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Doi-Koppinen Hopf Modules Versus Entwined Modules

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Abstract. A Hopf module is an A-module for an algebra A as well as a C-comodule for a coalgebra C, satisfying a suitable compatibility condition between the module and comodule structures. To formulate the compatibility condition one needs some kind of interaction between A and C. The most classical case arises when $A = C =: H$ is a bialgebra. Many subsequent variants of this were unified independently by Doi and Koppinen; in their version an auxiliary bialgebra H, over which A is a comodule algebra and C a module coalgebra, governs the compatibility. Another very general type of interaction between A and C is an entwining map as studied by Brzeziński — without an auxiliary bialgebra.

Every Doi-Koppinen datum induces an entwining structure, so Brzeziński's notion of an entwined module includes that of a Doi-Koppinen Hopf module. This paper investigates whether the inclusion is proper.

By work of Tambara, every entwining structure can be obtained from a suitable Doi-Koppinen datum whenever the algebra under consideration is finite dimensional.

We show by examples that this need not be true in general.

Contents

1. The definitions

Throughout the paper k is a field, and all algebras, coalgebras etc. are meant to be over k. We denote the multiplication map of an algebra A by $\nabla\colon A \otimes A \to A$, and the comultiplication of a coalgebra C by $\Delta\colon C \to C \otimes C$, using Sweedler's notation in the form $\Delta(c) = c_{(1)} \otimes c_{(2)}$. We refer to [4] for the general background on Hopf algebra theory.

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Let H be a Hopf algebra. A (right-right) Hopf module over H is a right H-module as well as right H-comodule M with the property that the comodule structure map $\rho\colon M \to M \otimes H$ is an H-module map; here $M \otimes H$ is endowed with the diagonal right H-module structure. This means that $(mh)_{(0)} \otimes (mh)_{(1)} = m_{(0)}h_{(1)} \otimes m_{(1)}h_{(2)}$ holds for $m \in M$ and $h \in H$, when we write the comodule structure as $\rho(m) = m_{(0)} \otimes m_{(1)}$. Equivalently, the H-module structure $\mu\colon M \otimes H \to M$ is an H-comodule map. This ur-notion of Hopf module has seen far-reaching generalizations. First of all, it makes perfect sense to replace the H-module structure by an A-module structure for an H-comodule algebra A, or, dually, to replace the H-comodule structure by a C-comodule structure for an H-module coalgebra C. Moreover, one has reason to study Hopf modules which are A-modules for an H-comodule algebra A, but only $\overline{H}$-comodules for some quotient coalgebra and right module $\overline{H}$ of H. To unify all these situations Doi [2] and Koppinen [3] have introduced what we will call Doi-Koppinen Hopf modules with respect to a Doi-Koppinen datum. By definition, a Doi-Koppinen datum is a triple (A, C, H) in which H is a bialgebra, A is a right H-comodule algebra, and C is a right H-module coalgebra whose module structure we denote by a dot. A Doi-Koppinen Hopf module with respect to the Doi-Koppinen datum (A, C, H) is a right A-module as well as right C-comodule M satisfying the compatibility condition $\rho(ma) = m_{(0)}a_{(0)} \otimes m_{(1)} \cdot a_{(1)}$ for all $a \in A$ and $m \in M$. Again, we may formulate this as the condition that the module structure is a C-colinear map, or, equivalently, the comodule structure is an A-module map, by making the obvious definitions.

As Brzeziński [1] realized, all that is really needed to write down a Hopf module-like compatibility between a right A-module and right C-comodule structure is a so-called entwining structure: This is by definition a triple (A, C, ψ) in which ψ is a map $\psi\colon C \otimes A \to A \otimes C$ satisfying

$$
\begin{aligned}
(A \otimes \Delta)\psi &= (\psi \otimes C)(C \otimes \psi)(\Delta \otimes A)\colon C \otimes A \to A \otimes C \otimes C, \\
\psi(C \otimes \nabla) &= (\nabla \otimes C)(A \otimes \psi)(\psi \otimes A)\colon C \otimes A \otimes A \to A \otimes C,
\end{aligned}
$$

$(A \otimes \varepsilon)\psi = \varepsilon \otimes A$, and $\psi(c \otimes 1) = 1 \otimes c$. An entwined module with respect to the entwining structure ψ is by definition a right A-module and right C-comodule M such that

$$\rho\mu = (\mu \otimes C)(M \otimes \psi)(\rho \otimes A)\colon M \otimes A \to M \otimes C.$$

Every Doi-Koppinen datum induces an entwining structure, namely

$$\psi\colon C \otimes A \ni c \otimes a \mapsto a_{(0)} \otimes c \cdot a_{(1)} \in A \otimes C.$$

The entwined modules with respect to this entwining structure are precisely the Doi-Koppinen Hopf modules for the Doi-Koppinen datum in consideration.

While entwining structures are sometimes harder to cope with notationally, they offer a clear conceptual advantage over Doi-Koppinen data: The auxiliary bialgebra that is needed to formulate the compatibility condition for a Doi-Koppinen Hopf module is no longer needed for entwining structures. However, the obvious question arises whether entwining structures are a truly more general notion than Doi-Koppinen data, or whether in fact every entwining structure is induced by a suitable Doi-Koppinen datum.

2. The finite dimensional case

If the algebra A in an entwining structure (A, C, ψ) is finite dimensional, then work of Tambara [5] shows that there is in fact a suitable bialgebra H and Doi-Koppinen datum (A, C, H) inducing the entwining map ψ.

Let us rephrase the results of Sections 1 and 2 of [5] that are relevant for our question: Let A be an algebra. A transition map through A is a pair (V, ψ) in which V is a vector space and $\psi\colon V \otimes A \to A \otimes V$ is a linear map satisfying

$$(\nabla \otimes V)(A \otimes \psi)(\psi \otimes A) = \psi(V \otimes \nabla)\colon V \otimes A \otimes A \to A \otimes V$$

and $\psi(v \otimes 1) = 1 \otimes v$. Transition maps through A form a monoidal category with $(V, \psi) \otimes (W, \phi) := (V \otimes W, (\psi \otimes W)(V \otimes \phi))$. Now if A is finite dimensional, then there exists a bialgebra $e(A) = a(A, A)$ and an $e(A)$-comodule algebra structure $A \to A \otimes e(A)$ with the following property: A category equivalence between the category of right $e(A)$-modules and the category of transition maps through A is given by assigning to an $e(A)$-module V the transition map (V, ψ) with $\psi(v \otimes a) = a_{(0)} \otimes v \cdot a_{(1)}$.

From the results of Tambara we have thus summed up we conclude immediately:

Proposition 2.1. *If the algebra A has finite dimension, then every entwining structure (A, C, ψ) is induced by a Doi-Koppinen datum.*

In fact we need only add the observation that an entwining structure (A, C, ψ) is the same as a coalgebra in the monoidal category of transition maps through A, hence the same as a coalgebra in the monoidal category of $e(A)$-modules; thus (A, C, ψ) is induced by a Doi-Koppinen datum $(A, C, e(A))$.

3. Counterexamples

There exist entwining structures (A, C, ψ) that cannot be obtained from a Doi-Koppinen datum (A, C, H); this can even happen for finite dimensional C.

The examples rely on the following easy observation.

Lemma 3.1. *Let (A, C, ψ) be an entwining structure. Fix $c \in C$ and $\gamma \in C^*$, and define the vector space endomorphism $T_{c,\gamma}$ of A by $T_{c,\gamma}(a) = (A \otimes \gamma)\psi(c \otimes a)$.*

If the entwining structure (A, C, ψ) is induced by a Doi-Koppinen datum, then every $a \in A$ is contained in a finite dimensional $T_{c,\gamma}$-invariant subspace of A.

Proof. If (A, C, ψ) is induced by the Doi-Koppinen datum (A, C, H), then we find $T_{c,\gamma}(a) = a_{(0)}\gamma(c \cdot a_{(1)})$. From this formula it is obvious that every H-subcomodule of A is a $T_{c,\gamma}$-invariant subspace. But every $a \in A$ is contained in a finite dimensional H-subcomodule of A. $\square$

Example 3.2. Let C be the two-dimensional coalgebra $k1 \oplus kt$ with grouplike element 1 and $(1,1)$-primitive t. Let A be the free algebra on generators X_i for $i \in \mathbb{Z}$. Define the entwining map $\psi\colon C \otimes A \to A \otimes C$ by $\psi(1 \otimes a) = a \otimes 1$ for all $a \in A$ and

$$\psi(t \otimes X_{i_1} X_{i_2} \ldots X_{i_n}) = X_{i_1+1} X_{i_2+1} \ldots X_{i_n+1} \otimes t.$$

Then (A, C, ψ) is an entwining structure that cannot be obtained from any Doi-Koppinen datum.

Proof. It is easy to check directly that ψ is an entwining map; see below for a more conceptual idea. Choose $\gamma \in C^*$ with $\gamma(t) = 1$. Then we have $T_{t,\gamma}(X_i) = X_{i+1}$. Thus the $T_{t,\gamma}$-invariant subspace of A generated by X_0 is infinite dimensional. We conclude that (A, C, ψ) cannot be obtained from any Doi-Koppinen datum. $\square$

While our counterexample cannot be obtained from a Doi-Koppinen datum, it can be derived from something very similar, which we will call an alternative Doi-Koppinen datum. This is by definition a triple (A, C, H) consisting of a bialgebra H, a left H-module algebra A, and a left H-comodule coalgebra C; we will write the comodule structure of the latter as $c \mapsto c_{[-1]} \otimes c_{[0]}$. One can check that every alternative Doi-Koppinen datum induces an entwining structure in a fashion very similar to the case of a Doi-Koppinen datum: One defines $\psi\colon C \otimes A \to A \otimes C$ by $\psi(c \otimes a) = c_{[-1]} \cdot a \otimes c_{[0]}$. We omit the necessary verifications as they are very similar to the ones for Doi-Koppinen data. Our example above can be obtained from an alternative Doi-Koppinen datum $(C, A, k\mathbb{Z})$: The necessary comodule structure on C corresponds to the $\mathbb{Z}$-grading of C in which 1 and t have degrees 0 and 1, respectively, and the module algebra structure on A is that for which the generator $1 \in \mathbb{Z}$ shifts the indices in all the free generators X_i by one.

Remark 3.3. If the coalgebra C has finite dimension, then every entwining structure (A, C, ψ) is induced by an alternative Doi-Koppinen datum.

We will not supply any details of the proof, which is very similar to that of Proposition 2.1; we only remark that Tambara's $a(C^*, C^*)$ can serve as the necessary bialgebra.

The question remains whether there exist entwining structures (A, C, ψ) that is induced neither by a Doi-Koppinen datum, nor by an alternative Doi-Koppinen datum.

Example 3.4. Let $C = k1 \oplus \bigoplus_{i\in\mathbb{Z}} kt_i$ where 1 is grouplike and each t_i is $(1,1)$-primitive, and we let $A = k[\tau_i \mid i \in \mathbb{Z}]/(\tau_i\tau_j \mid i, j \in \mathbb{Z})$. An entwining structure $\psi\colon C \otimes A \to A \otimes C$ can be defined by $\psi(1 \otimes \tau_j) = \tau_j \otimes 1$, $\psi(t_i \otimes 1) = 1 \otimes t_i$, and $\psi(t_i \otimes \tau_j) = \tau_{j+1} \otimes t_{i+1}$.

If $\gamma \in C^*$ satisfies $\gamma(t_1) = 1$, then we find $T_{t_0,\gamma}(\tau_j) = \tau_{j+1}$ for all $j \in \mathbb{Z}$, so that the $T_{t_0,\gamma}$-invariant subset of A generated by τ_0 is infinite dimensional. Thus ψ is not induced by a Doi-Koppinen datum. A similar argument using the endomorphism $T'\colon C \to C$ defined by $T'(c) = (\alpha \otimes C)\psi(c \otimes \tau_0)$ with $\alpha \in A^*$ satisfying $\alpha(\tau_1) = 1$ shows that ψ is not induced by an alternative Doi-Koppinen datum. We omit checking that ψ is an entwining map.

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