EXAMPLES OF EQUIVALENCES OF DOI-KOPPINEN HOPF MODULE CATEGORIES, INCLUDING YETTER-DRINFELD MODULES

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Abstract. Let H be a Hopf algebra. We exhibit the category equivalence ${}^H_H\mathcal{YD} \cong {}^H_H\mathcal{M}^H_H$ between Yetter-Drinfeld modules and two-sided two-cosided Hopf modules as an example of the adjunctions between categories of Doi-Koppinen unified Hopf modules studied by Caenepeel and Raianu. More generally, we study an induction functor ${}^D_B\mathcal{YD}(L) \to {}^D_R\mathcal{M}^H_T$, where L, H are Hopf algebras, D an L-bimodule coalgebra, T and R L-H-bicomodule algebras, and B a suitably constructed L-L-bicomodule algebra.

1. Introduction

Let A be a bialgebra, D a left A-module coalgebra, and B a left A-comodule algebra. In this situation (up to conventions like a choice of sides) Doi [4] and Koppinen [5] define a Hopf module in ${}^D_B\mathcal{M}(A)$ to be a left D-comodule and left B-module M satisfying a certain compatibility condition: The comodule structure is required to be given by a B-module map. These definitions unify several notions of Hopf modules in the literature as well as that of modules graded by sets with group actions. Caenepeel and Raianu [3] study induction and coinduction functors between categories of Doi-Koppinen Hopf modules and the question when these pairs of adjoint functors are equivalences. Their results unify and generalize the equivalences of Schneider [10] for ordinary (relative) Hopf modules over Hopf-Galois extensions and coextensions, and results of Menini [6] for modules graded by G-sets.

A Yetter-Drinfeld H-module for a Hopf algebra H is a (left) H-module and H-comodule M with the quite different compatibility condition $(h_{(1)} \rightharpoonup m)_{(-1)}h_{(2)} \otimes (h_{(1)} \rightharpoonup m)_{(0)} = h_{(1)}m_{(-1)} \otimes h_{(2)} \rightharpoonup m_{(0)}$ for $h \in H$ and $m \in M$, where $\rightharpoonup$ denotes the module structure. The definition appears in [11], the key property is that the category ${}^H_H\mathcal{YD}$ of Yetter-Drinfeld H-modules is braided if H has bijective antipode. There are two different connections between the notions of Yetter-Drinfeld and Hopf modules: In [7] it was shown that for a Hopf algebra H there is a category equivalence ${}^H_H\mathcal{YD} \cong {}^H_H\mathcal{M}^H_H$ between the category of Yetter-Drinfeld modules and that of two-sided two-cosided Hopf modules. Generalizations in [8, Thm. 3.5] and [1, Thm. 3.1] replace some of the four instances of H on the corners of the right hand side by more general objects. On the other hand, Caenepeel, Militaru and Zhu [2] have observed that Yetter-Drinfeld modules can be viewed as just a specific example of Doi-Koppinen unified Hopf modules: We have ${}_H^H\mathcal{YD} = {}_H^H\mathcal{M}(H \otimes H^{\mathrm{op}})$, in a sense we will recall below.

It is thus natural, and the purpose of this note, to incorporate the equivalences between Yetter-Drinfeld modules and two-sided two-cosided Hopf modules into the framework of [3].

More precisely, we will find a suitable triple (A', B', D') such that ${}_H^H\mathcal{M}_H^H \cong {}_{B'}^{D'}\mathcal{M}(A')$ (in fact, we will treat a more general setting). Two other ways of doing this were given for finitely generated projective H by Beattie, Dăscălescu, Raianu and Van Oystaeyen in [1], which also inspired the present paper. With our triple, we can show that the equivalence ${}_H^H\mathcal{YD} \cong {}_H^H\mathcal{M}_H^H$ coincides with one of the induction functors ${}_H^H\mathcal{M}(H \otimes H^{\mathrm{op}}) \to {}_{B'}^{D'}\mathcal{M}(A')$ from [3].

2. Preliminaries

Throughout the paper, k will denote a commutative ring, algebras, coalgebras etc. will be over k. We will make free use of Sweedler's notation (with the summation symbol omitted) for comultiplications of coalgebras and for comodules (for left comodules, we will use $v \mapsto v_{(-1)} \otimes v_{(0)}$ to denote the coaction).

A Doi-Hopf datum (A, B, D) consists of a bialgebra A, a left A-module coalgebra D and a left A-comodule algebra B. A left D-comodule and B-module M is said to be a Hopf (A, B, D)-module (an object of the category ${}_B^D\mathcal{M}(A)$) if the module structure map $B \otimes M \to M$ is D-colinear (with respect to the D-comodule structure of the left hand side induced by the canonical $A \otimes D$-comodule structure via the A-module coalgebra structure map $A \otimes D \to D$ of D), or, equivalently, if the comodule structure map $M \to D \otimes M$ is A-linear (with respect to the A-module structure on the right hand side induced by the canonical $A \otimes B$-module structure via the A-comodule algebra structure map $B \to A \otimes B$ of B). This simply means that the formula $(bm)_{(-1)} \otimes (bm)_{(0)} = b_{(-1)} \cdot m_{(-1)} \otimes b_{(0)} m_{(0)}$ holds for all $b \in B$ and $m \in M$, where $\cdot$ denotes the A-action on D.

Let (A, B, D) and (A', B', D') be Doi-Hopf data. Let $\alpha : A \to A'$ be a bialgebra map, $\beta : B \to B'$ an A'-comodule algebra map (that is, an algebra map satisfying $\beta(b)_{(-1)} \otimes \beta(b)_{(0)} = \alpha(b_{(-1)}) \otimes \beta(b_{(0)})$) and $\delta : D \to D'$ an A-module coalgebra map (that is, a coalgebra map satisfying $\delta(a \cdot d) = \alpha(a) \cdot \delta(d)$). We shall call $(\alpha, \beta, \delta) : (A, B, D) \to (A', B', D')$ a morphism of Doi-Hopf data. In this situation, Caenepeel and Raianu [3] study an induction functor $\mathcal{F} : {}_B^D\mathcal{M}(A) \to {}_{B'}^{D'}\mathcal{M}(A')$ defined as follows: $\mathcal{F}(M) = B' \underset{B}{\otimes} M$ with the obvious left B'-module structure and the D'-comodule structure λ defined by $\lambda(b' \otimes m) = b'_{(-1)} \cdot \delta(m_{(-1)}) \otimes b'_{(0)} \otimes m_{(0)}$. If D is k-flat, then $\mathcal{F}$ has a right adjoint $\mathcal{G}$ defined as follows: $\mathcal{G}(M') = D \underset{D'}{\square} M'$ with the obvious D-

comodule structure (which is where flatness of D is used) and the B-module structure defined by $b(\sum d_i \otimes m_i') = b_{(-1)} \cdot d_i \otimes \beta(b_{(0)})m_i'$.

We shall be needing the following variant of generalized Hopf modules: Let A be a bialgebra, B a left A-comodule algebra and D a right A-module coalgebra. The category ${}^D\mathcal{M}(A)_B$ consists of all left D-comodules and right B-modules M satisfying $(mb)_{(-1)} \otimes (mb)_{(0)} = m_{(-1)} \cdot b_{(-1)} \otimes m_{(0)}b_{(0)}$ for all $m \in M$ and $b \in B$ (so that, by definition, ${}^D\mathcal{M}(A)_B \cong {}^D_{B^{\mathrm{op}}}\mathcal{M}(A^{\mathrm{op}})$).

Let L be a bialgebra, B an L-bicomodule algebra and D an L-bimodule coalgebra; we say that (L, B, D) is a Yetter-Drinfeld datum. The category ${}^D_B\mathcal{YD}(L)$ of Yetter-Drinfeld (L, B, D)-modules was defined by Caenepeel, Militaru and Zhu, generalizing Yetter's [11] definition of crossed modules, which is the special case $B = D = H$. A Yetter-Drinfeld (L, B, D)-module is a left B-module and left D-comodule M satisfying the compatibility condition $(b_{(0)} \rightharpoonup m)_{(-1)} \leftharpoonup b_{(1)} \otimes (b_{(0)} \rightharpoonup m)_{(0)} = b_{(-1)} \rightharpoonup m_{(-1)} \otimes b_{(0)} \rightharpoonup m_{(0)}$ for all $b \in B$ and $m \in M$, where $\rightharpoonup$ and $\leftharpoonup$ are used to denote the left and right L-action on D, and $\rightharpoonup$ also to denote the B-action on M. If L has an antipode, then this condition is equivalent to $(b \rightharpoonup m)_{(-1)} \otimes (b \rightharpoonup m)_{(0)} = b_{(-1)} \rightharpoonup m_{(-1)} \leftharpoonup S(b_{(1)}) \otimes b_{(0)} \rightharpoonup m_{(0)}$ for all $b \in B$ and $m \in M$, which is the form found in [2].

Clearly, an L-bimodule coalgebra is the same thing as a left $L \otimes L^{\mathrm{op}}$-module coalgebra, and an L-bicomodule algebra is the same thing as a left $L \otimes L^{\mathrm{cop}}$-comodule algebra. If, moreover, L is a Hopf algebra, then any left $L \otimes L^{\mathrm{cop}}$-comodule algebra is also a left $L \otimes L^{\mathrm{op}}$-comodule algebra via the antipode. If the antipode is bijective, then in fact left $L \otimes L^{\mathrm{op}}$-comodule algebras and L-bicomodule algebras are equivalent notions. The following observation is also due to [2]: Let (L, B, D) be a Yetter-Drinfeld datum with L a Hopf algebra. Then, by the above, $(L \otimes L^{\mathrm{op}}, B, D)$ is a Doi-Hopf datum, and ${}^D_B\mathcal{YD}(L) = {}^D_B\mathcal{M}(L \otimes L^{\mathrm{op}})$.

3. An induction functor

We will set up a particular morphism between two Doi-Hopf data, one of which comes from a Yetter-Drinfeld datum, while the other has two-sided two-cosided Hopf modules of a certain type as its Doi-Hopf modules.

Throughout this section we will assume the following situation: Let L and H be bialgebras, D an L-bimodule coalgebra, R and T two L-H-bicomodule algebras. We will assume that L and D are flat over k.

Definition 3.1. Objects of the category ${}^D_R\mathcal{M}^H_T$ are by definition D-H-bicomodules and R-T-bimodules satisfying the four (generalized) Hopf module conditions for being an object of ${}^D_R\mathcal{M}(L)$, ${}^D\mathcal{M}(L)_T$, ${}_R\mathcal{M}^H$ and $\mathcal{M}^H_T$.

An R-T-bimodule is the same as a left B'-module for $B' = R \otimes T^{\mathrm{op}}$, and a D-H-bicomodule is the same as a left D'-comodule for $D' := D \otimes H^{\mathrm{cop}}$. Now the condition for a left B'-module and D'-comodule to be a Hopf module in each of the four ways in

Definition 3.1 can also be expressed as a unified Hopf module condition. We consider the Hopf algebra $A' = L \otimes L^{\mathrm{op}} \otimes H^{\mathrm{cop}} \otimes H^{\mathrm{op\ cop}}$, which has an obvious left action on D' and left coaction on B'. We have

$${}^{D}_{R}\mathcal{M}^{H}_{T} \cong {}^{D'}_{B'}\mathcal{M}(A').$$

In the case where H is finitely generated projective and $T = H$, two different descriptions of ${}^{D}_{R}\mathcal{M}^{H}_{H}$ as a category of Doi-Hopf modules were given in [1]. The basic idea used there is dualizing one or both of the (co-)actions of H; this makes the description more complicated in some respects, while we have to use a larger bialgebra A' in place of $H \otimes H^{*\mathrm{cop}}$ or $H \otimes H^{\mathrm{op}}$ that suffice in [1].

Definition 3.2. Put $B := (R \otimes T)^{\mathrm{co}\,H}$. Then B is a left $L \otimes L^{\mathrm{op}}$-subcomodule algebra of $R \otimes T^{\mathrm{op}}$.

In fact, $R \otimes T$ is an $L \otimes L^{\mathrm{op}}$-$H$-bicomodule, whose H-coinvariants form an $L \otimes L^{\mathrm{op}}$-subcomodule because $L \otimes L^{\mathrm{op}}$ is k-flat. It is straightforward to check that B is a subalgebra of $R \otimes T^{\mathrm{op}}$. In particular, we have a Doi-Hopf datum (A, B, D) for $A = L \otimes L^{\mathrm{op}}$.

In the case that L is a Hopf algebra with bijective antipode, B is an L-bicomodule algebra via

$$B \ni \sum r_i \otimes t_i \mapsto \sum r_{i(-1)} \otimes r_{i(0)} \otimes t_i \in L \otimes B$$
$$B \ni \sum r_i \otimes t_i \mapsto \sum r_i \otimes t_{i(0)} \otimes S^{-1}(t_{i(-1)}) \in B \otimes L$$

and for the resulting Yetter-Drinfeld datum (L, B, D) we have ${}^{D}_{B}\mathcal{YD}(L) \cong {}^{D}_{B}\mathcal{M}(A)$.

If L and H are Hopf algebras with bijective antipode, we define an H-L-bicomodule algebra T^{-1} as follows: As an algebra, $T^{-1} = T^{\mathrm{op}}$, the left H-comodule structure is given by $t \mapsto S^{-1}(t_{(1)}) \otimes t_{(0)}$, and the right L-comodule structure is given by $t \mapsto t_{(0)} \otimes S^{-1}(t_{(-1)})$. With this definition, we have

$$B \cong R \mathop{\square}_{H} T^{-1}$$

as an L-bicomodule subalgebra of $R \otimes T^{-1}$, by [10, Lem. 3.1].

Next, we define a morphism $(\alpha, \beta, \delta) : (A, B, D) \to (A', B', D')$ of Doi-Hopf data as follows:

$$\alpha : L \otimes L^{\mathrm{op}} \ni x \otimes y \mapsto x \otimes y \otimes 1 \otimes 1 \in L \otimes L^{\mathrm{op}} \otimes H^{\mathrm{cop}} \otimes H^{\mathrm{op\ cop}},$$
$$\delta : D \ni d \mapsto d \otimes 1 \in D \otimes H^{\mathrm{cop}},$$

and β is the inclusion.

Corollary 3.3. *There is a pair of adjoint functors of Caenepeel-Raianu*

$$
\begin{aligned}
{}^D_B\mathcal{M}(A) &\to {}^{D'}_{B'}\mathcal{M}(A') &\qquad {}^{D'}_{B'}\mathcal{M}(A') &\to {}^D_B\mathcal{M}(A)\\
V &\mapsto B' \underset{B}{\otimes} V & D \underset{D'}{\square} M &\leftarrow M
\end{aligned}
$$

We have an isomorphism ${}^{D'}_{B'}\mathcal{M}(A') \cong {}^D_R\mathcal{M}^H_T$ *and, if L is a Hopf algebra with bijective antipode, the equality* ${}^D_B\mathcal{M}(A) = {}^D_B\mathcal{YD}(L)$. *In these notations, the adjoint pair of Caenepeel-Raianu induces a pair of adjoint functors*

$$\mathcal{F} : {}^D_B\mathcal{YD}(L) \to {}^D_R\mathcal{M}^H_T \qquad \mathcal{G} : {}^D_R\mathcal{M}^H_T \to {}^D_B\mathcal{YD}(L).$$

We have $\mathcal{F}(V) = (R \otimes T^{\mathrm{op}}) \underset{B}{\otimes} V$ *with the R-T-bimodule structure induced by the obvious left $R \otimes T^{\mathrm{op}}$-module structure, the left D-comodule structure λ and right H-comodule structure ρ given by*

$$\lambda(r \otimes t \otimes v) = r_{(-1)} \rightharpoonup v_{(-1)} \leftharpoonup t_{(-1)} \otimes r_{(0)} \otimes t_{(0)} \otimes v_{(0)}$$

$$\rho(r \otimes t \otimes v) = r_{(0)} \otimes t_{(0)} \otimes v \otimes r_{(1)}t_{(1)},$$

and we have $\mathcal{G}(M) = M^{\mathrm{co}\,H}$, *which is a left D-subcomodule and B-submodule of M.*

4. Examples

The purpose of the definitions of the preceding section is that the pair of induction and coinduction functors resulting from them generalizes several examples of functors between Yetter-Drinfeld and two-sided two-cosided Hopf module categories. Thus, we see that, in view of [2] those examples can be incorporated as part of the theory developped in [3].

Let T be a right H-comodule algebra, and put $U := T^{\mathrm{co}\,H}$. Recall that T is called a right H-Galois extension of U if the Galois map $\beta : T \underset{U}{\otimes} T \to T \otimes H$ defined by $\beta(t \otimes t') = tt'_{(0)} \otimes t'_{(1)}$ is a bijection. We denote $\beta^{-1}(1 \otimes h) =: h^{[1]} \otimes h^{[2]} \in T \underset{B}{\otimes} T$. Assume in addition that T is a left faithfully flat U-module. Then by [10, Thm.I] we have an equivalence of categories

$$
\begin{aligned}
\mathcal{M}_U &\cong \mathcal{M}^H_T\\
N &\mapsto N \underset{U}{\otimes} T\\
M^{\mathrm{co}\,H} &\leftarrow M
\end{aligned}
$$

The isomorphism $N \cong (N \underset{U}{\otimes} T)^{\mathrm{co}\,H}$ for $N \in \mathcal{M}_U$ maps $n \in N$ to $n \otimes 1$. The isomorphism $M^{\mathrm{co}\,H} \underset{U}{\otimes} T \cong M$ for $M \in \mathcal{M}^H_T$ maps $m \otimes t \in M^{\mathrm{co}\,H} \underset{U}{\otimes} T$ to $mt \in M$, its inverse maps $m \in M$ to $m_{(0)}{m_{(1)}}^{[1]} \otimes {m_{(1)}}^{[2]} \in M^{\mathrm{co}\,H} \underset{U}{\otimes} T$.

Theorem 4.1. *Let L be a k-flat bialgebra, H a Hopf algebra with bijective antipode, T an L-H- bicomodule algebra which is a right H-Galois extension of $U := T^{\mathrm{co}\,H}$ and a faithfully flat left U-module, R an L-H-bicomodule algebra and D a k-flat L-L-bimodule coalgebra. Let $B := R \mathop{\square}\limits_{H} (T^{-1})$. Then we have an equivalence of categories*

$$\begin{aligned} {}^D_B\mathcal{YD}(L) &\to {}^D_R\mathcal{M}^H_T \\ M &\mapsto M^{\mathrm{co}\,H} \\ V \mathop{\otimes}\limits_U T &\leftarrow\!\shortmid V \end{aligned}$$

where $M^{\mathrm{co}\,H}$ has the B-module structure of a B-submodule of M, and $V \mathop{\otimes}\limits_U T$ has the following structures: The right T-module structure and H-comodule structure are induced by those of T. The left D-comodule structure maps $v \otimes t$ to $v_{(-1)} \leftharpoonup t_{(-1)} \otimes v_{(0)} \otimes t_{(0)}$, and the left R-module structure is given by $r(v\otimes t) = (r_{(0)} \otimes {r_{(1)}}^{[1]})v \otimes {r_{(1)}}^{[2]}t$.

Proof. Note that U^{op} is a subalgebra of B (via $u \mapsto 1 \otimes u$). We have inverse isomorphisms

$$\begin{aligned} (R \otimes T)^{\mathrm{co}\,H} \mathop{\otimes}\limits_U T &\cong R \otimes T \\ \sum r_i \otimes t_i \otimes t &\mapsto \sum r_i \otimes t_i t \\ r_{(0)} \otimes {r_{(1)}}^{[1]} \otimes {r_{(1)}}^{[2]} t &\leftarrow\!\shortmid r \otimes t. \end{aligned}$$

Hence, $T^{\mathrm{op}} \mathop{\otimes}\limits_{U^{\mathrm{op}}} B \cong R \otimes T^{\mathrm{op}}$, as T^{op}-B-bimodules. It follows that for $V \in {}^D_B\mathcal{YD}(L)$ we have an isomorphism

$$\alpha : V \mathop{\otimes}\limits_U T \ni v \otimes t \mapsto 1 \otimes t \otimes v \in (R \otimes T^{\mathrm{op}}) \mathop{\otimes}\limits_B V = \mathcal{F}(V)$$

with $\alpha^{-1}(r \otimes t \otimes v) = (r_{(0)} \otimes {r_{(1)}}^{[1]})v \otimes {r_{(1)}}^{[2]}t$. It is straightforward to check that the resulting structures making $V \mathop{\otimes}\limits_U T$ an object of ${}^D_R\mathcal{M}^H_T$ are as indicated. The adjunction morphisms are isomorphisms because of Schneider's theorem which we recalled just before the statement of the theorem. □

Corollary 4.2. *Assume the situation of Theorem* 4.1*.*

1. *Assume $D = k$. Then we have recovered* [8, Thm.3.2]*, a category equivalence ${}_R\mathcal{M}^H_T \cong {}_B\mathcal{M}$.*
2. *Assume $R = k$. Then we have a category equivalence ${}^D\mathcal{M}^H_T \cong {}^D\mathcal{M}_U$ (the category on the right hand side consists of D-comodules and U-modules M satisfying $(mu)_{(-1)} \otimes (mu)_{(0)} = m_{(-1)} \otimes m_{(0)}u$ for all $m \in M$ and $u \in U$.*

A special case of Theorem 4.1 occurs when $U = k$, that is, if T is a faithfully flat H-Galois extension of the base ring k. In that case we can make a special choice for L. By [9] there is a universal Hopf algebra $L := L(A, H)$ for which A is an L-H-bicomodule algebra. A is in fact also a left L-Galois extension of k in this case, that is, the Galois map $T \otimes T \ni x \otimes y \mapsto x_{(-1)} \otimes x_{(0)}y \in L \otimes T$ is a bijection. Let us denote the image of $\ell \otimes 1$ under the inverse of this map by $\ell^{(1)} \otimes \ell^{(2)} \in T \otimes T$. Since the Galois map is H-colinear with the codiagonal comodule structure on the domain and the comodule structure induced by that of T on the codomain, $L \ni \ell \mapsto \ell^{(1)} \otimes \ell^{(2)} \in (T \otimes T)^{\operatorname{co} H}$ is an isomorphism, which is an isomorphism of algebras with the right hand side considered a subalgebra of $T \otimes T^{\mathrm{op}}$. Under this isomorphism, the left L-comodule structure of T corresponds to the map

$$T \ni t \mapsto t_{(0)} \otimes {t_{(1)}}^{[1]} \otimes {t_{(1)}}^{[2]} \in (T \otimes T)^{\operatorname{co} H} \otimes T.$$

In this situation we can also give an answer to the following question: Starting with an L-H-bicomodule algebra R we have constructed an L-L-bicomodule algebra B to obtain the equivalence in Theorem 4.1. Which L-L-bicomodule algebras occur in this fashion?

Proposition 4.3. *Let H be a Hopf algebra with bijective antipode, T a faithfully flat right H-Galois extension of k and $L := L(T, H)$. Let G be a bialgebra. Then the assignment $R \mapsto R \mathbin{\underset{H}{\square}} T^{-1}$ defines a bijection between isomorphism classes of G-H-bicomodule algebras and isomorphism classes of G-L-bicomodule algebras, with the inverse given by $B \mapsto B \mathbin{\underset{H}{\square}} T$.*

In fact, this is a special case of [9, Thm. 5.5] which says that since T is an L-H-bigalois extension of k, cotensoring with T, respectively T^{-1}, defines inverse equivalences of monoidal categories $\mathcal{M}^L$ and $\mathcal{M}^H$.

In particular, every L-L-bicomodule algebra B occurs as $R \mathbin{\underset{H}{\square}} T^{-1}$ for a suitable L-R-bicomodule algebra R, and thus for every B there is a suitable R with ${}^{D}_{B}\mathcal{YD}(L) \cong {}^{D}_{R}\mathcal{M}^{H}_{T}$.

Corollary 4.4. *Let H and L be Hopf algebras with bijective antipodes and A an H-L-bicomodule algebra which is a faithfully flat left and right Hopf-Galois extension of k. Let B be an L-L-bicomodule algebra and R an L-H-bicomodule algebra with $B \cong R \mathbin{\underset{H}{\square}} T^{-1}$. Let D be an L-L-bimodule coalgebra.*

(1) *In the case that $R = T$, we have $B \cong L$, so that we get a category equivalence ${}^{D}_{L}\mathcal{YD}(L) \cong {}^{D}_{T}\mathcal{M}^{H}_{T}$, mapping V to $V \otimes T$, with the right module and comodule structures induced by those of T, the left module structure $t(v \otimes t') = t_{(-1)} \rightharpoonup v \otimes t_{(0)}t'$ and the left comodule structure mapping $v \otimes t$ to $v_{(-1)} \leftharpoonup t_{(-1)} \otimes v_{(0)} \otimes t_{(0)}$. The inverse equivalence maps $M \in {}^{D}_{T}\mathcal{M}^{H}_{T}$ to $M^{\operatorname{co} H}$, which is a D-subcomodule of M, and an L-module by $\ell \rightharpoonup m = \ell^{(1)}m\ell^{(2)}$ for $\ell \in L$ and $m \in M$. In the case that $D = L$, this is* [8, Thm. 3.5].

(2) *In the case that* $T = H$*, we have* $L \cong H$*, and* $B \cong R$*. The isomorphism* $R \to (R \otimes H)^{\mathrm{co}\,H}$ *is given by* $r \mapsto r_{(0)} \otimes S(r_{(1)})$*. Consequently, we have a category equivalence* ${}^D_R\mathcal{YD}(L) \cong {}^D_R\mathcal{M}^H_H$*, which maps* $V \in {}^D_R\mathcal{YD}(L)$ *to* $V \otimes H$*, with the right* H*-module and -comodule structure induced by those of* H*, the left* R*-module structure given by* $r(v \otimes h) = r_{(0)} \rightharpoonup v \otimes r_{(1)}h$*, and the left* D*-comodule structure mapping* $v \otimes h$ *to* $v_{(-1)} \leftharpoonup h_{(1)} \otimes v_{(0)} \otimes h_{(2)}$*. The inverse equivalence maps* $M \in {}^D_R\mathcal{M}^H_H$ *to* $M^{\mathrm{co}\,H}$*, a* D*-subcomodule of* M *with the* R*-module structure given by* $r \rightharpoonup m = r_{(0)} m S(r_{(1)})$ *for* $r \in R$ *and* $m \in M^{\mathrm{co}\,H}$*. This result is* [1, Thm. 3.1].

(3) *A common special case of the preceding two occurs when* $R = D = T = H$*, whence* $L = H$*. Here we get the equivalence* ${}^H_H\mathcal{YD} \cong {}^H_H\mathcal{M}^H_H$ *from* [7]

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