $\beta : B\otimes B \ni x\otimes y \mapsto x_{(-1)}\otimes x_{(0)}y \in L' \otimes B$ is well defined. To show that it is an isomorphism, it is sufficient to check that $\beta \underset{B}{\otimes} A$ is. But $\beta \underset{B}{\otimes} A : B \otimes A \to L' \otimes A$ identifies with ${}^{\mathrm{co}\, L/I}(\kappa_\ell)$. □

Corollary 3.12. *Let A be a faithfully flat L-H-bigalois extension of A, where H and L are Hopf algebras with bijective antipodes. Then every admissible L-H-subbicomodule algebra of A is an L'-H'-Bigalois extension for admissible Hopf subalgebras $L' \subset L$ and $H' \subset H$.*

As mentioned in section **??**, any faithfully flat H-Galois extension A of k determines uniquely a new Hopf algebra $L = L(A, H)$ for which A is L-H-Bigalois. This is an illustration of the fact, due to [**?**], that the Hopf algebra in a Hopf Galois extension is not unique. The following result, a direct consequence of **??** and **??** gives a certain relation between the structures of H and $L(A, H)$:

Corollary 3.13. *Let H be a Hopf algebra with bijective antipode and A a faithfully flat H-Galois extension of k. Then the lattices of admissible Hopf subalgebras of H and $L(A, H)$ are isomorphic.*

The following observation, which generalizes a result from [**?**] for normal separable field extensions, shows that the lattices of all admissible Hopf ideals of H and $L(A, H)$ are by contrast not in general isomorphic, even in reasonably simple cases.

We need to start with a little remark about Hopf subalgebras of group algebras. It is easy to check that any Hopf subalgebra of kG equals $\bigoplus_{g\in G} \mathfrak{a}_g g$ where a_g are idempotent ideals of k (with $\mathfrak{a}_e = k$ and $\mathfrak{a}_g\mathfrak{a}_h \subset \mathfrak{a}_{gh}$). Clearly such a Hopf subalgebra is k-split iff all $\mathfrak{a}_g$ are k-direct summands of k, hence generated by an idempotent element of k. In particular, if k has no nontrivial idempotents, then all the k-split Hopf subalgebras of kG are group algebras of subgroups of G.

Following [**?**, Chap.III] we call a k-algebra A normal if, for $G = \mathrm{Aut}_{\mathrm{Alg}}(A)$ we have $k = A^G := \{a \in A | \forall\sigma \in G : \sigma(a) = a\}$. Assume that 0 and 1 are the only idempotents in A. As a special case of [**?**, Thm.1.3], see also [**?**, Prop.III.1.2], A is an H-Galois extension of k with $H = k^G$ the dual of the group algebra of G if and only if A is normal and separable over k. Assume this is the case. By the fundamental theorem of the Galois theory of commutative rings, [**?**, Thm.2.3], a bijection between the subgroups of G and those k-subalgebras $B \subset A$ that are separable over k is given by $G' \mapsto A^{G'}$. Since every subgroup gives rise to a k-split quotient Hopf algebra of H, we see that A is faithfully projective over all its separable subalgebras (which it is no problem to prove directly). Conversely, assume that $B \subset A$ is a k-subalgebra with A_B faithfully projective. Since A is separable over B and finitely generated projective over k, it follows from [**?**, Thm.III.2.4] that B is separable over k. Put $L = L(A, H)$. Then **??** shows in particular that every separable subalgebra of A is an L-subcomodule. By [**?**, Thm.III.1.1] an admissible (hence separable) subalgebra of A is an H-subcomodule iff it is normal, iff the corresponding subgroup of G is normal. We summarize:

Example 3.14. Let A/k be a normal separable commutative k-algebra without non-trivial idempotents, G the automorphism group of A and $H = k^G$, so that A is a faithfully projective H-Galois extension of k. For $L := L(A, H)$ the k-split Hopf ideals of L correspond bijectively to the *normal* separable k-subalgebras $B \subset A$ which in turn correspond bijectively to the normal subgroups of G. In particular, all Hopf ideals of L are normal, which is not true for H.

Remark 3.15. In [?] we have shown that isomorphism classes of faithfully flat L-H-bigalois extensions correspond bijectively to isomorphism classes of monoidal equivalences between the categories ${}^L\mathcal{M}$ and ${}^H\mathcal{M}$ of left comodules. The equivalence corresponding to a Bigalois extension A is given by ${}^H\mathcal{M} \ni V \mapsto A \,\square_H\, V \in {}^L\mathcal{M}$. The lattice isomorphism between admissible Hopf subalgebras of H and admissible Hopf subalgebras of L has an interpretation in this framework, which amounts to an alternative proof (by Tannaka duality) at least if k is a field. Let A be an L-H-bigalois extension of k, let $H' \subset H$, $L' \subset L$ and $B \subset A$ be admissible Hopf subalgebras and an admissible L-H-bicomodule subalgebra, respectively, with $L' \,\square_L\, A \cong B \cong A \,\square_H\, H'$. The two Hopf subalgebras give rise to rigid monoidal subcategories ${}^{L'}\mathcal{M} \subset {}^L\mathcal{M}$ and ${}^{H'}\mathcal{M} \subset {}^H\mathcal{M}$ (in case k is a field, then by Tannaka duality any rigid monoidal k-linear abelian subcategory is of this form, which, in view of the equivalence ${}^L\mathcal{M} \cong {}^H\mathcal{M}$ afforded by A, accounts for the fact that H and L have isomorphic lattices of Hopf subalgebras). As can easily be checked, the L'-H'-Bigalois extension B of k (see **??**) induces an equivalence between these subcategories: For a left H'-comodule V, for example, we have $A \,\square_H\, V \cong A \,\square_H\, (H' \,\square_{H'}\, V) \cong (A \,\square_H\, H') \,\square_{H'}\, V \cong B \,\square_{H'}\, V$.

In fact, in the field case it follows that even the lattices of subcoalgebras of H and L are isomorphic, and for a subcoalgebra H' of H and corresponding subcoalgebra L' of L cotensoring with $L' \,\square_L\, A \cong L' \,\square_L\, A \,\square_H\, H' \cong A \,\square_H\, H'$ yields the equivalence between ${}^{L'}\mathcal{M}$ and ${}^{H'}\mathcal{M}$.

We have remarked following the definition of admissibility that in case that k is a field and the coradical of H is cocommutative, the condition of admissibility is automatic for Hopf subalgebras and quotient Hopf algebras. In view of this fact the question arises naturally, if, given a right H-Galois extension A of k, the new Hopf algebra $L(A, H)$ has cocommutative coradical if H does.

Proposition 3.16. *Assume k is a field, and let A be an L-H-Bigalois extension of k, where L and H are Hopf algebras. Assume the coradical H_0 of H is cocommutative. Then $L_0 \cong H_0$.*

Proof. First, note that the two coradicals are obviously Morita-Takeuchi equivalent, since the categories ${}^L\mathcal{M}$ and ${}^H\mathcal{M}$ are equivalent (see [?] for the Morita theory for comodule categories). It is therefore sufficient to show that L_0 is cocommutative (dually to the standard fact that Morita-equivalent commutative algebras are isomorphic; More generally, the cocenters of Morita-Takeuchi equivalent coalgebras are isomorphic by [?, Prop.2.6]). For this in turn it suffices to show by [?, Lem.1.1] that for the algebraic closure $\overline{k}$ of k the Hopf algebra $\overline{k} \otimes L$ is pointed. But $\overline{k} \otimes H$ is pointed, and $\overline{k} \otimes A$ is a $(\overline{k} \otimes L)$-$(\overline{k} \otimes H)$-Bigalois extension of $\overline{k}$. Since $(\overline{k} \otimes H)_0$ is a Hopf subalgebra, it follows from ??, ?? and ?? that the equivalence between ${}^{(\overline{k}\otimes L)_0}\mathcal{M}$ and ${}^{(\overline{k}\otimes H)_0}\mathcal{M}$ is induced by an $(\overline{k} \otimes L)_0$-$(\overline{k} \otimes H)_0$-Bigalois extension of $\overline{k}$. It follows from cocommutativity of $(\overline{k} \otimes H)_0$ that $(\overline{k} \otimes L)_0 \cong (\overline{k} \otimes H)_0$. $\square$

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