FACE ALGEBRAS ARE $\times_R$-BIALGEBRAS

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Abstract. Both R-face algebras and $\times_R$-bialgebras are defined by weakening the axioms of a k-bialgebra over a base field k. The first notion was defined and studied in a recent series of papers by T. Hayashi in the context of conformal field theory and Jones' index theory. The second notion was defined and studied in the late 70's by M. Takeuchi. The background are generalizations of the Brauer group, and cohomology of Hopf algebras.

The purpose of the present note is to point out that face algebras can be considered as special cases of $\times_R$-bialgebras.

We will also show that the notion of comodule defined over any $\times_R$-bialgebra, and the tensor product defined for such comodules, contain the truncated tensor product of comodules over a face algebra as a special case.

1. Introduction

Face algebras were studied by T. Hayashi in a recent series of papers [3, 4, 5, 1, 2]. They arise in the context of solvable lattice models, and of Jones' index theory of subfactors. Thus, face algebras are genuine examples of quantum groups as far as their area of applications is concerned, but they are not Hopf algebras, not even bialgebras. Recall that a k-bialgebra is a k-algebra and k-coalgebra H such that the comultiplication $\Delta : H \to H \otimes H$ and counit $\varepsilon : H \to k$ are algebra maps. The definition of a face algebra involves a fixed commutative separable k-algebra R. An R-face algebra is then a k-algebra and k-coalgebra, such that the comultiplication is a multiplicative map $\Delta : H \to H \otimes H$. The remaining bialgebra axioms are replaced by axioms involving an algebra map $R^* \otimes R \to H$, the special case of a bialgebra being $R = k$. Hayashi's first definition is included as the special case $R = k^{\mathcal{V}}$ with a finite set $\mathcal{V}$, and involves axioms pertaining to a square matrix of elements of H with rows and columns indexed by $\mathcal{V}$.

$\times_R$-Bialgebras were introduced by M. Takeuchi [8], generalizing a definition of M. Sweedler [7]. The applications in [7, 8] are generalizations of Brauer groups, and Hopf algebra cohomology. The notion of a $\times_R$-bialgebra is also a weakening of that of a k-bialgebra, but in a rather different way. The definition is made with reference to a fixed k-algebra R, which has to be neither commutative nor separable.

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Comultiplication and counit of a $\times_R$-bialgebra H are required to be algebra maps and compatible with a fixed algebra map $R\otimes\overline{R}\to H$ (where $\overline{R}$ denotes the opposite algebra of R.) However, this time the comultiplication and counit do not make H a coalgebra; rather than in $H\otimes H$, the comultiplication takes values in a different space, namely the centralizer of R in a certain R-R-bimodule $H\underset{R}{\otimes}H$, where the two copies of H in the last expression have different R-R-bimodule structures. The counit takes values in $\mathrm{End}(R)$ instead of k. A coassociativity and counit property are required.

We will show that the recent notion of an R-face algebra is in fact precisely the special case of separable commutative R of the older notion of a $\times_R$-bialgebra. More precisely, there is, for each k-algebra H and separable commutative k-algebra R, a bijection between R-face algebra structures and $\times_R$-bialgebra structures on H. The bijection is defined in such a way that, when the comultiplication of a face algebra is given by a formula, the same formula, considered in a different space, also defines the comultiplication in the corresponding $\times_R$-bialgebra.

For any k-algebra R and $\times_R$-bialgebra H there is a notion of left H-comodule (defined in the obvious way in [6]); these left H-comodules form a monoidal category with the tensor product over R. We will show that this monoidal category specializes to the category of comodules over a face algebra with truncated tensor product when R is commutative separable.

Throughout the paper k will be a fixed field; algebras, coalgebras, and tensor products are over k if not specified otherwise.

2. $\times_R$-BIALGEBRAS

We recall the axiomatics of $\times_R$-bialgebras as defined by Takeuchi [8]. Fix a k-algebra R. We denote the opposite algebra of R by $\overline{R}$, and write $R\ni r\mapsto\overline{r}\in\overline{R}$ for the canonical algebra antiisomorphism. For two $R\otimes\overline{R}$-bimodules M and N we denote

$$\int^r {}_{\overline{r}}M\otimes{}_rN := \{\sum m_i\otimes n_i\in M\otimes N|\forall r\in R: \sum\overline{r}m_i\otimes n_i=\sum m_i\otimes rn_i\}$$

$$\int_r M_{\overline{r}}\otimes N_r := M\otimes N/k\text{-span}\{m\overline{r}\otimes nr|r\in R, m\in M, n\in N\}$$

$$\int^s\int_r {}_{\overline{r}}M_{\overline{s}}\otimes{}_rN_s := \{\sum m_i\otimes n_i\in\int_r {}_{\overline{r}}M\otimes{}_rN|\forall r\in R: \sum\overline{r}m_i\otimes n_i=\sum m_i\otimes rn_i\}$$

We will use other instances of the symbols $\int^r$ and $\int_r$ whose meaning will be an obvious variation of the above examples. We write $M\times_R N := \int^s\int_r {}_{\overline{r}}M_{\overline{s}}\otimes{}_rN_s$.

An algebra over $R\otimes\overline{R}$ is a k-algebra H with a k-algebra map $\iota: R\otimes\overline{R}\to H$, which we will write as $\iota(r\otimes\overline{s}) =: r\overline{s}\in H$. A map of algebras over $R\otimes\overline{R}$ will mean an algebra map compatible with the respective maps ι.

If M and N are algebras over $R\otimes\overline{R}$, then so is $M\times_R N$ with the multiplication $(\sum m_i\otimes n_i)(\sum m'_j\otimes n'_j)=\sum m_i m'_j\otimes n_i n'_j$ and $\iota(r\otimes\overline{s})=r\otimes\overline{s}\in M\times_R N$.

For three $R\otimes\overline{R}$-bimodules M,P,N we define

$$M\times_R P\times_R N:=\int^{s,q}\int_{r,p}{}_{\overline{r}}M_{\overline{s}}\otimes{}_{r,\overline{p}}P_{s,\overline{q}}\otimes{}_pN_q$$

(where we have abbreviated $\int^{s,q}:=\int^s\int^q=\int^q\int^s$. There are associativity maps

$$\begin{aligned}(M\times_R P)\times_R N&\xrightarrow{\alpha}M\times_R P\times_R N\\ \sum_j(\sum_i m_{ij}\otimes p_{ij})\otimes n_j&\mapsto\sum_{i,j}m_{ij}\otimes p_{ij}\otimes n_j\\ M\times_R(P\times_R N)&\xrightarrow{\alpha'}M\times_R P\times_R N\\ \sum_i m_i\otimes\sum_j p_{ij}\otimes n_{ij}&\mapsto\sum_{i,j}m_i\otimes p_{ij}\otimes n_{ij}\end{aligned}$$

If M and N are algebras over $R\otimes\overline{R}$-bimodules, then so are $(M\times_R P)\times_R N$, $M\times_R(P\times_R N)$ and $M\times_R P\times_R N$, and the maps α,α' are maps of algebras over $R\otimes\overline{R}$.

The algebra $\mathrm{End}(R)$ is an algebra over $R\otimes\overline{R}$ with $\iota(q\otimes\overline{r})(s):=qsr$. For any $R\otimes\overline{R}$-bimodule M we have $R\otimes\overline{R}$-bimodule maps

$$\begin{aligned}\theta:M\times_R\mathrm{End}(R)&\to M\\ \sum m_i\otimes f_i&\mapsto\sum\overline{f_i(1)}m_i\\ \theta':\mathrm{End}(R)\times_R M&\to M\\ \sum f_i\otimes m_i&\mapsto\sum f_i(1)n_i\end{aligned}$$

which are maps of algebras over $R\otimes\overline{R}$ if M is an algebra over $R\otimes\overline{R}$.

Now a $\times_R$-bialgebra is an algebra H over $R\otimes\overline{R}$ together with two maps of algebras over $R\otimes\overline{R}$, a comultiplication $\Gamma:H\to H\times_R H$ and a counit $C:H\to\mathrm{End}(R)$, for which the diagrams

$$\begin{tikzcd}H \arrow[r, "\Gamma"] \arrow[d, "\Gamma"'] & H\times_R H \arrow[d, "H\times_R\Gamma"]\\ H\times_R H \arrow[d, "\Gamma\times_R H"'] & H\times_R(H\times_R H) \arrow[d, "\alpha'"]\\ (H\times_R H)\times_R H \arrow[r, "\alpha"'] & H\times_R H\times_R H\end{tikzcd}$$

$$\begin{tikzcd} H \arrow[r, "\Gamma"] \arrow[d, "\mathrm{id}"'] & H \times_R H \arrow[d, "H \times_R \varepsilon"] \\ H & H \times_R \mathrm{End}(R) \arrow[l, "\theta"] \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "\Gamma"] \arrow[d, "\mathrm{id}"'] & H \times_R H \arrow[d, "\varepsilon \times_R H"] \\ H & \mathrm{End}(R) \times_R H \arrow[l, "\theta'"] \end{tikzcd}$$

commute.

We will use the notation $\Gamma(h) =: h_{[1]} \otimes h_{[2]}$ for comultiplication in a $\times_R$-algebra.

3. Face algebras

Next, we recall the axiomatics of face algebras, due to Hayashi, see [5].

An almost bialgebra H is a k-algebra and k-coalgebra whose comultiplication $\Delta : H \to H \otimes H$ is multiplicative, that is, fulfills

$$\Delta(gh) = \Delta(g)\Delta(h)$$

for all $g, h \in H$. None of the usual compatibility conditions of a bialgebra involving the unit of the algebra and the counit ε of the coalgebra are assumed.

Let R be a commutative separable k-algebra with separability idempotent $e = \sum e_i \otimes e'_i \in (R \otimes R)^R$. Let $t : R \to k$ be the trace functional. Then t is nondegenerate; that is, the bilinear form $R \times R \ni (r, s) \mapsto t(rs) \in k$ is nondegenerate and induces an isomorphism $R \ni r \mapsto \mathring{r} := t(r-) \in R^*$. The tensor $\sum e_i \otimes e'_i$ is a dual basis with respect to this bilinear form: We have $\sum t(re_i)e'_i = r = \sum e_i t(e'_i r)$ for all $r \in R$. In particular, the separabilty element e is unique, and $\sum e_i \otimes e'_i = \sum e'_i \otimes e_i$ holds. The algebra R is an almost bialgebra with comultiplication $\Delta(r) = \sum re_i \otimes e'_i$ and counit t. The almost bialgebra R is selfdual: The isomorphism $R \ni r \mapsto \mathring{r} := t(r-) \in R^*$ is an algebra and coalgebra isomorphism, with inverse $R^* \ni \varphi \mapsto \mathring{\varphi} := \sum \varphi(e_i)e'_i \in R$. The algebra $\mathfrak{E}(R) := R^* \otimes R$ is an almost bialgebra with the comultiplication $\Delta_{\mathfrak{E}(R)}(\varphi \otimes r) := \sum \varphi \otimes e_i \otimes \mathring{e}'_i \otimes r$.

An R-face algebra is an almost bialgebra H together with an algebra and coalgebra homomorphism $j : \mathfrak{E}(R) \to H$, which we will denote by $j(\varphi \otimes r) =: \varphi\overline{r} \in H$, satisfying

$$\forall g, h \in H : \qquad \sum \varepsilon(g\overline{e}_i)\varepsilon(\mathring{e}'_i h) = \varepsilon(gh) \tag{3.1}$$

(The notation $j(1 \otimes r) = \overline{r}$ instead of just r is an unnecessary complication for dealing with face algebras alone, but will be convenient for the comparison with $\times_R$-bialgebras.)

Important examples of R-face algebras occur when $R = k^{\mathcal{V}}$ is a direct prouct of a finite number of copies of the base field, indexed by a finite set $\mathcal{V}$. In this case the axiomatics were defined separately and previously to the above definition, also by Hayashi, see [4]. The structure of an R-face algebra on H for $R = k^{\mathcal{V}}$ is equivalent to the structure of a $\mathcal{V}$-face algebra, which in turn is defined as an algebra and coalgebra, with multiplicative comultiplication, together with a square matrix of elements $e\binom{i}{j} \in H$, with rows and columns indexed by $\mathcal{V}$, such that the equations

$$
\begin{aligned}
e\begin{pmatrix} i \\ j \end{pmatrix} e\begin{pmatrix} i' \\ j' \end{pmatrix} &= \delta_{ii'}\delta_{jj'} e\begin{pmatrix} i \\ j \end{pmatrix} \\
\sum_{k,\ell} e\begin{pmatrix} k \\ \ell \end{pmatrix} &= 1 \\
\Delta\left(e\begin{pmatrix} i \\ j \end{pmatrix}\right) &= \sum_k e\begin{pmatrix} i \\ k \end{pmatrix} \otimes e\begin{pmatrix} k \\ j \end{pmatrix} \\
\varepsilon\left(e\begin{pmatrix} i \\ j \end{pmatrix}\right) &= \delta_{ij} \\
\sum_k \varepsilon(ge_i)\varepsilon(\mathring{e}_k h) &= \varepsilon(gh)
\end{aligned}
$$

hold, where $e_j := \sum_i e\binom{i}{j}$ and $\mathring{e}_i := \sum_j e\binom{i}{j}$.

4. Face algebras are $\times_R$-bialgebras

In this section we show that R-face algebras can be considered as the special case of $\times_R$-bialgebras for commutative separable R. While, of course, face algebra structures and $\times_R$-bialgebra structures are given by maps with different codomains, the two sets of data are in bijective correspondence for such special R.

One of the basic ideas of the proof is the following very easy observation.

Lemma 4.1. *Let R be a commutative separable k-algebra and M an R-bimodule. Then the identity on M induces an isomorphism*

$$\int^r {}_rM_r \to \int_r {}_rM_r$$

whose inverse maps $m \in \int_r {}_rM_r$ to $\sum e_i m e_i'$.

The inverse map is well defined because $\sum e_i \otimes e_i'$ is in the centralizer of R in the bimodule $R \otimes R$. That the two maps are inverse to each other is straightforward to check.

A first consequence is that, in the case of a separable commutative k-algebra R, the $\times_R$-product is very well behaved.

Corollary 4.2. *Let R be a commutative separable k-algebra. Then for any three $R \otimes \overline{R}$-bimodules M, P, N the maps*

$$\begin{aligned} \alpha &: (M \times_R P) \times_R N \to M \times_R P \times_R N \\ \alpha' &: M \times_R (P \times_R N) \to M \times_R P \times_R N \\ \theta &: M \times_R \operatorname{End}(R) \to M \\ \theta' &: \operatorname{End}(R) \times_R M \to M \end{aligned}$$

are isomorphisms; in particular, any triple of $R \otimes \overline{R}$-bimodules associates in the sense of [8]*; the category of $R \otimes \overline{R}$-bimodules is a monoidal category with tensor product $\times_R$.*

In fact, the products $(M \times_R P) \times_R N$, $M \times_R (P \times_R N)$ and $M \times_R N \times_R P$ all identify naturally with $\int_{r,s,p,q} {}_{\overline{r}}M_{\overline{s}} \otimes {}_{r,\overline{p}}P_{s,\overline{q}} \otimes {}_pN_q$ in view of the Lemma. Moreover,

$$\begin{aligned} \operatorname{End}(R) \times_R M &= \int^s \int_r {}_{\overline{r}}\operatorname{End}(R)_{\overline{s}} \otimes {}_rM_s \\ &= \int^{r,s} \operatorname{Hom}(R_s, R_r) \otimes {}_rM_s \\ &\cong \int^{r,s} \operatorname{Hom}(R_s, R_r \otimes {}_rM_s) \\ &\cong \int_r R_r \otimes {}_rM \cong M \end{aligned}$$

and similarly for $M \times_R \operatorname{End}(R)$.

Theorem 4.3. *Let H be a k-algebra, and let R be a commutative separable k-algebra. Then there is a bijection between R-face algebra structures on H and $\times_R$-bialgebra structures on H.*

Proof. The underlying bijection between algebra maps $j : \mathfrak{E}(R) \to H$ and algebra maps $\iota : R \otimes \overline{R} \to H$ will be the trivial one: $j(\varphi \otimes r) := \iota(\mathring{\varphi} \otimes \overline{r})$. Thanks to the choice of notation for j in section 3 we can write $\iota(r \otimes \overline{s}) = r\overline{s}$ and $j(\varphi \otimes r) = \mathring{\varphi}\overline{r}$ consistently. We fix corresponding algebra maps ι and j for the rest of the proof

Assume we are given a $\times_R$-bialgebra structure on H, with comultiplication $\Gamma : H \to H \times_R H$ and counit $C : H \to \operatorname{End}(R)$. Define

$$\begin{aligned} \Delta(h) &:= \sum \overline{e}_i h_{[1]} \otimes e'_i h_{[2]} \\ \varepsilon(h) &:= t(C(h)(1)) \end{aligned}$$

for $h \in H$. Note that $\Delta(h)$ is a well defined element of $H \otimes H$ by a variation of the Lemma, so that we have defined maps $\Delta : H \to H \otimes H$ and $\varepsilon : H \to k$ We claim that these maps give H the structure of an R-face algebra.

(1) Coassociativity of Δ:

$$\begin{aligned}(\Delta\otimes H)\Delta(h) &= \sum \Delta(\overline{e}_i h_{[1]})\otimes e_i' h_{[2]}\\ &= \sum \overline{e}_j(\overline{e}_i h_{[1]})_{[1]}\otimes e_j'(\overline{e}_i h_{[1]})_{[2]}\otimes e_i' h_{[2]}\\ &= \sum \overline{e}_j h_{[1]}\otimes \overline{e}_i e_j' h_{[2]}\otimes e_i' h_{[3]}\\ &= \sum \overline{e}_j h_{[1]}\otimes \overline{e}_i(e_j' h_{[2]})_{[1]}\otimes e_i'(e_j' h_{[2]})_{[2]}\\ &= (H\otimes\Delta)\Delta(h)\end{aligned}$$

where the third and fourth equality use that Γ is a map of algebras over $R\otimes\overline{R}$.

(2) Comultiplicativity of Δ:

$$\begin{aligned}\Delta(gh) &= \sum \overline{e}_i(gh)_{[1]}\otimes e_i'(gh)_{[2]}\\ &= \sum \overline{e}_i g_{[1]}h_{[1]}\otimes e_i' g_{[2]}h_{[2]}\\ &= \sum \overline{e}_i g_{[1]}h_{[1]}\otimes e_i' g_{[2]}e_j e_j' h_{[2]}\\ &= \sum \overline{e}_i g_{[1]}\overline{e}_j h_{[1]}\otimes e_i' g_{[2]}e_j' h_{[2]}\\ &= \Delta(g)\Delta(h)\end{aligned}$$

the fourth equality being a consequence of $h_{[1]}\otimes h_{[2]}\in H\times_R H$.

(3) ε is a counit:

$$\begin{aligned}(H\otimes\varepsilon)\Delta(h) &= (H\otimes\varepsilon)(\sum \overline{e}_i h_{[1]}\otimes e_i' h_{[2]})\\ &= \sum \overline{e}_i h_{[1]} t(C(e_i' h_{[2]})(1))\\ &= \sum \overline{e}_i h_{[1]} t(e_i' C(h_{[2]})(1))\\ &= \sum \overline{C(h_{[2]})(1)}h_{[1]}\\ &= \theta(h_{[1]}\otimes C(h_{[2]})) = h,\end{aligned}$$

which shows the right counit property. The proof on the other side is analogous.

(4) j is a coalgebra map:

$$\begin{aligned}\Delta_H j(\varphi\otimes r) &= \sum \overline{e}_i(\mathring{\varphi}\otimes\overline{r})_{[1]}\otimes e_i'(\mathring{\varphi}\otimes\overline{r})_{[2]}\\ &= \sum \overline{e}_i\mathring{\varphi}\otimes e_i'\overline{r}\\ &= j(\varphi\otimes e_i)\otimes j(\mathring{e}_i'\otimes r)\\ &= (j\otimes j)\Delta_R(\varphi\otimes r)\end{aligned}$$

(5) Finally, the face algebra axiom (3.1) holds:

$$\begin{aligned}\sum \varepsilon(g\overline{e}_i)\varepsilon(\mathring{e}'_i h) &= \sum t(C(g\overline{e}_i)(1))t(C(\mathring{e}'_i h)(1))\\ &= \sum t(C(g)(e_i))t(\mathring{e}'_i C(h)(1))\\ &= t(C(g)(C(h)(1)))\\ &= t(C(gh)(1)) = \varepsilon(gh).\end{aligned}$$

We are done constructing a face algebra structure from a $\times_R$-bialgebra structure. Now assume that we are given an R-face algebra structure on H, with comultiplication Δ and counit ε. First, we observe that the composition

$$H \xrightarrow{\Delta} H \otimes H \to \int_r {}_{\overline{r}}H \otimes {}_r H$$

has its image in $H \times_R H$. In fact the image of Δ is even contained in $\int^{r,s} {}_{\overline{r}}H_{\overline{s}} \otimes {}_r H_s$, since $\Delta(1) = \sum \overline{e}_i \otimes e'_i$ because j is a coalgebra map, and hence

$$\Delta(h) = \Delta(1 \cdot h \cdot 1) = \Delta(1)\Delta(h)\Delta(1) = \sum \overline{e}_i h_{(1)}\overline{e}_j \otimes e'_i h_{(2)} e'_j$$

and $e \in (R \otimes R)^R$. Thus, we can define a map $\Gamma : H \to H \times_R H$ by letting $\Gamma(h)$ be the image of $\Delta(h)$ in $H \times_R H$. That Γ is multiplicative and coassociative is then trivial. Moreover,

$$\begin{aligned}\Gamma(r\overline{s}) &= j(\mathring{r} \otimes s)_{(1)} \otimes j(\mathring{r} \otimes s)_{(2)}\\ &= \sum j(\mathring{r} \otimes e_i) \otimes j(\mathring{e}'_i \otimes s)\\ &= \sum r\overline{e}_i \otimes e'_i\overline{s}\\ &= \sum r \otimes e_i e'_i \overline{s} = r \otimes \overline{s}\end{aligned}$$

holds in $H \times_R H$, so that Γ is a map of algebras over $R \otimes \overline{R}$.

Define $C : H \to \operatorname{End}(R)$ by

$$C(h)(r) = \sum \varepsilon(\overline{e}_i hr)e'_i = \sum \varepsilon(\mathring{e}_i hr)e'_i = \sum \varepsilon(e_i hr)e'_i,$$

where we have used the equation $\varepsilon(\mathring{\varphi} h) = \varepsilon(\varphi h)$, see [5].

Clearly, $C(1) = \mathrm{id}$, and C is multiplicative because $C(gh)(r) = \sum \varepsilon(e_i ghr)e'_i = \varepsilon(e_i g e_j)\varepsilon(e'_j hr)e'_i = \varepsilon(e_i g e'_j)\varepsilon(e_j hr)e'_i = C(g)(\sum \varepsilon(e_j hr)e'_j = C(g)(C(h)(r))$.

Moreover, C is a map of algebras over $R \otimes \overline{R}$, because

$$C(r\overline{s})(q) = \sum \varepsilon(e_i rsq)e'_i = rsq = \iota_{\operatorname{End}(R)}(r \otimes \overline{s})(q).$$

Finally, C is a counit for Γ:

$$\theta'(C \times_R H)\Gamma(h) = C(h_{(1)})(1)h_{(2)} = \sum \varepsilon(\overline{e}_i h_{(1)})e'_i h_{(2)} = \varepsilon(h_{(1)})h_{(2)} = h$$

shows one of the two relevant equations, the other can be proved in the same manner.

We have constructed a face algebra structure from each $\times_R$-bialgebra structure on H and vice versa. We omit the calculations checking that the two constructions are inverse to each other. □

Remark 4.4. Note that the $\times_R$-bialgebra comultiplication on a face algebra is "given by the same formula" as the face algebra comultiplication.

5. Comodules and their tensor products

An important feature of face algebras is that their comodules admit a tensor product. More precisely, let M, N be two left H-comodules over an R-face algebra H. Then the truncated tensor product of M and N is

$$\begin{aligned} M\overline{\otimes}N &:= k\text{-span}\{\varepsilon(m_{(-1)}n_{(-1)})m_{(0)}\otimes n_{(0)}\} \\ &= \{\sum m_i\otimes n_i\in M\otimes N|\sum m_i\otimes n_i=\sum\varepsilon(m_{i(-1)}n_{i(-1)})m_{i(0)}\otimes n_{i(0)}\}. \end{aligned}$$

This tensor product is associative and makes the category ${}^H\mathcal{M}$ of left H-comodules a monoidal category.

It is also an important feature of $\times_R$-bialgebras over an arbitrary k-algebra R that their comodules form a monoidal category, as we observed in [6]. Here, the notion of comodule has to be defined in a different way: Let H be a $\times_R$-bialgebra. Then a left H-comodule M is an R-bimodule together with an R-bimodule map $\Lambda : M\to H\times_R M$ (where the product $H\times_R M$ is defined as in section 2 and gets the R-bimodule structure from the left tensorand) which fulfills a coassociativity and counit condition analogous to the ones we recalled in section 2 for $\times_R$-bialgebras. We will write $\Lambda(m) =: m_{[-1]}\otimes m_{[0]}$. Let us denote the category of left H-comodules of a $\times_R$-bialgebra by ${}^{H\times_R}\mathcal{M}$. This is a monoidal category, where the tensor product of two comodules $M, N\in {}^{H\times_R}\mathcal{M}$ is defined to be $M\underset{R}{\otimes}N$ with the left comodule structure defined by $\Lambda(m\otimes n) := m_{[-1]}n_{[-1]}\otimes m_{[0]}\otimes n_{[0]}$.

Theorem 5.1. *Let R be a commutative separable k-algebra and H a k-algebra with corresponding structures of R-face algebra and $\times_R$-bialgebra. Then for any k-vector space M there is a bijection between left H-comodule structures on M with respect to the face algebra comultiplication, and left H-comodule structures on M in the $\times_R$-bialgebra sense. This induces an equivalence of monoidal categories ${}^H\mathcal{M}\cong {}^{H\times_R}\mathcal{M}$.*

Proof. We use the same notations for a face algebra and corresponding $\times_R$-bialgebra structure on H as in section 4.

Assume that $M\in {}^{H\times_R}\mathcal{M}$. Using Lemma 4.1 we can define a map $\lambda : M\to H\otimes M$ by $\lambda(m)=\sum\overline{e}_i m_{[-1]}\otimes e'_i m_{[0]}$. The verification that this gives M the structure of an object of ${}^H\mathcal{M}$ is analogous to the proof of coassociativity and counit property of (H,Δ,ε) in Theorem 4.3.

Conversely, assume given $M \in {}^H\mathcal{M}$. Then we can define an R-bimodule structure on M by setting $rm := \varepsilon(rm_{(-1)})m_{(0)}$ and $mr := \varepsilon(m_{(-1)}r)m_{(0)}$ for $r \in R$ and $m \in M$. In fact, we have

$$\begin{aligned}(mr)s &= \varepsilon(m_{(-1)}r)m_{(0)}s\\ &= \varepsilon(m_{(-2)}r)\varepsilon(m_{(-1)}s)m_{(0)}\\ &= \sum \varepsilon(m_{(-2)}e_i r)\varepsilon(m_{(-1)}e'_i s)m_{(0)}\\ &= \sum \varepsilon(m_{(-2)}(rs)_{(1)})\varepsilon(m_{(-1)}(rs)_{(2)})m_{(0)}\\ &= \sum \varepsilon((m_{(-1)}rs)_{(1)})\varepsilon((m_{(-1)}rs)_{(2)})m_{(0)}\\ &= \sum \varepsilon(m_{(-1)}rs)m_{(0)} = m(rs)\end{aligned}$$

for $r, s \in R$ and $m \in M$. Here, the third equality uses $\Delta(h) = \Delta(h)\Delta(1) = \sum h_{(1)}\overline{e}_i \otimes h_{(2)}e'_i$ for $h \in H$. The equation $m \cdot 1_R = m$ is trivial. The proofs that we also have a left module and a bimodule structure are analogous. Now we observe that the image of λ is contained in $\int^r H_{\overline{r}} \otimes M_r$. In fact

$$\begin{aligned}m_{(-1)}\overline{r} \otimes m_{(0)} &= (m_{(-1)}\overline{r})_{(1)}\varepsilon(m_{(-1)}\overline{r})_{(2)} \otimes m_{(0)}\\ &= m_{(-2)}\overline{e}_i \otimes \varepsilon(m_{(-1)}e'_i\overline{r})m_{(0)}\\ &= m_{(-2)} \otimes \varepsilon(m_{(-1)}r)m_{(0)}\\ &= m_{(-1)} \otimes m_{(0)}r.\end{aligned}$$

Thus, λ induces a map $\Lambda : M \to H \times_R M$, which is trivially coassociative. That Λ is an R-bimodule map follows from the fact that $\Delta(h) \in \int^{r,s} {}_{\overline{r}}H_{\overline{s}} \otimes {}_rH_s$ for all $h \in H$, see the proof of Theorem 4.3:

$$\begin{aligned}\Lambda(rms) = \varepsilon(rm_{(-1)}s)\Lambda(m_{(0)}) &= \varepsilon(\overline{r}m_{(-2)}\overline{s})m_{(-1)} \otimes m_{(0)}\\ &= \varepsilon(m_{(-2)})rm_{(-1)}s \otimes m_{(0)} = rm_{(-1)}s \otimes m_{(0)} = r\Lambda(m)s.\end{aligned}$$

We omit the proof that Λ fulfills the counit property for a comodule in the $\times_R$-bialgebra sense.

We also skip the proof that we have constructed mutually inverse bijections between structures making M an object of ${}^H\mathcal{M}$ and ${}^{H\times_R}\mathcal{M}$, respectively.

Consider two objects $M, N \in {}^H\mathcal{M}$. By Lemma 4.1 their tensor product in ${}^{H\times_R}\mathcal{M}$ is isomorphic to $\int^r M_r \otimes {}_rN$. Now let $\sum m_i \otimes n_i \in \int^r M_r \otimes {}_rN$. Then

$$\begin{aligned}\sum \varepsilon(m_{i(-1)}n_{i(-1)})m_{i(0)} \otimes n_{i(0)} &= \sum \varepsilon(m_{i(-1)}e_j)\varepsilon(e'_j m_{i(-1)})m_{i(0)} \otimes n_{i(0)}\\ &= \sum m_i e_j \otimes e'_j n_i = \sum m_i \otimes n_i\end{aligned}$$

so that $\sum m_i \otimes n_i \in M\overline{\otimes}N$. Conversely, assuming that $\sum m_i \otimes n_i \in M\overline{\otimes}N$, we have, for all $r \in R$,

$$\begin{aligned}\sum m_i r \otimes n_i &= \sum \varepsilon(m_{i(-1)}n_{1(-))})m_{i(0)}r \otimes n_{i(0)} \\ &= \sum \varepsilon(m_{i(-1)}\overline{r}n_{i(-1)})m_{i(0)} \otimes n_{i(0)} \\ &= \sum \varepsilon(m_{i(-1)}n_{i(-1)})m_{i(0)} \otimes rn_{i(0)} \\ &= \sum m_i \otimes rn_i\end{aligned}$$

using $\lambda(m_i) \in \int^r H_{\overline{r}} \otimes M_r$ as proved above, and the analogous fact $\lambda(n_i) \in \int^r {}_{\overline{r}}M \otimes_r N$. This proves that the equivalence ${}^H\mathcal{M} \cong {}^{H\times_R}\mathcal{M}$ that we constructed is monoidal. □

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