FAITHFUL FLATNESS OVER HOPF SUBALGEBRAS — COUNTEREXAMPLES

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1. Introduction

Throughout the paper we will assume that k is a field, and all tensor products, algebras, coalgebras, etc. are over k.

By the famous Nichols-Zoeller Theorem [10], every finite dimensional Hopf algebra is free over its Hopf subalgebras—a Hopf algebra version of Lagrange's theorem in group theory. In general, the first of Kaplansky's conjectures [3] was that *every* Hopf algebra H is a free module over each Hopf subalgebra $B \subset H$. This is known to be true also when B is semisimple—by another theorem of Nichols and Richmond (Zoeller) [9]—, and when B is finite dimensional and normal by Schneider [13]. However, the conjecture in its general form had already been refuted by Oberst and Schneider [11] when it appeared.

For more general Hopf algebras, a weaker condition, namely, faithful flatness over Hopf subalgebras, seems to be satisfied very often (and is used as a technical tool in many situations [5, 12, 13]). In view of many positive results, the question was asked formally in Montgomery's book [7, Question 3.5.4]: If B is a Hopf subalgebra of a Hopf algebra H, is then H always left and right faithfully flat as a B-module?

For commutative H, this is a classical result on algebraic groups [1, III, 3, no.7]; a purely Hopf algebraic proof including the stronger assertion that H is a projective B-module was given by Takeuchi [16]. Schneider [13] shows that H is B-faithfully flat if B is central in H and H is left or right noetherian.

For the cocommutative case, the assertion is proved in the theory of formal groups [2, Section 2.4.], again, a Hopf algebraic proof is in [16]. Masuoka [4] shows that it is enough to assume that the coradical of H is cocommutative.

Summing up, there are many interesting results showing that a Hopf algebra is faithfully flat over all of its Hopf subalgebras under some additional assumptions; the general question seems to remain open.

In the present paper we discuss a class of pathological counterexamples to faithful flatness over Hopf subalgebras. These are based on the observation that faithful flatness fails when we have certain pathological behavior of the antipodes of the Hopf

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(sub)algebras in consideration. In fact, we shall show that the Hopf algebra inclusions $B \subset H$ we study do not even have the following property:

(P) If f is an endomorphism of a finitely generated free B-module such that $f \underset{B}{\otimes} H$ is an isomorphism, then f is an isomorphism.

Property (P) is clearly implied by faithful flatness.

Takeuchi [14] constructed the free Hopf algebra $H(C)$ over a coalgebra C and showed that Hopf algebras with non-bijective antipodes exist. Nichols [8] computed a k-basis for $H(M_n(k)^*)$, where $M_n(k)^*$ is the dual coalgebra of the algebra of n-by-n matrices, and showed that this is a Hopf algebra whose antipode is injective but not surjective. We will show that such a Hopf algebra has a Hopf subalgebra over which it is not faithfully flat. This leads us to modify the original question and ask if at least Hopf algebras *with bijective antipode* are always faithfully flat over all their Hopf subalgebras. However, we show that whenever a Hopf algebra with bijective antipode has a Hopf subalgebra whose antipode is not bijective, the inclusion does not have property (P). In view of results of Masuoka and Wigner [6] such an inclusion is also not flat.

This reduces our problem to the first of a pair of dual questions of general Hopf algebra theory: Does every Hopf subalgebra (resp. quotient) of a Hopf algebra with bijective antipode in turn have bijective antipode?

A counterexample to the first question is obtained by a construction, Proposition 2.7, which maps every Hopf algebra H to a universal Hopf algebra $\hat{H}$ with bijective antipode, the map from H to $\hat{H}$ being injective if and only if the antipode of H is.

Obtaining counterexamples to the dual question is slightly more complicated. It seems to have been an open question wether Hopf algebras with surjective, but not injective antipodes exist (but cf. [15]). Note that the results on quotient bialgebras of Hopf algebras obtained in Nichols' paper [8] also imply that under rather mild assumptions on a Hopf algebra H with bijective antipode any of its quotient bialgebras will automatically be a Hopf algebra with bijective antipode. On the other hand, Nichols [8] considers the biideal I in $H(M_2(k)^*)$ generated by the basis element x_{21} of $M_2(k)^*$, uses the diamond lemma to calculate a basis for the quotient and shows that I is not a Hopf ideal, that is, the quotient does not have an antipode. We will modify this example, using the free Hopf algebra with bijective antipode $\hat{H}(C)$ over a coalgebra C, and find a Hopf ideal such that the quotient does not have a skew antipode. Of course, the quotient has surjective, but not bijective antipode.

2. Faithful flatness over Hopf subalgebras—counterexamples

Takeuchi [17, Cor.1.5(b)] observed that if H is a Hopf algebra with bijective antipode and $B \subset H$ is a subbialgebra such that H is a faithfully flat left and right B-module, then B is necessarily a Hopf algebra itself. We will be using variants of this observation to provide counterexamples to [7, Question 3.5.4]. In fact, we will use a property of ring inclusions, called property (P) below, which is much weaker than faithful flatness, and give a direct proof that inclusions of bialgebras into Hopf algebras, and of Hopf algebras whose antipode is not bijective into Hopf algebras with bijective antipodes, cannot have property (P).

Definition 2.1. Let $B \subset A$ be a ring inclusion. We say that the inclusion has property (P) if, for each endomorphism f of a finitely generated free right B-module M such that $f \underset{B}{\otimes} A$ is an ismorphism, f is also an isomorphism.

Remark 2.2. Clearly a left faithfully flat ring inclusion has property (P). Moreover, any ring inclusion $B \subset A$ such that B is a left B-direct summand of A has property (P). Also, if A is a generator left B-module, then the inclusion $B \subset A$ has property (P).

In terms of matrices, property (P) can be reformulated as saying that for every square matrix with entries in B which is invertible as a matrix with entries in A, the inverse has entries in B as well. Note that this implies that the condition is left-right symmetric.

Lemma 2.3. *Let $\iota\colon B \to A$ be an inclusion of algebras having property (P). Let C be a coalgebra and $\phi\colon C \to B$ a k-linear map. If $\iota\phi\colon C \to A$ is convolution invertible, then so is ϕ, that is, the convolution inverse of $\iota\phi$ has image in B.*

Proof. Note first that ϕ is convolution invertible if and only if its restriction to every finite dimensional subcoalgebra of C is. Thus we may assume that C is finite dimensional to begin with. By the Lemma on page 91 of [7] we have, for every k-algebra R, an isomorphism of k-algebras

$$T_R^C\colon \operatorname{Hom}_k(C, R) \cong \operatorname{End}^{C-}_{-R}(C \otimes R),$$

where the left hand side is an algebra by convolution, and the right hand side, which denotes the set of all right R-linear and left C-colinear endomorphisms, is an algebra by composition. The isomorphism is given explicitly by $T_R^C(\phi)(c \otimes r) = c_{(1)} \otimes \phi(c_{(2)})r$. A short computation shows that $T_A^C(\iota\phi) = T_B^C(\phi) \underset{B}{\otimes} A$ as endomorphisms of $C \otimes A = C \otimes B \underset{B}{\otimes} A$. From this, the claim follows immediately. $\square$

Our first corollary contains a direct proof of [17, Cor.1.5(b)].

Corollary 2.4. *Let H be a Hopf algebra and B a subbialgebra such that $B \subset H$ has property (P). Then B is a Hopf subalgebra. If the antipode of H is bijective, then so is the antipode of B.*

In fact the first claim follows by applying the preceding Lemma with ϕ the inclusion of B into H, the second claim follows by applying the first to $B^{\mathrm{op}} \subset H^{\mathrm{op}}$.

Since we will encounter below examples of Hopf subalgebras with non-bijective antipodes in Hopf algebras with bijective antipodes, the Corollary is sufficient to answer negatively our original question. Still, we list another Corollary which will show that a counterexample is already at hand in the literature:

Corollary 2.5. *Let H be a Hopf algebra and $B \subset H$ a Hopf subalgebra such that the inclusion $S(B) \subset H$ has property (P). Then $S(B) = B$.*

Proof. Consider the map $\phi \colon B \ni b \mapsto S(b) \in S(B)$. Writing $\iota \colon S(B) \to H$ for the inclusion, $\iota\phi$ is convolution invertible, and its inverse is the inclusion $B \to H$, which thus has its image, B, contained in $S(B)$ by Lemma 2.3. □

Remark 2.6. Nichols [8] showed that there is a Hopf algebra H whose antipode is injective, but not surjective. Applying the preceding Lemma to $B = H$ we see that the inclusion $S(H) \subset H$ does not have property (P); in particular, the answer to [7, Question 3.5.4] is negative.

Next, we want to find a Hopf algebra H *with bijective antipode* and a Hopf subalgebra B in it such that the inclusion $B \subset H$ is not faithfully flat. By Lemma 2.4 it suffices to find an example where the antipode of B is not bijective.

Proposition 2.7. *The underlying functor from the category of Hopf algebras with bijective antipode to the category of Hopf algebras has a left adjoint.*

In other words: Let H be a Hopf algebra. Then there is a Hopf algebra $\hat{H}$ with bijective antipode and a Hopf algebra map $\iota \colon H \to \hat{H}$ such that every Hopf algebra map $H \to F$ to a Hopf algebra F with bijective antipode factors uniquely through ι.

The map $\iota : H \to \hat{H}$ is injective if and only if the antipode of H is injective.

Proof. One can check in general that the colimit of a directed system of bialgebras (Hopf algebras) is again a bialgebra (resp. Hopf algebra) in the obvious way.

Let $\hat{H}$ be the colimit of the directed system of Hopf algebras $(H_i, i \in \mathbb{N}_0)$ and Hopf algebra maps $f_i \colon H_i \to H_{i+1}$ for $i \in \mathbb{N}_0$, where $H_i = H$ and $f_i = S^2$ for all i. We denote by $[h]_i$ the class in $\hat{H}$ of $h \in H = H_i$. We have a Hopf algebra map $\iota \colon H \to \hat{H}$ with $\iota(h) = [h]_0$. We define $\overline{S} \colon \hat{H} \to \hat{H}$ by $\overline{S}([h]_i) = [S(h)]_{i+1}$ (to see that this is well defined, let $t_i \colon H_i \to \hat{H}^{\mathrm{op}}$ be defined by $t_i(h) = [S(h)]_{i+1}$ and check $t_{i+1}f_i(h) = [S(f_i(h))]_{i+1} = [f_i(S(h))]_{i+1} = [S(h)]_i = t_i(h)$). Then $\overline{S}S([h]_i) = \overline{S}([S(h)]_i) = [S^2(h)]_{i+1} = [h]_i$ and $S\overline{S}([h]_i) = S([S(h)]_{i+1}) = [S^2(h)]_{i+1} = [h]_i$, so that the antipode of $\hat{H}$ is bijective with inverse $\overline{S}$.

To check the universal mapping property of $\hat{H}$, let F be a Hopf algebra with bijective antipode and $\phi \colon H \to F$. We define Hopf algebra maps $\alpha_i \colon H_i \to F$ by $\alpha_i(h) = S^{-2i}\phi(h)$ and check $\alpha_{i+1}f_i(h) = S^{-2(i+1)}S^2(h) = S^{-2i}(h) = \alpha_i(h)$, so that there exists a Hopf algebra map $\psi \colon \hat{H} \to F$ with $\psi([h]_i) = S^{-2i}\phi(h)$ for all $i \in \mathbb{N}_0$ and $h \in H$. In particular, $\psi\iota = \phi$. If $\psi' \colon \hat{H} \to F$ is another Hopf algebra map with $\psi'\iota = \phi$, then $\psi'([h]_i) = \psi'(S^{-2i}S^{2i}([h]_i)) = S^{-2i}\psi([S^{2i}(h)]_i) = S^{-2i}\psi([h]_0) = S^{-2i}\phi(h)$ and hence $\psi' = \psi$.

Clearly the antipode of H is injective if ι is injective. On the other hand $\iota(h) = [h]_0 = 0$ holds in $\hat{H}$ if and only if there exists $n \in \mathbb{N}$ with $0 = S^{2n}(h)$ in H. If the antipode of H is injective, this implies $h = 0$. □

Corollary 2.8. *There exists a Hopf algebra H with bijective antipode and a Hopf subalgebra $B \subset H$ such that the inclusion $B \subset H$ does not satisfy property (P); in particular H is not faithfully flat over B.*

Proof. Let B be a Hopf algebra with injective, but not surjective antipode. Using Proposition 2.7 we view B as a Hopf subalgebra of $H := \hat{B}$. The embedding does not have property (P) by Lemma 2.4. □

Remark 2.9. Masuoka and Wigner [6, Cor.2.9.] show that when H is a Hopf algebra with bijective antipode and $B \subset H$ is a Hopf subalgebra, then H is left B-flat iff H is left B-faithfully flat iff H is a progenerator left B-module. (Note that while the statement of [6, Cor.2.9] requires B to also have bijective antipode, the part of the Corollary just cited is proved without that hypothesis.) Hence, we have in fact proved that there exist Hopf algebra inclusions which are not flat.

One should be aware that the counterexample to (faithful) flatness thus obtained still relies entirely upon pathological behaviour of the antipode, in this case of the Hopf subalgebra in question. Thus, an important and certainly very relevant weaker version of the original question still remains open:

Question. Is every Hopf algebra with bijective antipode faithfully flat over all its Hopf subalgebras with bijective antipode?

3. There exists a Hopf algebra whose antipode is surjective, but not bijective

The following simple observation is obtained by combining Takeuchi's construction of the free Hopf algebra generated by a coalgebra [14] and Lemma 2.7:

Lemma 3.1. *The underlying functor from the category of Hopf algebras with bijective antipode to the category of coalgebras has a left adjoint, which we denote by $\hat{H}$. We call $\hat{H}(C)$ the free Hopf algebra with bijective antipode generated by the coalgebra C.*

In fact, we can construct $\hat{H}(C) := \widehat{H(C)}$, where $H(C)$ is the free Hopf algebra generated by C. Alternatively, we can copy the construction of $H(C)$ in [14]: We define coalgebras V_r for $r \in \mathbb{Z}$ by $V_r := C$ if r is even, and $V_r := C^{\mathrm{cop}}$ if r is odd. We let $V := \bigoplus_{r\in\mathbb{Z}} V_r$. We consider the map $S\colon V \to V$ given by $S((v_r)_{r\in\mathbb{Z}}) := (v_{r-1})_{r\in\mathbb{Z}}$. It induces an algebra map $S\colon T(V) \to T(V)^{\mathrm{op}}$. Let N be the ideal of $T(V)$ generated by $\{v_{(1)}S(v_{(2)}) - \varepsilon(v)\cdot 1 | v \in V\} \cup \{S(v_{(1)})v_{(2)} - \varepsilon(v)\cdot 1 | v \in V\}$. One checks that N is a biideal and hence $H(C) := T(V)/N$ is a bialgebra. Moreover, N is stable under both S and its inverse, hence S induces an invertible map $S\colon H(C) \to H(C)^{\mathrm{op}}$. One proves as in [14] that S is an antipode for $H(C)$.

Theorem 3.2. *Let (t_{ij}) be the standard k-basis of $M_4(k)^*$. Then the Hopf ideal I in $H := \hat{H}(M_4(k)^*)$ generated by the images of t_{31}, t_{32}, t_{41} and t_{42} is not stable under the inverse of the antipode. The antipode of H/I is surjective, but not injective.*

Proof. We first align our notations with those of Nichols [8]: We introduce the matrix (t_{ij}) of standard basis elements in $M_n(k)^*$. We denote basis elements in the coalgebras V_r by $x^r_{ij} := t_{ij} \in V_r$ if r is even, and $x^r_{ij} := t_{ji} \in V_r$ if r is odd. We let $X := \{x^r_{ij} | i, j \in \{1, \dots, n\}, r \in \mathbb{Z}\}$, write $\langle X\rangle$ for the free monoid generated by X and $k\langle X\rangle$ for its monoid algebra (that is, the polynomial ring with noncommuting variables x^r_{ij}). Then we have

$$\hat{H}(M_n(k)^*) \cong k\langle X\rangle \big/ N$$

with comultiplication and antipode given by $\Delta(x^r_{ij}) = \sum_{a=1}^n x^r_{ia} \otimes x^r_{aj}$ and $S(x^r_{ij}) = x^{r+1}_{ji}$, where

$$N = (\sum_{a=1}^{n} x^r_{ia}x^{r+1}_{ja} - \delta_{ij}, \sum_{a=1}^{n} x^r_{ai}x^{r-1}_{aj} - \delta_{ij} | i, j \in \{1, \dots, n\}, r \in \mathbb{Z}).$$

Next, we wish to compute a k-basis of $\hat{H}(M_n(k)^*)$ by applying Bergman's diamond lemma. Nichols did the same for $H(M_n(k)^*)$, and our calculations and results are the same as his. Namely, the following four sets of reductions, for all $i, j, k \in \{1, \dots, n\}$ and $r \in \mathbb{Z}$, satisfy the conditions of the diamond lemma:

$$\text{(I)} \qquad x^r_{in}x^{r+1}_{jn} \to \delta_{ij} - \sum_{a<n} x^r_{ia}x^{r+1}_{ja}$$

$$\text{(II)} \qquad x^r_{ni}x^{r-1}_{nj} \to \delta_{ij} - \sum_{a<n} x^r_{ai}x^{r-1}_{aj}$$

$$\text{(III)} \quad x^r_{in}x^{r+1}_{j,n-1}x^{r+2}_{k,n-1} \to \delta_{jk}x^r_{in} - \delta_{ij}x^{r+2}_{kn} + \sum_{a<n} x^r_{ia}x^{r+1}_{ja}x^{r+2}_{kn} - \sum_{a<n-1} x^r_{in}x^{r+1}_{ja}x^{r+2}_{ka}$$

$$\text{(IV)} \quad x^r_{ni}x^{r-1}_{n-1,j}x^{r-2}_{n-1,k} \to \delta_{jk}x^r_{ni} - \delta_{ij}x^{r-2}_{nk} + \sum_{a<n} x^r_{ai}x^{r-1}_{aj}x^{r-2}_{nk} - \sum_{a<n-1} x^r_{ni}x^{r-1}_{aj}x^{r-2}_{ak}$$

We also have to provide a semigroup partial order on $\langle X\rangle$ compatible with these reductions and satisfying the descending chain condition. We define $x^{r_1}_{i_1j_1} \cdots x^{r_a}_{i_aj_a} \geq x^{s_1}_{k_1\ell_1} \cdots x^{s_b}_{k_b\ell_b}$ if and only if either $a > b$ or we have $a = b$, $r_\alpha = s_\alpha$ for all α, and $(i_1, j_1, \dots, i_a, j_a) \geq (k_1, \ell_1, \dots, k_a, \ell_a)$ in lexicographical order.

We shall not prove that the conditions of the diamond lemma are really satisfied for (I) through (IV), but rather refer to Nichols' paper [8] where this is asserted for $H(M_n(k)^*)$ instead of $\hat{H}(M_n(k)^*)$ (where we have generators x^r_{ij} and corresponding relations only for $r \geq 0$).

Now to obtain the example in Theorem 3.2, we have to divide out the Hopf ideal I in $H = \hat{H}(M_4(k)^*)$ generated by the elements x^0_{ij} for $i \geq 3$, $j \leq 2$. This is the ideal generated by all the $S^r(x^0_{ij}) = x^r_{ij}$ for $r \geq 0$ even, $i \geq 3$ and $j \leq 2$, and the $S^r(x^0_{ji}) = x^r_{ij}$ for $r \geq 1$ odd, $i \leq 2$ and $j \geq 3$. The claim of the theorem reduces to $S^{-1}(I) \not\subset I$; we will show, using the diamond lemma, that (for example) $x^{-1}_{13} \neq 0$ in H/I, that is, $x^{-1}_{13} = S^{-1}(x^0_{31}) \notin I$.

The idea for the counterexample comes from Nichols' example that the biideal $(x^0_{21}) \subset H(M_2(k)^*)$ is not a Hopf ideal. Since so far all our computations in $\hat{H}(M_n(k)^*)$ looked exactly the same as the calculations for $H(M_n(k)^*)$ in [8], we would like to point out that the example does not carry over directly:

Remark 3.3. The ideal generated by x^0_{21} in $\hat{H}(M_2(k)^*)$ is a Hopf ideal stable under the inverse of the antipode.

Proof. To see this, it is sufficient to show that in the quotient $\hat{H}(M_2(k)^*)/(x^0_{21})$ we have $x^r_{21} = 0$ whenever r is even, and $x^r_{12} = 0$ whenever r is odd. Assume for example that $x^r_{21} = 0$. Then we have $1 = x^r_{11}x^{r-1}_{11} + x^r_{21}x^{r-1}_{21} = x^r_{11}x^{r-1}_{11}$ and $1 = x^{r+1}_{11}x^r_{11} + x^{r+1}_{21}x^r_{21} = x^{r+1}_{11}x^r_{11}$, whence x^r_{11} is a unit. Furthermore $0 = x^{r+1}_{12}x^r_{11} + x^{r+1}_{22}x^r_{21} = x^{r+1}x^r_{11}$ and $0 = x^r_{11}x^{r-1}_{12} + x^r_{21}x^{r-1}_{22} = x^r_{11}x^{r-1}_{12}$, whence $x^{r+1}_{12} = x^{r-1}_{12} = 0$. Similarly, one shows that $x^r_{12} = 0$ implies $x^{r+1}_{21} = x^{r-1}_{21} = 0$. □

Let us now finish the proof of Theorem 3.2. It will be sufficient that the following system of reductions satisfies the conditions of the diamond lemma (with respect to the order introduced above) and defines $\hat{H}(M_4(k)^*)/I$ as a quotient of $k\langle X\rangle$.

$$x^r_{i4}x^{r+1}_{j4} \to \delta_{ij} - \sum_{a<4} x^r_{ia}x^{r+1}_{ja} \tag{I}$$

$$x^r_{4i}x^{r-1}_{4j} \to \delta_{ij} - \sum_{a<4} x^r_{ai}x^{r-1}_{aj} \tag{II}$$

$$x^r_{i4}x^{r+1}_{j3}x^{r+2}_{k3} \to \delta_{jk}x^r_{i4} - \delta_{ij}x^{r+2}_{k4} + \sum_{a<4} x^r_{ia}x^{r+1}_{ja}x^{r+2}_{k4} - \sum_{a<3} x^r_{i4}x^{r+1}_{ja}x^{r+2}_{ka} \tag{III}$$

$$x^r_{4i}x^{r-1}_{3j}x^{r-2}_{3k} \to \delta_{jk}x^r_{4i} - \delta_{ij}x^{r-2}_{4k} + \sum_{a<4} x^r_{ai}x^{r-1}_{aj}x^{r-2}_{4k} - \sum_{a<3} x^r_{4i}x^{r-1}_{aj}x^{r-2}_{ak} \tag{IV}$$

$$(\text{for } r \geq 0 \text{ even}, i \geq 3, j \leq 2) \quad x^r_{ij} \to 0 \tag{V}$$

$$(\text{for } r \geq 1 \text{ odd}, i \leq 2, j \geq 3) \quad x^r_{ij} \to 0 \tag{VI}$$

$$(\text{for } r \geq 1, i, j \leq 2) \quad x^r_{2i}x^{r-1}_{2j} \to \delta_{ij} - x^r_{1i}x^{r-1}_{1j} \tag{VII}$$

$$(\text{for } i \leq 2) \quad x^0_{2i}x^{-1}_{2j} \to \delta_{ij} - x^0_{1i}x^{-1}_{1j} \tag{VII'}$$

$$(\text{for } r \geq 0, i, j \leq 2) \quad x^r_{i2}x^{r+1}_{j2} \to \delta_{ij} - x^r_{i1}x^{r+1}_{j1} \tag{VIII}$$

(IX) (for $r \geq 2, i, j, k \leq 2$) $x^r_{2i}x^{r-1}_{1j}x^{r-2}_{1k} \to \delta_{jk}x^r_{2i} - \delta_{ij}x^{r-2}_{2k} + x^r_{1i}x^{r-1}_{1j}x^{r-2}_{2k}$

(IX') (for $i, j \leq 2$) $x^1_{2i}x^0_{1j}x^{-1}_{1k} \to \delta_{jk}x^1_{2i} - \delta_{ij}x^{-1}_{2k} + x^1_{1i}x^0_{1j}x^{-1}_{2k}$

(X) (for $r \geq 0, i, j, k \leq 2$) $x^r_{i2}x^{r+1}_{j1}x^{r+2}_{k1} \to \delta_{jk}x^r_{i2} - \delta_{ij}x^{r+2}_{k2} + x^r_{i1}x^{r+1}_{j1}x^{r+2}_{k2}$

Note that the first six of these reductions define $\hat{H}(M_4(k)^*)/I$ as a quotient of $k\langle X\rangle$. From our computations, it will be obvious that the rules (VII) to (X) are only introduced to resolve ambiguities, so that the complete set of reductions still defines the same algebra. Since x^{-1}_{ij} are irreducible with respect to (I) through (X), the proof of Theorem 3.2 will be complete when we show that all inclusion and overlap ambiguities between (I) through (X) resolve.

(V) can be contained in (II) as the left factor: For $r \geq 0$ even and $i \leq 2$ the word $x^r_{4i}x^{r-1}_{4j}$ reduces via (V) to zero; on the other hand we have

$$x^r_{4i}x^{r-1}_{4j} \xrightarrow{\text{(II)}} \delta_{ij} - \sum_{a<4} x^r_{ai}x^{r-1}_{aj} \xrightarrow{\text{(V)}} \delta_{ij} - \sum_{a<3} x^r_{ai}x^{r-1}_{aj}.$$

This reduces to zero by (VI) if $r \geq 2$ and $j \geq 3$. To resolve the ambiguity for $r \geq 2$ and $j \leq 2$ we add one half of (VII), for $r = 0$ we add (VII').

Similarly, if $r \geq 1$ is odd and $j \leq 2$, we have $x^r_{4i}x^{r-1}_{4j} \xrightarrow{\text{(V)}} 0$ and

$$x^r_{4i}x^{r-1}_{4j} \xrightarrow{\text{(II)}} \delta_{ij} - \sum_{a<4} x^r_{ai}x^{r-1}_{aj} \xrightarrow{\text{(V)}} \delta_{ij} - \sum_{a<3} x^r_{ai}x^{r-1}_{aj}$$

which reduces to zero by (VI) if $i \geq 3$ and leads us to add the odd half of (VII) if $i \leq 2$.

(V) can occur as the first factor of (IV): For $r \geq 0$ even and $i \leq 2$ we have

$$x^r_{4i}x^{r-1}_{3j}x^{r-2}_{3k} \xrightarrow{\text{(V)}} 0$$

and

$$\begin{aligned} x^r_{4i}x^{r-1}_{3j}x^{r-2}_{3k} &\xrightarrow{\text{(IV)}} \delta_{jk}x^r_{4i} - \delta_{ij}x^{r-2}_{4k} + \sum_{a<4} x^r_{ai}x^{r-1}_{aj}x^{r-2}_{4k} - \sum_{a<3} x^r_{4i}x^{r-1}_{aj}x^{r-2}_{ak} \\ &\xrightarrow{\text{(V)}} -\delta_{ij}x^{r-2}_{4k} + \sum_{a<3} x^r_{ai}x^{r-1}_{aj}x^{r-2}_{4k} \end{aligned}$$

This reduces to zero by (VII) or (VII') if $j \leq 2$ or $r = 0$. If $r \geq 2$ and $j \geq 3$ it reduces to zero by (VI).

(V) can occur as the second factor of (IV): For $r \geq 1$ odd and $j \leq 2$ we have

$$x^r_{4i}x^{r-1}_{3j}x^{r-2}_{3k} \xrightarrow{\text{(V)}} 0$$

and

$$x_{4i}^r x_{3j}^{r-1} x_{3k}^{r-2} \xrightarrow{\text{(IV)}} \delta_{jk} x_{4i}^r - \delta_{ij} x_{4k}^{r-2} + \sum_{a<4} x_{ai}^r x_{aj}^{r-1} x_{4k}^{r-2} - \sum_{a<3} x_{4i}^r x_{aj}^{r-1} x_{ak}^{r-2}$$
$$\xrightarrow{\text{(V)}} \delta_{jk} x_{4i}^r - \delta_{ij} x_{4k}^{r-2} + \sum_{a<3} x_{ai}^r x_{aj}^{r-1} x_{4k}^{r-2} - \sum_{a<3} x_{4i}^r x_{aj}^{r-1} x_{ak}^{r-2}$$

Using (VI) if $i \geq 3$ or (VII) if $i \leq 2$, this reduces further to

$$\delta_{jk} x_{4i}^r - \sum_{a<3} x_{4i}^r x_{aj}^{r-1} x_{ak}^{r-2}.$$

If $r = 1$ this reduces to zero by (VII'), if $r \geq 3$ and $k \leq 2$ it reduces to zero by (VII), and if $r \geq 3$ and $k \geq 3$, it reduces to zero by (VI).

(V) can occur as the third factor of (IV): For $r \geq 2$ even and $k \leq 2$, we have

$$x_{4i}^r x_{3j}^{r-1} x_{3k}^{r-2} \xrightarrow{\text{(V)}} 0$$

and

$$x_{4i}^r x_{3j}^{r-1} x_{3k}^{r-2} \xrightarrow{\text{(IV)}} \delta_{jk} x_{4i}^r - \delta_{ij} x_{4k}^{r-2} + \sum_{a<4} x_{ai}^r x_{aj}^{r-1} x_{4k}^{r-2} - \sum_{a<3} x_{4i}^r x_{aj}^{r-1} x_{ak}^{r-2}$$
$$\xrightarrow{\text{(V)}} \delta_{jk} x_{4i}^r - \sum_{a<3} x_{4i}^r x_{aj}^{r-1} x_{ak}^{r-2},$$

which reduces to zero by (VII) if $j \leq 2$, and by (VI) if $j \geq 3$.

(VI) can occur as the left factor of (I): If $r \geq 1$ is odd and $i \leq 2$,

$$x_{i4}^r x_{j4}^{r+1} \xrightarrow{\text{(VI)}} 0,$$

while

$$x_{i4}^r x_{j4}^{r+1} \xrightarrow{\text{(I)}} \delta_{ij} - \sum_{a<4} x_{ia}^r x_{ja}^{r+1} \xrightarrow{\text{(VI)}} \delta_{ij} - \sum_{a<3} x_{ia}^r x_{ja}^{r+1}$$

If $j \geq 3$, this reduces to zero by (V), otherwise it forces us to introduce the odd half of (VIII).

(VI) can occur as the right factor of (I), leading in the same way to the even half of (VIII).

(VI) can occur as the leftmost factor of (III): If $r \geq 1$ is odd and $i \leq 2$,

$$x_{i4}^r x_{j3}^{r+1} x_{k3}^{r+2} \xrightarrow{\text{(VI)}} 0,$$

and

$$x_{i4}^r x_{j3}^{r+1} x_{k3}^{r+2} \xrightarrow{\text{(III)}} \delta_{jk} x_{i4}^r - \delta_{ij} x_{k4}^{r+2} + \sum_{a<4} x_{ia}^r x_{ja}^{r+1} x_{k4}^{r+2} - \sum_{a<3} x_{i4}^r x_{ja}^{r+1} x_{ka}^{r+2}$$
$$\xrightarrow{\text{(VI)}} -\delta_{ij} x_{k4}^{r+2} + \sum_{a<3} x_{ia}^r x_{ja}^{r+1} x_{k4}^{r+2}.$$

This reduces to zero by (V) if $j \geq 3$ or by (VIII) if $j \leq 2$.

(VI) can occur as the middle factor of (III): If $r \geq 0$ is even and $j \leq 2$,

$$x^r_{i4}x^{r+1}_{j3}x^{r+2}_{k3} \xrightarrow{\text{(VI)}} 0,$$

and

$$\begin{aligned} x^r_{i4}x^{r+1}_{j3}x^{r+2}_{k3} &\xrightarrow{\text{(III)}} \delta_{jk}x^r_{i4} - \delta_{ij}x^{r+2}_{k4} + \sum_{a<4} x^r_{ia}x^{r+1}_{ja}x^{r+2}_{k4} - \sum_{a<3} x^r_{i4}x^{r+1}_{ja}x^{r+2}_{ka} \\ &\xrightarrow{\text{(VI)}} \delta_{jk}x^r_{i4} - \delta_{ij}x^{r+2}_{k4} + \sum_{a<3} x^r_{ia}x^{r+1}_{ja}x^{r+2}_{k4} - \sum_{a<3} x^r_{i4}x^{r+1}_{ja}x^{r+2}_{ka}. \end{aligned}$$

By applying (VIII) if $i \leq 2$ or (V) if $i \geq 3$ we can reduce this to

$$\delta_{jk}x^r_{i4} - \sum_{a<3} x^r_{i4}x^{r+1}_{ja}x^{r+2}_{ka},$$

which in turn reduces to zero using (VIII) if $k \leq 2$ or (V) if $k \geq 3$.

(VI) can occur as the rightmost factor in (III): If $r \geq -1$ is odd and $k \leq 2$,

$$x^r_{i4}x^{r+1}_{j3}x^{r+2}_{k3} \xrightarrow{\text{(VI)}} 0,$$

and

$$\begin{aligned} x^r_{i4}x^{r+1}_{j3}x^{r+2}_{k3} &\xrightarrow{\text{(III)}} \delta_{jk}x^r_{i4} - \delta_{ij}x^{r+2}_{k4} + \sum_{a<4} x^r_{ia}x^{r+1}_{ja}x^{r+2}_{k4} - \sum_{a<3} x^r_{i4}x^{r+1}_{ja}x^{r+2}_{ka} \\ &\xrightarrow{\text{(VI)}} \delta_{jk}x^r_{i4} - \sum_{a<3} x^r_{i4}x^{r+1}_{ja}x^{r+2}_{ka}. \end{aligned}$$

This reduces to zero by (V) if $j \geq 3$ and by (VIII) if $j \leq 2$.

(VII) cannot overlap any of (I)...(VI).

The right factor of (VII') may overlap (I): For $i \leq 2$

$$x^0_{2i}x^{-1}_{24}x^0_{j4} \xrightarrow{\text{(VII')}} -x^0_{1i}x^{-1}_{14}x^0_{j4} \xrightarrow{\text{(I)}} -\delta_{1j}x^0_{1i} + \sum_{a<4} x^0_{1i}x^{-1}_{1a}x^0_{ja}$$

and

$$\begin{aligned} x^0_{2i}x^{-1}_{24}x^0_{j4} &\xrightarrow{\text{(I)}} \delta_{2j}x^0_{2i} - \sum_{a<4} x^0_{2i}x^{-1}_{2a}x^0_{ja} \\ &\xrightarrow{\text{(VII')}} \delta_{2j}x^0_{2i} - \sum_{a<4}(\delta_{ia}x^0_{ja} - x^0_{1i}x^{-1}_{1a}x^0_{ja}) \\ &= \delta_{2j}x^0_{2i} - x^0_{ji} + \sum_{a<4} x^0_{1i}x^{-1}_{1a}x^0_{ja} \end{aligned}$$

The results of the two reductions differ by $x^0_{ji} - \delta_{1j}x^0_{1i} - \delta_{2j}x^0_{2i}$, which vanishes for $j \leq 2$, and equals x^0_{ji} for $j \geq 3$. The latter reduces to zero by (V).

The right hand factor of (VII') can overlap (III). For $i \leq 2$

$$
\begin{aligned}
&x^0_{2i}x^{-1}_{24}x^0_{j3}x^1_{k3}\\
&\quad\xrightarrow{\text{(III)}} \delta_{jk}x^0_{2i}x^{-1}_{24} - \delta_{2j}x^0_{2i}x^1_{k4} + \sum_{a<4} x^0_{2i}x^{-1}_{2a}x^0_{ja}x^1_{k4} - \sum_{a<3} x^0_{2i}x^{-1}_{24}x^0_{ja}x^1_{ka}\\
&\quad\xrightarrow{\text{(VII')}} -\delta_{jk}x^0_{1i}x^{-1}_{14} - \delta_{2j}x^0_{2i}x^1_{k4} + \sum_{a<4}(\delta_{ia} - x^0_{1i}x^{-1}_{1a})x^0_{ja}x^1_{k4} + \sum_{a<3} x^0_{1i}x^{-1}_{14}x^0_{ja}x^1_{ka}\\
&\quad = -\delta_{jk}x^0_{1i}x^{-1}_{14} - \delta_{2j}x^0_{2i}x^1_{k4} + x^0_{ji}x^1_{k4} - \sum_{a<4} x^0_{1i}x^{-1}_{1a}x^0_{ja}x^1_{k4} + \sum_{a<3} x^0_{1i}x^{-1}_{14}x^0_{ja}x^1_{ka}
\end{aligned}
$$

and on the other hand

$$
\begin{aligned}
x^0_{2i}x^{-1}_{24}x^0_{j3}x^1_{k3} &\xrightarrow{\text{(VII')}} -x^0_{1i}x^{-1}_{14}x^0_{j3}x^1_{k3}\\
&\xrightarrow{\text{(III)}} -\delta_{jk}x^0_{1i}x^{-1}_{14} + \delta_{1j}x^0_{1i}x^1_{k4} - \sum_{a<4} x^0_{1i}x^{-1}_{1a}x^0_{ja}x^1_{k4} + \sum_{a<3} x^0_{1i}x^{-1}_{14}x^0_{ja}x^1_{ka}
\end{aligned}
$$

The results differ by $x^0_{ji}x^1_{k4} - \delta_{1j}x^0_{1i}x^1_{k4} - \delta_{2j}x^0_{2i}x^1_{k4}$, which vanishes if $j \leq 2$, and reduces to zero by (V) if $j \geq 3$.

(VII) can overlap with itself as well as with (VII'): If $r \geq 2$ and $i, j, k \leq 2$ or $r = 1$ and $i, j \leq 2$, we have

$$
\begin{aligned}
x^r_{2i}x^{r-1}_{2j}x^{r-2}_{2k} &\xrightarrow{\text{(VII)}} \delta_{ij}x^{r-2} - x^r_{1i}x^{r-1}_{2k}\\
x^r_{2i}x^{r-1}_{2j}x^{r-2}_{2k} &\xrightarrow{\text{(VII) or (VII')}} \delta_{jk}x^r_{2i} - x^r_{2i}x^{r-1}_{1j}x^{r-2}_{1k}
\end{aligned}
$$

which forces us to introduce rules (IX) and (IX'), respectively.

Similarly, the possible overlaps of (VIII) with itself lead to (X).

Note that (VII), (VIII), (IX) and (X) are precisely the reduction rules that define $H(M_2(k)^*)$. In particular, we need not consider any overlaps involving only those reduction rules.

By a short inspection we see that the only overlaps left to deal with are those involving (IX') with (I), (III) or (VIII). We omit treating the overlap of (the rightmost factor of) (VIII) with (the leftmost factor of) (IX') since it resolves in the same way as that of (VIII) with (IX).

The rightmost factor of (IX') may overlap (I): For $i, j \leq 2$ we have

$$
\begin{aligned}
x^1_{2i}x^0_{1j}x^{-1}_{14}x^0_{k4} &\xrightarrow{\text{(IX')}} -\delta_{ij}x^{-1}_{24}x^0_{k4} + x^1_{1i}x^0_{1j}x^{-1}_{24}x^0_{k4}\\
&\xrightarrow{\text{(I)}} -\delta_{ij}\delta_{2k} + \delta_{ij}\sum_{a<4} x^{-1}_{2a}x^0_{ka} + \delta_{2k}x^1_{1i}x^0_{1j} - \sum_{a<4} x^1_{1i}x^0_{1j}x^{-1}_{2a}x^0_{ka},
\end{aligned}
$$

and, on the other hand,

$$\begin{aligned} x^1_{2i}x^0_{1j}x^{-1}_{14}x^0_{k4} &\xrightarrow{\text{(I)}} \delta_{1k}x^1_{2i}x^0_{1j} - \sum_{a<4} x^1_{2i}x^0_{1j}x^{-1}_{1a}x^0_{ka} \\ &\xrightarrow{\text{(IX')}} \delta_{1k}x^1_{2i}x^0_{1j} - \sum_{a<4}(\delta_{ja}x^1_{2i} - \delta_{ij}x^{-1}_{2a} + x^1_{1i}x^0_{1j}x^{-1}_{2a})x^0_{ka} \\ &= \delta_{1k}x^1_{2i}x^0_{1j} - x^1_{2i}x^0_{kj} + \sum_{a<4}\delta_{ij}x^{-1}_{2a}x^0_{ka} - \sum_{a<4} x^1_{1i}x^0_{1j}x^{-1}_{2a}x^0_{ka}. \end{aligned}$$

The results of the two reductions differ by

$$x^1_{2i}x^0_{kj} - \delta_{1k}x^1_{2i}x^0_{1j} + \delta_{2k}x^1_{1i}x^0_{1j} - \delta_{ij}\delta_{2k}.$$

If $k=1$, this vanishes. If $k=2$, it equals $x^1_{2i}x^0_{2j} + x^1_{1i}x^0_{1j} - \delta_{ij}$ and reduces to zero by (VII). If $k \geq 3$, we have $x^1_{2i}x^0_{kj}$ and can reduce to zero by (V).

The last, and most complicated, overlap to deal with is that of the right factor of (XI') with (III). For $i, j \leq 2$ we have the reductions

$$\begin{aligned} & x^1_{2i}x^0_{1j}x^{-1}_{14}x^0_{k3}x^1_{\ell 3} \\ &\xrightarrow{\text{(III)}} \delta_{k\ell}x^1_{2i}x^0_{1j}x^{-1}_{14} - \delta_{1k}x^1_{2i}x^0_{1j}x^1_{\ell 4} + \sum_{a<4} x^1_{2i}x^0_{1j}x^{-1}_{1a}x^0_{ka}x^1_{\ell 4} - \sum_{a<3} x^1_{2i}x^0_{1j}x^{-1}_{14}x^0_{ka}x^1_{\ell a} \\ &\xrightarrow{\text{(IX')}} \ -\delta_{k\ell}\delta_{ij}x^{-1}_{24} + \delta_{k\ell}x^1_{1i}x^0_{1j}x^{-1}_{24} - \delta_{1k}x^1_{2i}x^0_{1j}x^1_{\ell 4} \\ &\qquad + \sum_{a<4}(\delta_{ja}x^1_{2i} - \delta_{ij}x^{-1}_{2a} + x^1_{1i}x^0_{1j}x^{-1}_{2a})x^0_{ka}x^1_{\ell 4} - \sum_{a<3}(-\delta_{ij}x^{-1}_{24} + x^1_{1i}x^0_{1j}x^{-1}_{24})x^0_{ka}x^1_{\ell a} \\ &= (x^1_{1i}x^0_{1j} - \delta_{ij})(\delta_{k\ell}x^{-1}_{24} + \sum_{a<4} x^{-1}_{2a}x^0_{ka}x^1_{\ell 4} - \sum_{a<3} x^{-1}_{24}x^0_{ka}x^1_{\ell a}) - \delta_{1k}x^1_{2i}x^0_{1j}x^1_{\ell 4} + x^1_{2i}x^0_{kj}x^1_{\ell 4} \end{aligned}$$

and

$$\begin{aligned} x^1_{2i}x^0_{1j}x^{-1}_{14}x^0_{k3}x^1_{\ell 3} &\xrightarrow{\text{(IX')}} (-\delta_{ij}x^{-1}_{24} + x^1_{1i}x^0_{1j}x^{-1}_{24})x^0_{k3}x^1_{\ell 3} \\ &\xrightarrow{\text{(III)}} (x^1_{1i}x^0_{1j} - \delta_{ij})(\delta_{k\ell}x^{-1}_{24} - \delta_{2k}x^1_{\ell 4} + \sum_{a<4} x^{-1}_{2a}x^0_{ka}x^1_{\ell 4} - \sum_{a<3} x^{-1}_{24}x^0_{ka}x^1_{\ell a}) \end{aligned}$$

The results differ by

$$x^1_{2i}x^0_{kj}x^1_{\ell 4} - \delta_{1k}x^1_{2i}x^0_{1j}x^1_{\ell 4} - \delta_{2k}\delta_{ij}x^1_{\ell 4} + \delta_{2k}x^1_{1i}x^0_{1j}x^1_{\ell 4}.$$

This vanishes if $k=1$. If $k=2$, it equals

$$(x^1_{2i}x^0_{2j} + x^1_{1i}x^0_{1j} - \delta_{ij})x^1_{\ell 4}$$

and reduces to zero by (VII). For $k \geq 3$ the result is $x^1_{2i}x^0_{kj}x^1_{\ell 4}$ and reduces to zero by (V). $\square$

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