Duals and doubles of quantum groupoids ($\times_R$-Hopf algebras)

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ABSTRACT. We generalize the construction of Drinfeld's quantum double to the theory of quantum groupoids, or, more precisely, $\underset{R}{\times}$-bialgebras as introduced by Takeuchi.

1. Introduction

$\times_R$-bialgebras were introduced by M. Takeuchi [**19**] as a generalization of the notion of bialgebra. The definition of a $\times_R$-bialgebra involves a k-algebra R (over some commutative base ring k) and usual k-bialgebras are recovered as the case $R = k$. A predecessor was defined by M. Sweedler [**18**] for the case that R is commutative. The motivation in both [**18**] and [**19**] is the study of isomorphism classes of (simple) algebras.

With a completely different motivation, $\times_R$-bialgebras were introduced again by Jiang-Hua Lu [**4**] under the name of bialgebroids; more precisely, the axioms in [**4**] were pointed out by Ping Xu [**11, 10**] to be equivalent to those of Takeuchi (although neither author seems to know Takeuchi's previous work.) A predecessor in this line is Maltsiniotis [**7**] where the algebra R is commutative.

The motivation in Lu's and Xu's papers is that the notion of a $\times_R$-bialgebra (perhaps equipped with some sort of antipode) is related to the notion of a groupoid in much the same manner as Hopf algebras relate to groups. For this reason (certain) $\times_R$-bialgebras could perhaps rightly be called quantum groupoids.

Recall that a groupoid consists of a set X of objects, a set Y of arrows, two maps (source and target) $s, t\colon Y \to X$, and a partially defined product from $\{(y, y') \in Y \times Y | s(y) = t(y')\}$ to Y (satisfying axioms that we'll skip altogether). If we assume that we are talking about an affine groupoid scheme (i. e. a situation as just described, but with all sets and maps replaced by affine schemes and morphisms of affine schemes), this corresponds to the following setup: One is given a commutative algebra R (the algebra of functions on the space of objects) and a commutative algebra L (the algebra of functions on the space of arrows), and two algebra maps $\alpha, \omega\colon R \to L$; we write $\alpha(r) = r$ and $\omega(r) = \overline{r}$. Composition corresponds to an algebra map

$$\Delta\colon L \to L \diamond L := ({}_\omega L) \underset{R}{\otimes} ({}_\alpha L) = L \otimes L/\langle \overline{r}\ell \otimes \ell' - \ell \otimes r\ell' | r \in R, \ell, \ell' \in L\rangle.$$

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When we try to pass from this setup to noncommutative algebraic geometry or the notion of a quantum groupoid, we would formally like to replace both R and L by noncommutative algebras. It turns out that it is reasonable to assume that one of the two maps from R to L, say ω, is an *anti*algebra map, and that the images of α and ω commute. However, one cannot expect that $L \diamond L$ defined as above is a quotient algebra of $L \otimes L$, and thus one cannot define a comultiplication on L to be an algebra map $L \to L \diamond L$. Now on the subspace $L \times_R L \subset L \diamond L$ consisting of those $\sum \ell_i \otimes \ell'_i \in L \diamond L$ satisfying $\sum \ell_i \overline{r} \otimes \ell'_i = \sum \ell_i \otimes \ell'_i r$ for all $r \in R$ one does have an algebra structure induced by that of $L \otimes L$, and the definition of a $\times_R$-bialgebra says that comultiplication is an algebra map $L \to L \times_R L$.

Independent evidence that the (quite technical) definition of a bialgebroid is the "right" one is given in [**14**]: Recall that a k-algebra H is a bialgebra if and only if the category ${}_H\mathcal{M}$ of left H-modules is a monoidal category in such a way that the underlying functor to the category of k-modules is monoidal (i. e. tensor product preserving). Stated more simply, the key fact about a bialgebra is that the tensor product over k of two of its modules is again a module. This is perhaps the starting point of Tannaka duality or reconstruction theory for quantum groups, where one reconstructs bialgebras and additional structures (like antipodes or quasitriangular structures) from their monoidal module categories and additional structures of those. In much the same way $\times_R$-bialgebras can be characterized by the fact that the category of modules over a $\times_R$-bialgebra L is a monoidal category. Only this time the underlying functor to the category of k-modules is not monoidal, but one has a monoidal underlying functor to the category of R-bimodules. Stated more simply, the tensor product over R of two L-modules is again an L-module.

Furthermore, the notion of a $\times_R$-bialgebra contains that of a face algebra as defined by Hayashi [**3**] as a special case [**15**]. Close relations to the notions of weak Hopf algebra or quantum groupoid considered by several authors [**1, 9, 8**], mainly in connection to inclusions of von Neumann algebras, should be expected but have not been worked out in detail.

The topic of the present paper are $\times_R$-bialgebra analogs of some constructions and facts well-known in the theory of usual k-bialgebras.

In section 3 we will discuss a suitable notion of $\times_R$-Hopf algebra. Formally the definition will say that a $\times_R$-bialgebra is a $\times_R$-Hopf algebra if and only if it is a left Hopf-Galois object over itself. Although this is well known to characterize ordinary k-Hopf algebras among bialgebras, we will provide better evidence that it is the "right" definition for the $\times_R$-bialgebra case: We will show that a $\times_R$-Hopf algebra structure on a $\times_R$-bialgebra L can be reconstructed from an appropriate property of the monoidal category of L-modules. One version of this approach is known for ordinary k-Hopf algebras if k is a field: The category of finite dimensional right H-comodules over a k-bialgebra H is rigid if and only if H is a Hopf algebra [**21**]. Here a monoidal category is called rigid if one has a well-behaved notion of dual objects within the category. Now this characterization of Hopf algebras cannot be generalized to the $\times_R$-bialgebra situation: Even for ordinary bialgebras over a ring k rather than a field, one cannot extract enough information from just the finitely generated projective comodules (or the finitely generated projective modules), and a comodule (or module) that is not finitely generated projective over the base ring k can never have a dual in the sense of the definition of a rigid monoidal category. However, if H is a Hopf algebra over a commutative ring k, then for any two H-modules V, W one can endow the module $\mathrm{Hom}_k(V, W)$ of k-linear maps with a good

H-module structure. In particular, the category of H-modules is (like every rigid category as well) closed in the sense of Mac Lane: One has, within the category, so-called inner hom-objects $\mathrm{hom}(V, W)$ for all V, W, satisfying a formal analog of the well-known adjointness between tensor product and hom-functors for bimodules. Now while rigidity was too strong for our purposes, requiring inner hom-functors is too weak: It turns out that the category of L-modules is always closed, even for any $\times_R$-bialgebra L. But our "reconstruction" theorem for $\times_R$-Hopf algebras will say that a $\times_R$-bialgebra L is a $\times_R$-Hopf algebra if and only if the underlying functor from ${}_L\mathcal{M}$ to the category of R-bimodules is compatible with the inner hom-functors. In the k-bialgebra case this is precisely the observation that inner hom-functors in ${}_H\mathcal{M}$ are modelled on the modules of k-linear maps, not on some other object.

In the rest of the paper we will construct a $\times_R$-bialgebra analog of Drinfeld's quantum double. Let us recall first the construction of the double $D(H)$ of a finite dimensional k-Hopf algebra H (a reference is Kassel's book [**6**], with different conventions from ours): One has $D(H) = H \otimes H^*$ as coalgebras; the algebra structure of $D(H)$ is more complicated and perhaps explained most easily by Doi and Takeuchi [**2**]: In any bialgebra k, multiplication can be altered by a two-cocycle $\tau\colon B \otimes B \to k$. Such a two-cocycle can be found very easily if $B = H \otimes \Lambda$ is the tensor product of two Hopf algebras H and Λ, and one has a skew pairing $\sigma\colon \Lambda \otimes H \to k$, that is, a bilinear form with respect to which multiplication (resp. comultiplication) of H and comultiplication (resp. multiplication) of Λ are adjoint to each other (and called a *skew* pairing because in one of the equations opposite comultiplication is used). The equations for a skew pairing are easier to treat than those of a two-cocycle; the most obvious example of a skew pairing is evaluation, as a skew pairing between $H^{*\mathrm{op}}$ and H. In this situation the twist of the usual tensor product Hopf algebra $H \otimes H^{*\mathrm{op}}$ by the cocycle obtained from the canonical skew pairing is the Drinfeld double $D(H)$. A categorical explanation of $D(H)$ is as follows: To any monoidal category one can associate a braided monoidal category called its center. The center of the category ${}_H\mathcal{M}$ is the category ${}_H^H\mathcal{YD}$ of (left-left) Yetter-Drinfeld modules; these are by definition H-modules as well as comodules satisfying a compatibility condition. Now by dualizing the comodule structure to give an $H^{*\mathrm{op}}$-module structure, a Yetter-Drinfeld module is the same as a module over the Drinfeld double.

If we want to generalize all of this to the $\times_R$-bialgebra setting, the first step is to define dual $\times_R$-bialgebras. We will give a definition of skew pairings of $\times_R$-bialgebras in Section 5, as well as the definition of a "skew" dual $\times_R$-bialgebra $L^\vee$ of a $\times_R$-bialgebra L that is finite projective as R-module. The skew dual is a $\times_R$-bialgebra analog of the opposite bialgebra of the dual of an ordinary k-bialgebra. Oddly enough, neither Hopf pairings (rather than skew ones) nor duals (rather than opposite duals) seem to exist for $\times_R$-bialgebras (note that the opposite of a $\times_R$-bialgebra is not a $\times_R$-bialgebra). A predecessor of our duality construction can be found in [**12**], where two subclasses of $\times_R$-bialgebras for commutative R are shown to be in duality with each other. As the referee kindly pointed out, the duality between $\times_R$-coalgebras and R^e-rings was already treated completely (with a trivial variation in notations that is not suited well to the $\times_R$-bialgebra case) by Takeuchi in [**20**].

Luckily a *skew* pairing is what we need to generalize Doi-Takeuchi's generalized Drinfeld double in Section 6. Here another difficulty arises. Doi and Takeuchi first take the ordinary tensor product bialgebra, and then twist it by a skew pairing in a second step. However, the first step is impossible in our setting. For two $\times_R$-bialgebras L and Λ, there does not seem to be a reasonable tensor product $\times_R$-bialgebra: While $L \otimes_{R\otimes\overline{R}} \Lambda$ does not have an algebra structure, $L \times_R \Lambda$ does not have a comultiplication. Luckily, once we are given a skew pairing between Λ and L, with L a $\times_R$-Hopf algebra, we can do both steps in one and endow $L \bowtie \Lambda := L \otimes_{R\otimes\overline{R}} \Lambda$ with a $\times_R$-bialgebra structure that generalizes Doi-Takeuchi's twist of the tensor product bialgebra.

To complete the picture, we show that modules over the double $D(L) := L \bowtie L^\vee$ of a finite projective $\times_R$-Hopf algebra L are precisely the Yetter-Drinfeld modules over L, a notion we have defined in Section 4.

2. Bialgebroids and their modules

In this section we will very briefly recall some definitions and notations concerning $\times_R$-bialgebras, and the monoidal categories of their modules and comodules. For more details we refer to [**18, 19, 14**]. We will not make explicit reference to the closely related notions in [**4, 11, 10**].

For the rest of the paper k denotes a commutative ring and R denotes a fixed k-algebra. We denote the opposite algebra by $\overline{R}$, we let $R \ni r \mapsto \overline{r} \in \overline{R}$ denote the obvious k-algebra antiisomorphism, and abbreviate the enveloping algebra $R^e := R \otimes \overline{R}$. We consider R and $\overline{R}$ as subalgebras of R^e in the obvious way.

Let $M, N \in {}_{R^e}\mathcal{M}_{R^e}$. We use the following notations, due to Mac Lane, see [**18, 19**]: We let

$$\int_r {}_{\overline{r}}M \otimes {}_r N := M \otimes N / \langle \overline{r}m \otimes n - m \otimes rn | r \in R, m \in M, n \in N \rangle$$

We let $\displaystyle\int^r M_{\overline{r}} \otimes N_r \subset M \otimes N$ denote the k-submodule consisting of all elements $\sum m_i \otimes n_i \in M \otimes N$ satisfying $\sum m_i \overline{r} \otimes n_i = \sum m_i \otimes n_i r$ for all $r \in R$. In blatant abuse of notation, we will most of the time write $m \otimes n$ for a general element of $\int^r M_{\overline{r}} \otimes N_r$, although of course this k-module needs not be generated by decomposable tensors in $M \otimes N$.

In the instance above the $\int_r$ symbol is just a tensor product over R, with the advantage of indicating precisely the module structures we have in mind. The symbols $\int_r$ and $\int^r$ can be mixed to indicate iterated constructions of submodules and quotient modules; it should be noted that the two operations will in general not commute.

For two left R^e-modules M, N we abbreviate $M \diamond N := \int_r {}_{\overline{r}}M \otimes {}_r N$, which is again in ${}_{R^e}\mathcal{M}$ with the left $\overline{R}$-module structure induced by the structure of N, and the left R-module structure induced by the structure of M. In this way, $({}_{R^e}\mathcal{M}, \diamond)$ is a monoidal category, naturally isomorphic to $({}_R\mathcal{M}_R, \otimes_R)$.

For two R^e-bimodules M and N we abbreviate

$$M \underset{R}{\times} N := \int^s \int_r {}_{\overline{r}}M_{\overline{s}} \otimes {}_r N_s.$$

This is again an R^e-bimodule with $r\overline{s}(m \otimes n)t\overline{u} = rmt \otimes \overline{s}n\overline{u}$ for $r, s, t, u \in R$ and $m \otimes n \in M \times_R N$, i. e. R acts on the left, $\overline{R}$ on the right factor. The product functor $\times_R$ thus defined on the category ${}_{R^e}\mathcal{M}_{R^e}$ needs neither be associative nor unitary.

For $M,N,P \in {}_{R^e}\mathcal{M}_{R^e}$ one defines

$$M\underset{R}{\times}P\underset{R}{\times}N := \int^{s,u}\int_{r,t} {}_{\overline{r}}M_{\overline{s}} \otimes {}_{r,\overline{t}}P_{s,\overline{u}} \otimes {}_{t}N_{u}$$

(where $\int^{s,u} := \int^{s}\int^{u} = \int^{u}\int^{s}$). There are associativity maps

$$(M\underset{R}{\times}P)\underset{R}{\times}N \xrightarrow{\alpha} M\underset{R}{\times}P\underset{R}{\times}N$$

$$M\underset{R}{\times}(P\underset{R}{\times}N) \xrightarrow{\alpha'} M\underset{R}{\times}P\underset{R}{\times}N$$

which are (with our abuse of notations!) given on elements by the formula $m\otimes p\otimes n \mapsto m\otimes p\otimes n$, but need not be isomorphisms.

An R^e-bimodule structure on $\mathrm{End}(R)$ is induced by the left R^e-module structure on R. We have, for any $M \in {}_{R^e}\mathcal{M}_{R^e}$, two R^e-bimodule maps

$$\theta : M\underset{R}{\times}\mathrm{End}(R) \to M; \qquad m\otimes f \mapsto \overline{f(1)}m$$

$$\theta' : \mathrm{End}(R)\underset{R}{\times}M \to M; \qquad f\otimes m \mapsto f(1)m.$$

We will write $\mathrm{E} := \mathrm{End}(R)$ below.

A $\times_R$-coalgebra L is an R^e-bimodule equipped with a comultiplication, a map $\Delta\colon L \to L\times_R L$ of R^e-bimodules over R^e, and a counit, a map $\varepsilon : L \to \mathrm{E}$ of R^e-bimodules, such that

$$\alpha(\Delta\underset{R}{\times}L)\Delta = \alpha'(L\underset{R}{\times}\Delta)\Delta\colon L \to L\underset{R}{\times}L\underset{R}{\times}L \tag{2.1}$$

$$\theta(L\underset{R}{\times}\varepsilon)\Delta = \mathrm{id}_L = \theta(\varepsilon\underset{R}{\times}L)\Delta. \tag{2.2}$$

Note that a map $\Delta\colon L \to L\times_R L$ in ${}_{R^e}\mathcal{M}_{R^e}$ induces a map $\Delta_0\colon L \to L\diamond L$ in ${}_{R^e}\mathcal{M}$. A map $\varepsilon : L \to \mathrm{E}$ in ${}_{R^e}\mathcal{M}_{R^e}$ induces a map $\varepsilon_0 : L \ni \ell \mapsto \varepsilon(\ell)(1) \in R$ in ${}_{R^e}\mathcal{M}$. One checks that Δ and ε fulfill the equations (2.1) and (2.2) if and only if

$$(\Delta_0\diamond L)\Delta_0 = (L\diamond\Delta_0)\Delta_0\colon L \to L\diamond L\diamond L$$

and $(\varepsilon_0\diamond L)\Delta_0 = \mathrm{id}_L = (L\diamond\varepsilon_0)$ hold. These mean that L, considered as an R-R-bimodule via the left R^e-module structure, is an R-coring.

By an R^e-ring we will simply mean an algebra L equipped with an algebra map $i\colon R^e \to L$; a map of R^e-rings will be an algebra map commuting with the respective maps from R^e. Of course, any R^e-ring is an R^e-bimodule in a natural way, and it turns out that if L, M are R^e-rings, then $L\times_R M$ is an R^e-ring with $(\ell\otimes\ell')(m\otimes m') = \ell m\otimes\ell' m'$. Similarly, the triple $\times_R$-product of R^e-rings is an R^e-ring.

By definition a $\times_R$-bialgebra L is an R^e-ring equipped which is a $\times_R$-coalgebra such that comultiplication $\Delta\colon L \to L\times_R L$ and counit $\varepsilon\colon L \to E$ are maps of R^e-rings. The following equivalent definition was pointed out by the referee: The category of $\times_R$-coalgebras is a monoidal category with tensor product $\otimes_{R^e}$, by [**19**, Prop. 4.7]. The neutral object is the $\times_R$-coalgebra R^e with comultiplication $\Delta(r\otimes\overline{s}) = (r\otimes\overline{1})\otimes(1\otimes\overline{s})$ and counit $\varepsilon(r\otimes\overline{s})(t) = rts$. Now a $\times_R$-bialgebra is the same as an algebra in the monoidal category of $\times_R$-coalgebras by [**19**, Prop. 4.8]. In particular R^e is a $\times_R$-bialgebra with comultiplication $\Delta(r\otimes\overline{s}) = (r\otimes\overline{1})\otimes(1\otimes\overline{s})$.

For $\times_R$-bialgebras we will make use of the usual Sweedler notation, writing $\Delta(\ell) =: \ell_{(1)}\otimes\ell_{(2)} \in L\times_R L$, and

$$\alpha(\Delta\underset{R}{\times}L)\Delta(\ell) =: \ell_{(1)}\otimes\ell_{(2)}\otimes\ell_{(3)} \in L\underset{R}{\times}L\underset{R}{\times}L.$$

However, a word of warning is necessary. When using Sweedler notation for ordinary k-coalgebras, one can substitute expressions and equalities like $c_{(1)} \otimes c_{(2)} \otimes c_{(3)} = c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)}$ freely into multilinear expressions. When doing calculations with $\times_R$-bialgebras, complications arise: Although coassociativity reads

$$\ell_{(1)} \otimes \ell_{(2)(1)} \otimes \ell_{(2)(2)} = \ell_{(1)} \otimes \ell_{(2)} \otimes \ell_{(3)} = \ell_{(1)(1)} \otimes \ell_{(1)(2)} \otimes \ell_{(2)}$$

(as elements of $L \diamond L \diamond L$), the elements $\ell_{(1)} \otimes \ell_{(2)(1)} \otimes \ell_{(2)(2)} \in L \times_R (L \times_R L)$, $\ell_{(1)} \otimes \ell_{(2)} \otimes \ell_{(3)} \in L \times_R L \times_R L$, and $\ell_{(1)(1)} \otimes \ell_{(1)(2)} \otimes \ell_{(2)} \in (L \times_R L) \times_R L$ must not be confused, even if (as is true in many interesting cases) both α and α' are isomorphisms.

Example 2.1. Let $R = k \times k$ with the canonical idempotents $e_1, e_2 \in k \times k$, and consider $L = R \otimes R$ with basis $e_{ij} := e_i \otimes e_j$. We let the action of R on the left tensor factor of L give rise to both the left R- and $\overline{R}$-module structure, and similarly on the right. In this way, L is an R^e-bimodule.

Consider the map

$$\begin{aligned} \varphi_{(2.3)}\colon L \otimes L \otimes L &\to k \\ e_{11} \otimes e_{12} \otimes e_{21} &\mapsto 1 \\ e_{ij} \otimes e_{k\ell} \otimes e_{mn} &\mapsto 0 \text{ for other sets of indices.} \end{aligned} \tag{2.3}$$

We claim that

$$\varphi_{(2.4)}\colon (L \underset{R}{\times} L) \underset{R}{\times} L \to k;\ \ell \otimes \ell' \otimes \ell'' \mapsto \varphi_{(2.3)}(\ell \otimes \ell' \otimes \ell'') \tag{2.4}$$

is well defined. In fact it is immediate to check that $\varphi_{(2.3)}$ factors over

$$\varphi_{(2.5)}\colon L \diamond L \otimes L \to k, \tag{2.5}$$

and the restriction of $\varphi_{(2.5)}$ to $L \times_R L \otimes L$ vanishes, since $e_{11} \otimes e_{12} \notin L \times_R L$.

Clearly the canonical morphisms $\alpha'\colon L \times_R (L \times_R L) \to L \times_R L \times_R L$ as well as $\alpha\colon (L \times_R L) \times_R L \to L \times_R L \times_R L$ are isomorphisms, so one would expect that one also has a well defined map

$$\varphi_{(2.6)}\colon L \underset{R}{\times} (L \underset{R}{\times} L) \to k;\ \ell \otimes \ell' \otimes \ell'' \mapsto \varphi_{(2.3)}(\ell \otimes \ell' \otimes \ell'') \tag{2.6}$$

However, we have $0 = e_{12} \otimes e_{21} \in L \diamond L$, hence $0 = e_{11} \otimes e_{12} \otimes e_{21} \in L \times_R (L \times_R L)$, but $\varphi_{(2.6)}$ would have to map this to $1 \in k$.

The example shows that whenever we use equalities like coassociativity within a computation, where they occur substituted into some kind of multilinear formula, we have to make sure that all the expressions we consider are well defined, and we even have to do this when, inside a well-defined expression, we wish to replace some tensor by another one, which is "equal". Due to this necessity, we have to use a large number of auxiliary maps, and since it seems pointless to invent meaningful names for all of them, we shall often take refuge in the notation already employed above: The auxiliary map defined in equation (2.3) is called simply $\varphi_{(2.3)}$.

The category of modules over a $\times_R$-bialgebra L has the structure of a monoidal category. The tensor product in ${}_L\mathcal{M}$ of two L-modules M and N is $M \diamond N$ equipped with the left L-module structure $\ell \triangleright (m \otimes n) = \ell_{(1)} \triangleright m \otimes \ell_{(2)} \triangleright n$. In particular the underlying functor ${}_L\mathcal{M} \to {}_{R^e}\mathcal{M}$ is a monoidal functor; essentially $\times_R$-bialgebras can be characterized by this behavior.

The suitable definition of comodules over a $\times_R$-bialgebra L is as follows: A left L-comodule is an R-bimodule M together with a map $\lambda\colon M \to L\times_R M$ of R-bimodules such that

$$\alpha'(L\underset{R}{\times}\lambda)\lambda = \alpha(\Delta\underset{R}{\times}M)\lambda\colon M \to L\underset{R}{\times}L\underset{R}{\times}M$$

and $\theta'(\varepsilon\times_R M)\lambda = \mathrm{id}_M$ hold. If we denote by $\lambda_0\colon M \to L\diamond M$ the composition of λ with the inclusion of $L\diamond M$ into $L\times_R M$, then coassociativity is equivalent to $(L\diamond\lambda_0)\lambda_0 = (\Delta_0\diamond M)\lambda_0\colon M \to L\diamond L\diamond M$ and $(\varepsilon_0\diamond M)\lambda_0 = \mathrm{id}_M$. We will denote by ${}^L\mathcal{M}$ the category of left L-comodules. We will use Sweedler notation in the form $\lambda(m) = m_{(-1)}\otimes m_{(0)}$ and $\alpha(\Delta\times_R M)(m) = m_{(-2)}\otimes m_{(-1)}\otimes m_{(0)}$ for L-comodules.

The category ${}^L\mathcal{M}$ of left L-comodules over a $\times_R$-bialgebra is monoidal. The tensor product of $M, N \in {}^L\mathcal{M}$ is their tensor product $M\otimes_R N$ over R, equipped with the comodule structure

$$\begin{aligned} M\underset{R}{\otimes}N &\to L\underset{R}{\times}(M\underset{R}{\otimes}N)\\ m\otimes n &\mapsto m_{(-1)}n_{(-1)}\otimes m_{(0)}\otimes n_{(0)}\end{aligned}$$

3. $\times_R$-Hopf algebras and duality for modules

We will give a definition of $\times_R$-Hopf algebras in the spirit of Tannaka duality, that is, we will try to find a definition for which the property of being a Hopf algebra rather than just a bialgebra reflects properties of the monoidal category of representations.

It is well known that an ordinary bialgebra H over a field k is a Hopf algebra if and only if the category of its finite dimensional comodules is rigid [**21**]. Here a monoidal category $\mathcal{C}$ is said to be rigid if every object $X\in\mathcal{C}$ has a left dual object, which in turn is a triple $(X^*, \mathrm{ev}, \mathrm{db})$ where X^* is an object of $\mathcal{C}$, $\mathrm{ev}\colon X^*\otimes X \to I$ and $\mathrm{db}\colon I \to X\otimes X^*$ are morphisms satisfying $(X\otimes\mathrm{ev})(\mathrm{db}\otimes X) = \mathrm{id}_X$ and $(\mathrm{ev}\otimes X^*)(X^*\otimes\mathrm{db}) = \mathrm{id}_{X^*}$. In the category of finite dimensional right comodules over a Hopf algebra H over a field, the dual vector space, equipped with the unique comodule structure satisfying $\mathrm{ev}(\phi_{(0)}\otimes v)\phi_{(1)} = \mathrm{ev}(\phi\otimes v_{(0)})S(v_{(1)})$ is a dual in the above sense. However, infinite dimensional H-comodules can never have a dual object. In particular, since finite modules alone are not enough to say much about an algebra (the situation being different for comodules due to the finiteness theorems), we cannot hope to copy this result for modules over bialgebras, or even less for modules over $\times_R$-bialgebras.

A suitable weaker concept of duality for monoidal categories is the notion of an inner hom-functor.

Definition 3.1. Let $\mathcal{C}$ be a monoidal category.

Let Y be an object of $\mathcal{C}$. If the functor $\mathcal{C}\ni X\mapsto X\otimes Y\in\mathcal{C}$ is left adjoint, we denote its right adjoint by by $\mathcal{C}\ni X\mapsto \hom(Y,X)\in\mathcal{C}$ and call it a (right) inner hom-functor.

If the functor $-\otimes Y$ is left adjoint for every $Y\in\mathcal{C}$, then we say that $\mathcal{C}$ is (right) closed.

By the standard machinery of adjoint functors, an inner hom-functor $\hom(X,-)$ comes with an adjunction morphism $\mathrm{ev}\colon \hom(X,Y)\otimes X \to Y$ with the following universal property: For each $T\in\mathcal{C}$ and $e\colon T\otimes X\to Y$ there is a unique $f\colon T\to \hom(X,Y)$ such that $e = \mathrm{ev}(f\otimes X)$.

If X has a left dual, then $-\otimes X$ is left adjoint with $\hom(X,Y) = Y \otimes X^*$

Of course, the notion of inner hom-functor is modelled on the standard hom-tensor adjunction for (bi)modules. In particular, the category $({}_{R^e}\mathcal{M}, \diamond)$, which is isomorphic to $({}_R\mathcal{M}_R, \otimes_R)$, is closed. We note the details for later use:

REMARK 3.2. The monoidal category $({}_{R^e}\mathcal{M}, \diamond)$ is right closed with inner hom-functors $\hom_{{}_{R^e}\mathcal{M}}(N,P) = \mathrm{Hom}_{\overline{R}}(N,P)$ for $N, P \in {}_{R^e}\mathcal{M}$, where the left R^e-module structure is given by $((r \otimes \overline{s})f)(n) = rf(sn)$.

If H is a Hopf algebra, then the k-module $\mathrm{Hom}_k(V,W)$ of k-linear maps has a canonical H-module structure, defined by $(hf)(v) = h_{(1)}f(S(h_{(2)})v)$ for $h \in H$, $f \in \mathrm{Hom}_k(V,W)$ and $v \in V$. With this structure, $\mathrm{Hom}_k(V,W)$ defines a right inner Hom-functor in the category of left H-modules.

Now while a Hopf algebra allows us to construct inner hom-functors in its module category, the converse is not true: a bialgebra needs not be a Hopf algebra for its module category to be closed. On the contrary the module category over any bialgebra and even $\times_R$-bialgebra is closed.

PROPOSITION 3.3. *Let L be a $\times_R$-bialgebra. Then the category ${}_L\mathcal{M}$ is right closed with right inner hom-functors*

$$\hom_{{}_L\mathcal{M}}(N,P) = \mathrm{Hom}_L(L \diamond N, P)$$

where $L \diamond N$ is an L-L-bimodule with the left L-module structure as above, and the right L-module structure induced by that of the factor L.

PROOF. Using that

$$M \diamond N \ni m \otimes n \mapsto 1 \otimes n \otimes m \in (L \diamond N) \underset{L}{\otimes} M$$

is an isomorphism with inverse $(\ell \otimes n) \otimes m \mapsto \ell m \otimes n$, we obtain

$$\mathrm{Hom}_L(M \diamond N, P) \cong \mathrm{Hom}_L((L \diamond N) \underset{L}{\otimes} M, P) \cong \mathrm{Hom}_L(M, \mathrm{Hom}_L(L \diamond N, P))$$

from the standard hom-tensor adjunction. □

Now to save the idea that being a Hopf algebra is related to the category of modules being closed, we need to observe that the inner hom-functors for modules over a Hopf algebra are modelled on the inner hom-functors in the underlying category of k-modules, while this is not so in the bialgebra case. In a more categorical language, the underlying functor ${}_H\mathcal{M} \to {}_k\mathcal{M}$ for a Hopf algebra H preserves hom-functors, in the following sense (cf. [**13**, I.4.3]):

DEFINITION 3.4. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ be a monoidal functor between right closed monoidal categories, and let

$$\zeta\colon \mathcal{F}(\hom(X,Y)) \to \hom(\mathcal{F}(X), \mathcal{F}(Y))$$

for $X, Y \in \mathcal{C}$ be the unique morphism for which

$$\begin{tikzcd}
\mathcal{F}(\hom(X,Y)) \otimes \mathcal{F}(X) \arrow[r, "\zeta \otimes \mathcal{F}(X)"] \arrow[d, "\cong"'] & \hom(\mathcal{F}(X), \mathcal{F}(Y)) \otimes \mathcal{F}(X) \arrow[d, "\mathrm{ev}"] \\
\mathcal{F}(\hom(X,Y) \otimes X) \arrow[r, "\mathcal{F}(\mathrm{ev})"'] & \mathcal{F}(Y)
\end{tikzcd}$$

commutes. We say that $\mathcal{F}$ preserves right inner hom-functors if ζ an isomorphism for all $X, Y \in \mathcal{C}$.

THEOREM AND DEFINITION 3.5. *Let L be a $\times_R$-bialgebra L. Then the following are equivalent:*

(1) *The underlying functor ${}_L\mathcal{M} \to {}_{R^e}\mathcal{M}$ preserves right inner hom-functors.*
(2) *The map $\tilde{\beta}\colon L \otimes_{\overline{R}} L \ni \ell \otimes m \mapsto \ell_{(1)} \otimes \ell_{(2)} m \in L \diamond L$ is a bijection.*

If these equivalent conditions hold, we say that L is a $\times_R$-Hopf algebra, and write

$$\ell_+ \otimes \ell_- := \tilde{\beta}^{-1}(\ell \otimes 1) \in L \underset{\overline{R}}{\otimes} L.$$

PROOF. Note first that if $\tilde{\beta}$ is a bijection, then so is

$$\tilde{\beta}_N : L \underset{\overline{R}}{\otimes} N \ni \ell \otimes n \mapsto \ell_{(1)} \otimes \ell_{(2)} \triangleright n \in L \diamond N$$

for every $N \in {}_L\mathcal{M}$, since we can identify $\tilde{\beta}_N$ with $\tilde{\beta} \otimes_L N$.

Let $N, P \in {}_L\mathcal{M}$. One checks that the evaluation

$$\mathrm{ev}\colon \operatorname{Hom}_L(L \diamond N, P) \diamond N \to P$$

for the inner hom-functor in ${}_L\mathcal{M}$ is given by $\mathrm{ev}(f \otimes n) = f(1 \otimes n)$. On the other hand, we can identify the inner hom-functor $\operatorname{Hom}_{\overline{R}}(N, P)$ in ${}_{R^e}\mathcal{M}$ canonically with $\operatorname{Hom}_L(L \otimes_{\overline{R}} N, P)$; under this identification, the evaluation map

$$\mathrm{ev}'\colon \operatorname{Hom}_L(L \underset{\overline{R}}{\otimes} N, P) \diamond N \to P$$

is given by $\mathrm{ev}'(f \otimes n) = f(1 \otimes n)$. Then the unique map

$$T\colon \operatorname{Hom}_L(L \diamond N, P) \to \operatorname{Hom}_L(L \underset{\overline{R}}{\otimes} N, P)$$

compatible with the respective evaluation maps is given by

$$\begin{aligned} T(f)(\ell \otimes n) = \ell \triangleright T(f)(1 \otimes n) = \ell \triangleright f(1 \otimes n) = f(\ell \triangleright (1 \otimes n)) \\ = f(\ell_{(1)} \otimes \ell_{(2)} \triangleright n) = f\tilde{\beta}_N(\ell \otimes n). \end{aligned}$$

Yoneda's Lemma now says that T is a bijection for all P if and only if $\tilde{\beta}_N$ is a bijection. $\square$

EXAMPLE 3.6. Surely the (identical) underlying functor ${}_{R^e}\mathcal{M} \to {}_{R^e}\mathcal{M}$ preserves inner hom-functors. Hence R^e is a $\times_R$-Hopf algebra; one checks $(r \otimes \overline{s})_+ \otimes (r \otimes \overline{s})_- = (r \otimes \overline{1}) \otimes (s \otimes \overline{1})$ for $r, s \in R$.

In the case $R = k$ of an ordinary bialgebra L, it is in fact well-known that L is a Hopf algebra if and only if $\tilde{\beta}$ is a bijection. The inverse of $\tilde{\beta}$ in this case is given by $\tilde{\beta}^{-1}(\ell \otimes \ell') = \ell_{(1)} \otimes S(\ell_{(2)})\ell'$, where S denotes the antipode of L. In particular, we have $\ell_+ \otimes \ell_- = \ell_{(1)} \otimes S(\ell_{(2)})$. The antipode can be obtained from $\tilde{\beta}^{-1}$ by the formula $S(\ell) = (\varepsilon \otimes L)\tilde{\beta}^{-1}(\ell \otimes 1)$. In the general $\times_R$-bialgebra case, there does not appear to be an analogous way to obtain an endomorphism of L (a more direct analog of an antipode) from $\tilde{\beta}^{-1}$: Applying ε to the first factor of $\ell_+ \otimes \ell_-$ yields an element of $E \otimes_{\overline{R}} L$, which does not come with a canonical map to L. Nevertheless we can keep in mind that $\ell_+ \otimes \ell_-$ as a $\times_R$-bialgebra analog of the Hopf algebraic expression $\ell_{(1)} \otimes S(\ell_{(2)})$; at any rate, this analogy will make the formulas for "anticomultiplication" $\ell \mapsto \ell_+ \otimes \ell_-$ in the following Proposition obviously desirable. More generally, for a left L-comodule algebra A over a k-Hopf algebra L bijectivity of a map $A\otimes A\to L\otimes A$ analogous to $\tilde\beta$ is the key part of the definition of Hopf-Galois extensions; one can compare the formulas in the following Proposition to the ones obtained for the inverse of the Galois map in [**17**].

PROPOSITION 3.7. *Let L be a $\times_R$-Hopf algebra. Then we have*

$$\ell_{+(1)}\otimes\ell_{+(2)}\ell_- = \ell\otimes 1\in L\diamond L \tag{3.1}$$

$$\ell_{(1)+}\otimes\ell_{(1)-}\ell_{(2)} = \ell\otimes 1\in L\underset{\overline{R}}{\otimes} L \tag{3.2}$$

$$\ell_+\otimes\ell_-\in\int^s\int_r {}_{\overline{s}}L_{\overline{r}}\otimes {}_{\overline{r}}L_{\overline{s}} \tag{3.3}$$

$$(\ell m)_+\otimes(\ell m)_- = \ell_+m_+\otimes m_-\ell_-\in L\underset{\overline{R}}{\otimes} L \tag{3.4}$$

$$1_+\otimes 1_- = 1\otimes 1 \tag{3.5}$$

$$\ell_{+(1)}\otimes\ell_{+(2)}\otimes\ell_- = \ell_{(1)}\otimes\ell_{(2)+}\otimes\ell_{(2)-}\in\int^{tu}\int_{rs} {}_{\overline{r}}L_{\overline{t}}\otimes {}_{r\overline{u}}L_{t\overline{s}}\otimes {}_{\overline{s}}L_{\overline{u}} \tag{3.6}$$

$$\ell_+\otimes\ell_{-(1)}\otimes\ell_{-(2)} = \ell_{++}\otimes\ell_-\otimes\ell_{+-}\in\int^{tu}\int_{rs} {}_{\overline{t}}L_{\overline{r}}\otimes {}_{\overline{s}}L_{\overline{u}}\otimes {}_{s\overline{r}}L_{u\overline{t}} \tag{3.7}$$

$$\ell_+\overline{\varepsilon(\ell_-)(1)} = \ell \tag{3.8}$$

$$\ell_+\ell_- = \varepsilon(\ell)(1) \tag{3.9}$$

PROOF. (3.1) and (3.2) are just the definition of $\ell_+\otimes\ell_-$. For (3.3), apply $\tilde\beta_L$ to $\overline{s}\ell_+\otimes\ell_-$ and $\ell_+\otimes\ell_-\overline{s}$ for $s\in R$ to get $\ell\otimes\overline{s}$ each time. Note that the right hand side of (3.4) is well-defined by (3.3). Apply $\tilde\beta_L$ to the right hand side to get $\ell m\otimes 1$. (3.5) is trivial. For (3.6), consider both sides as elements of $L\diamond L\otimes_{\overline{R}} L$, and apply $L\diamond\tilde\beta_L$ (note that this is well-defined!) to get, for the right hand side, $\ell_{(1)}\otimes\ell_{(2)}\otimes 1$ by definition, and for the left hand side

$$\begin{aligned}\ell_{+(1)}\otimes\ell_{+(2)(1)}\otimes\ell_{+(2)(2)}\ell_- &= \ell_{+(1)}\otimes\ell_{+(2)}\otimes\ell_{+(3)}\ell_-\\ &= \ell_{+(1)(1)}\otimes\ell_{+(1)(2)}\otimes\ell_{+(2)}\ell_- = \ell_{(1)}\otimes\ell_{(2)}\otimes 1\end{aligned}$$

noting that the second expression is well-defined.

For (3.7) we consider the left L-module $L\diamond L$. The left hand side of (3.7) is $\tilde\beta^{-1}_{L\diamond L}(\ell\otimes 1\otimes 1)$, and

$$\begin{aligned}\tilde\beta_{L\diamond L}(\ell_{++}\otimes\ell_-\otimes\ell_{+-}) &= \ell_{++(1)}\otimes\ell_{++(2)(1)}\ell_-\otimes\ell_{++(2)(2)}\ell_{+-}\\ &= \ell_{++(1)}\otimes\ell_{++(2)}\ell_-\otimes\ell_{++(3)}\ell_{+-} = \ell_{++(1)(1)}\otimes\ell_{++(1)(2)}\ell_-\otimes\ell_{++(2)}\ell_{+-}\\ &= \ell_{+(1)}\otimes\ell_{+(2)}\ell_-\otimes 1 = \ell\otimes 1\otimes 1\end{aligned}$$

since it is easy to see that the third expression is well-defined.

For (3.8), apply the map $\ell\otimes\ell'\mapsto\overline{\varepsilon(\ell')(1)}\ell$ to (3.1) and find

$$\begin{aligned}\ell = \overline{\varepsilon(1)(1)}\ell &= \overline{\varepsilon(\ell_{+(2)}\ell_-)(1)}\ell_{+(1)} = \overline{\varepsilon(\ell_{+(2)})(\varepsilon(\ell_-)(1))}\ell_{+(1)}\\ &= \overline{\varepsilon(\ell_{+(2)})(\varepsilon(\ell_-)(1)\cdot 1)}\ell_{+(1)} = \overline{(\varepsilon(\ell_{+(2)})\varepsilon(\ell_-)(1))(1)}\ell_{+(1)}\\ &= \overline{\varepsilon(\ell_{+(2)}\varepsilon(\ell_-)(1))(1)}\ell_{+(1)} = \overline{\varepsilon(\ell_{+(2)})(1)}\ell_{+(1)}\overline{\varepsilon(\ell_-)(1)} = \ell_+\overline{\varepsilon(\ell_-)(1)}.\end{aligned}$$

Finally, (3.9) follows from (3.1) by applying $\ell\otimes\ell'\mapsto\varepsilon(\ell)(1)\ell'$. $\square$

REMARK 3.8. The identities (3.3), (3.4) and (3.5) can be summarized as saying that

$$\mathcal{S}\colon L \ni \ell \mapsto \overline{\ell_- \otimes \overline{\ell_+}} \in \overline{L \underset{R}{\times} \overline{L}}$$

is a map of R^e-rings, or, equivalently,

$$\mathcal{S}'\colon \overline{L} \ni \overline{\ell} \mapsto \overline{\overline{\ell_-} \otimes \ell_+} \in \overline{\overline{L} \underset{\overline{R}}{\times} L}$$

is a map of $\overline{R}^e \cong R^e$-rings.

The equations (3.6), (3.7) and (3.8) then say that $\mathcal{S}'$ is an Ess map for the $\underset{\overline{R}}{\times}$-bialgebra $\overline{L}$ in the sense of [**19**, Def. 6.8]

4. Yetter-Drinfeld modules

In this section we will generalize the notion of a Yetter-Drinfeld module or crossed bimodule [**22**] and its close connection to the construction of the center of a monoidal category [**5**], see also [**6**], to the theory of $\times_R$-bialgebras. After we manage to show that the condition for a module and comodule to be a Yetter-Drinfeld module is well-defined in our setting, almost everything will be "as usual", except that we will have to replace the center construction by the weak center, where the braiding maps are not necessarily isomorphisms.

LEMMA 4.1. *Let L be a $\times_R$-bialgebra and V a left L-comodule as well as a left L-module such that the underlying left R-module structures are the same, and $\overline{r}v = vr$ holds for all $r \in R$ and $v \in V$. Then the maps*

$$\varphi_{(4.1)}\colon L \otimes V \ni \ell \otimes v \mapsto \ell_{(1)} v_{(-1)} \otimes \ell_{(2)} \triangleright v_{(0)} \in L \underset{R}{\times} V \tag{4.1}$$

$$\varphi_{(4.2)}\colon L \otimes V \ni \ell \otimes v \mapsto (\ell_{(1)} \triangleright v)_{(-1)} \ell_{(2)} \otimes (\ell_{(1)} \triangleright v)_{(0)} \in L \diamond V \tag{4.2}$$

are well-defined.

PROOF. Clearly the map

$$\varphi_{(4.3)}\colon L \otimes L \otimes V \ni \ell \otimes m \otimes v \mapsto \ell_{(1)} m \otimes \ell_{(2)} \triangleright v \in L \diamond V \tag{4.3}$$

is well-defined. Now

$$\varphi_{(4.3)}(\ell \otimes \overline{r}m \otimes v) = \ell_{(1)}\overline{r}m \otimes \ell_{(2)} \triangleright v = \ell_{(1)} m \otimes \ell_{(2)} r \triangleright v = \varphi_{(4.3)}(\ell \otimes m \otimes rv)$$

holds for $\ell, m \in L$, $v \in V$ and $r \in R$, whence $\varphi_{(4.3)}$ factors through $L \otimes (L \diamond V)$, and $\varphi_{(4.3)}(\ell \otimes v_{(-1)} \otimes v_{(0)}) = \ell_{(1)} v_{(-1)} \otimes \ell_{(2)} \triangleright v_{(0)} = \varphi_{(4.1)}(\ell \otimes v)$ is well-defined. Further, $v_{(-1)} \otimes v_{(0)} \in L \times_R V$ implies $\ell_{(1)} v_{(-1)} \otimes \ell_{(2)} \triangleright v_{(0)} \in L \times_R V$.

Now define

$$\varphi_{(4.4)}\colon L \otimes L \otimes V \to L \diamond V \tag{4.4}$$

by $\varphi_{(4.4)}(\ell \otimes m \otimes v) = (\ell \triangleright v)_{(-1)} m \otimes (\ell \triangleright v)_{(0)}$. Then

$$\begin{aligned}\varphi_{(4.4)}(\overline{r}\ell \otimes m \otimes v) &= (\overline{r}\ell \triangleright v)_{(-1)} m \otimes (\overline{r}\ell \triangleright v)_{(0)} \\ &= ((\ell \triangleright v) r)_{(-1)} m \otimes ((\ell \triangleright v) r)_{(0)} = (\ell \triangleright v)_{(-1)} r m \otimes (\ell \triangleright v)_{(0)} \\ &= \varphi_{(4.4)}(\ell \otimes rm \otimes v)\end{aligned}$$

for $\ell, m \in L$ and $r \in R$, so that $\varphi_{(4.4)}$ factors through a map

$$\varphi_{(4.5)}\colon L \diamond L \otimes V \to L \diamond V, \tag{4.5}$$

and $\varphi_{(4.5)}(\ell_{(1)} \otimes \ell_{(2)} \otimes v) = (\ell_{(1)} \triangleright v)_{(-1)}\ell_{(2)} \otimes (\ell_{(1)} \triangleright v)_{(0)} = \varphi_{(4.2)}(\ell \otimes v)$ is well-defined. $\square$

Definition 4.2. Let L be a $\times_R$-bialgebra. A (left-left) Yetter-Drinfeld L-module is a left L-module and left L-comodule V such that the respective underlying left R-module structures are the same, $\overline{r}v = vr$ for all $r \in R$ and $v \in V$ holds, and we have

$$\ell_{(1)}v_{(-1)} \otimes \ell_{(2)} \triangleright v_{(0)} = (\ell_{(1)} \triangleright v)_{(-1)}\ell_{(2)} \otimes (\ell_{(1)} \triangleright v)_{(0)}$$

in $L \diamond V$ for all $\ell \in L$ and $v \in V$

We denote the category of Yetter-Drinfeld L-modules by ${}^L_L\mathcal{YD}$.

One can check directly that the category ${}^L_L\mathcal{YD}$ is monoidal: When $V, W \in {}^L_L\mathcal{YD}$, then $V \diamond W = V \otimes_B W$, equipped with the diagonal module structure and codiagonal comodule structure, is also a Yetter-Drinfeld L-module. However this will also follow from Proposition 4.4 below.

Definition 4.3. Let $\mathcal{C}$ be a monoidal category. The (left) weak center of $\mathcal{C}$, denoted $\mathcal{Z}_0(\mathcal{C})$ is a category whose objects are pairs (X, σ_{X-}) in which $X \in \mathcal{C}$, and $\sigma_{XY} \colon X \otimes Y \to Y \otimes X$ is natural in $Y \in \mathcal{C}$ and satisfies $\sigma_{X,Y\otimes Z} = (Y \otimes \sigma_{XZ})(\sigma_{XY} \otimes Z)$ for all $Y, Z \in \mathcal{C}$ and $\sigma_{XI} = \mathrm{id}_X$. The category $\mathcal{Z}_0(\mathcal{C})$ is monoidal with $(X, \sigma_{X-}) \otimes (Y, \sigma_{Y-}) = (X \otimes Y, \sigma_{X\otimes Y,-})$, where $\sigma_{X\otimes Y,Z} = (\sigma_{XZ} \otimes Y)(X \otimes \sigma_{YZ})$.

Proposition 4.4. *A category equivalence of ${}^L_L\mathcal{YD}$ with the weak center of the monoidal category ${}_L\mathcal{M}$ is given as follows: To $V \in {}^L_L\mathcal{YD}$ assign the underlying L-module of V together with the natural transformation*

$$\sigma_{VW} \colon V \diamond W \ni v \otimes w \mapsto v_{(-1)} \triangleright w \otimes v_{(0)} \in W \diamond V$$

for $W \in {}_L\mathcal{M}$. In particular, the category ${}^L_L\mathcal{YD}$ is a monoidal category by the usual tensor products for L-modules and L-comodules, and the equivalence ${}^L_L\mathcal{YD} \cong \mathcal{Z}_0({}_L\mathcal{M})$ is a monoidal equivalence.

Proof. We omit checking that one has a functor ${}^L_L\mathcal{YD} \to \mathcal{Z}_0({}_L\mathcal{M})$ as claimed. Assume $(V, \sigma_{V,-}) \in \mathcal{Z}_0({}_L\mathcal{M})$. Endow V with the right R-module structure that identifies with its underlying left $\overline{R}$-module structure, and put

$$\lambda_0(v) := v_{(-1)} \otimes v_{(0)} := \sigma_{VL}(v \otimes 1) \in L \diamond V.$$

Let $\rho_{\overline{r}} \colon L \to L$ denote right multiplication by $\overline{r}$ for $r \in R$. Then

$$\begin{aligned} v_{(-1)}\overline{r} \otimes v_{(0)} &= (\rho_{\overline{r}} \diamond V)\sigma_{VL}(v \otimes 1) = \sigma_{VL}(V \diamond \rho_{\overline{r}})(v \otimes 1) = \sigma_{VL}(v \otimes \overline{r}) \\ &= \sigma_{VL}(\overline{r}(v \otimes 1)) = \overline{r}\sigma_{VL}(v \otimes 1) = v_{(-1)} \otimes \overline{r}v_{(0)} = v_{(-1)} \otimes v_{(0)}r \end{aligned}$$

shows that λ_0 takes values in $L \times_R V$. To show that λ_0 makes V a left L-comodule, consider the left L-module map $\Delta_0 \colon L \to L \diamond L$. We have

$$\begin{aligned} \sigma_{V,L\diamond L}(v \otimes 1 \otimes 1) &= (L \diamond \sigma_{VL})(\sigma_{VL} \diamond L)(v \otimes 1 \otimes 1) \\ &= (L \diamond \sigma_{VL})(v_{(-1)} \otimes v_{(0)} \otimes 1) = v_{(-1)} \otimes v_{(0)(-1)} \otimes v_{(0)(0)}. \end{aligned}$$

Moreover, we have $(\Delta_0 \diamond V)\sigma_{VL} = \sigma_{V,L\diamond L}(V \diamond \Delta_0)$ by naturality of σ, so that

$$\begin{aligned} v_{(-1)} \otimes v_{(0)(-1)} \otimes v_{(0)(0)} &= \sigma_{V,L\diamond L}(V \diamond \Delta_0)(v \otimes 1) \\ &= (\Delta_0 \diamond V)\sigma_{VL}(v \otimes 1) = \Delta_0(v_{(-1)}) \otimes v_{(0)}. \end{aligned}$$

Next, for any $W \in {}_L\mathcal{M}$ and $w \in W$, consider $\rho_w : L \ni \ell \mapsto \ell \triangleright w \in W$, which is a left L-module map. By naturality of σ, we have

$$\sigma_{VW}(v \otimes w) = \sigma_{VW}(V \diamond \rho_w)(v \otimes 1) = (\rho_w \diamond V)\sigma_{VW}(v \otimes 1) = v_{(-1)} \triangleright w \otimes v_{(0)}.$$

Finally, σ_{VL} is an L-linear map, so that

$$\begin{aligned}\ell_{(1)}v_{(-1)} \otimes \ell_{(2)} \triangleright v_{(0)} &= \ell \triangleright \sigma_{VL}(v \otimes 1) = \sigma_{VL}(\ell \triangleright (v \otimes 1)) \\ &= \sigma_{VL}(\ell_{(1)}v_{(-1)} \otimes \ell_{(2)} \triangleright v_{(0)}) = (\ell_{(1)} \triangleright v)_{(-1)}\ell_{(2)} \otimes (\ell_{(1)} \triangleright v)_{(0)}\end{aligned}$$

and V is a Yetter-Drinfeld module.

It is clear that by the equivalence with the monoidal category $\mathcal{Z}_0({}_L\mathcal{M})$, the category ${}_L^L\mathcal{YD}$ is also monoidal, with the module structure of the tensor product of $V, W \in {}_L^L\mathcal{YD}$ as in ${}_L\mathcal{M}$. For the comodule structure we compute

$$\begin{aligned}\lambda_0(v \otimes w) &= \sigma_{V \diamond W, L}(v \otimes w \otimes 1) = (\sigma_{VL} \diamond W)(V \diamond \sigma_{WL})(v \otimes w \otimes 1) \\ &= \sigma_{VL}(v \otimes w_{(-1)} \otimes w_{(0)}) = v_{(-1)}w_{(-1)} \otimes v_{(0)} \otimes w_{(0)}. \quad \square\end{aligned}$$

5. Skew pairings and Duality

In this section we will define skew pairings between $\times_R$-bialgebras, as well as duals of a $\times_R$-bialgebra L, which will have skew pairings with L. In the k-bialgebra case, our definitions specialize to (trivial variations of) the well-known definition, see [**2**]. It is not surprising that the duals of $\times_R$-bialgebras are modelled on the dual R-modules of the algebra in question, but it is a curious point that we cannot define Hopf algebra pairings (not skew) between $\times_R$-bialgebras, nor duals that would directly generalize the duals of k-bialgebras. It appears that taking the opposite algebra or coalgebra structure has to be built in from the start.

Definition 5.1. Let R be a k-algebra, Λ and L two $\times_R$-bialgebras. A skew pairing between Λ and L is a map $\tau \colon \Lambda \otimes L \to R$ satisfying

$$\tau((r \otimes \overline{s})\xi(t \otimes \overline{u})|\ell)v = r\tau(\xi|(t \otimes \overline{v})\ell(u \otimes \overline{s})) \tag{5.1}$$

$$\tau(\xi|\ell m) = \tau(\xi_{(1)}|\ell\overline{\tau(\xi_{(2)}|m)}) = \tau(\overline{\tau(\xi_{(2)}|m)}\xi_{(1)}|\ell) \tag{5.2}$$

$$\tau(\xi\zeta|\ell) = \tau(\xi\tau(\zeta|\ell_{(1)})|\ell_{(2)}) = \tau(\xi|\tau(\zeta|\ell_{(1)})\ell_{(2)}) \tag{5.3}$$

$$\tau(\xi|1) = \varepsilon(\xi)(1) \tag{5.4}$$

$$\tau(1|\ell) = \varepsilon(\ell)(1) \tag{5.5}$$

for all $r, s, t, u, v \in R$, $\xi, \zeta \in \Lambda$ and $\ell, m \in L$.

Remark 5.2. Assume τ satisfies (5.1) in the preceding definition. Then the right hand sides of (5.2) and (5.3) are well-defined (and the second equalities in these equations are satisfied.)

Proof. For $\xi, \xi' \in \Lambda$, $\ell, m \in L$ and $r \in R$ we have

$$\tau(\overline{r}\xi|\ell\overline{\tau(\xi'|m)}) = \tau(\xi|\ell\overline{\tau(\xi'|m)}\,\overline{r}) = \tau(\xi|\ell\overline{r\tau(\xi'|m)}) = \tau(\xi|\ell\overline{\tau(r\xi'|m)})$$

which shows the middle term of (5.2) to be well defined. The reasoning for (5.3) is similar. $\square$

Lemma 5.3. *Let L and Λ be two $\times_R$-bialgebras, and $\tau \colon \Lambda \otimes L \to R$ a skew pairing. Then a monoidal functor ${}^L\mathcal{M} \to {}_\Lambda\mathcal{M}$ is defined by endowing a left L-comodule V with the left Λ-module structure defined by $\xi \bullet v = \tau(\xi|v_{(-1)})v_{(0)}$ for $\xi \in \Lambda$ and $v \in V$. The underlying left R-module structure of V as a Λ-module is* *the same as the original one, and the underlying left $\overline{R}$-module structure is given by $\overline{r}v = vr$.*

PROOF. $\xi \bullet v$ is well-defined because of (5.1). We have

$$\xi\zeta \bullet v = \tau(\xi\zeta|v_{(-1)})v_{(0)} = \tau(\xi|\tau(\zeta|v_{(-1)(1)})v_{(-1)(2)})v_{(0)} \tag{5.6}$$

and

$$\xi \bullet \zeta \bullet v = \xi \bullet \tau(\zeta|v_{(-1)})v_{(0)} = \tau(\xi|\tau(\zeta|v_{(-1)})v_{(0)(-1)})v_{(0)(0)}. \tag{5.7}$$

The two expressions coincide because $\tau(\xi|\tau(\zeta|v_{(-2)})v_{(-1)})v_{(0)}$ is well-defined as an easy consequence of (5.1). We have

$$\begin{aligned}(r\overline{s}) \bullet v = \tau(r\overline{s}|v_{(-1)})v_{(0)} = r\tau(1|v_{(-1)}\overline{s})v_{(0)} &= r\tau(1|v_{(-1)})v_{(0)}s\\ &= r\varepsilon(v_{(-1)})(1)v_{(0)}s = rvs\end{aligned}$$

which shows unitality of the module structure as well as the claim on the underlying R^e-module structure of the left Λ-module V.

Now consider two left L-comodules V and W. Of course $V \otimes_R W = V \diamond W$ as left R^e-modules. The left Λ-module structure induced by the left L-comodule structure of $V \otimes_R W$ satisfies

$$\begin{aligned}\xi \bullet (v \otimes w) = \tau(\xi|(v \otimes w)_{(-1)})(v \otimes w)_{(0)} &= \tau(\xi|v_{(-1)}w_{(-1)})v_{(0)} \otimes w_{(0)}\\ &= \tau(\xi_{(1)}|v_{(-1)}\overline{\tau(\xi_{(2)}|w_{(-1)})})v_{(0)} \otimes w_{(0)}\\ &= \tau(\xi_{(1)}|v_{(-1)})v_{(0)}\tau(\xi_{(2)}|w_{(-1)}) \otimes w_{(0)} = \xi_{(1)} \bullet v \otimes \xi_{(2)} \bullet w\end{aligned}$$

which shows that the functor from left L-comodules to left Λ-modules that we have constructed is indeed monoidal. □

It will now be our aim to construct dual $\times_R$-bialgebras from (suitably "finite") $\times_R$-bialgebras. In more detail, we will construct a "left dual" $\times_R$-bialgebra $L^\vee$ from a "left finite" $\times_R$-bialgebra L which will act by evaluation on the left of L like Λ in the definition of a skew pairing; from a "right finite" $\times_R$-bialgebra Λ in turn we will define a "right dual" $\times_R$-bialgebra ${}^\vee\Lambda$ which will act by evaluation on the right of Λ like L in the definition of a skew pairing.

The duality between R^e-rings and $\times_R$-coalgebras that underlies our duality between $\times_R$-bialgebras was worked out already by Takeuchi in [**20**]. In fact, [**20**, Prop. 5.3] gives a duality between the categories of left R-finite projective $S|R$-corings and left $\overline{R}$-finite projective $S|R$-rings. But an R^e ring is an $R|R$-ring, and a $\times_R$-coalgebra L is the same as an $R|R$-coring, provided that we identify the two R-bimodule structures needed for an $R|R$-coring in the right way with the R^e-bimodule structure of the $\times_R$-coalgebra: The R-bimodule structure with respect to which an $R|R$-coring is an R-coring has to be obtained from the left R^e-module structure, and the second R-bimodule structure of the $R|R$-coring from the right R^e-module structure of the $\times_R$-coalgebra. The switching of module structures is of course entirely trivial, but essential, since for $\times_R$-bialgebras the module structures to be used in the definition of the coalgebra structure are already fixed by the R^e-ring structure. Instead of performing the (trivial but confusing) task of translating the duality from [**20**] to our situation, we will develop it again from scratch.

We start by defining duality on the level of mere R^e-bimodules. The definition of left and right dual R^e-bimodules will be guided by the definition of a skew pairing: The evaluations $L^\vee \otimes L \to R$ as well as $\Lambda \otimes {}^\vee\Lambda \to R$ should satisfy (5.1).

Definition 5.4. Let L and Λ be two R^e-bimodules. We define the left dual R^e-bimodule of L to be

$$L^\vee := \mathrm{Hom}_{\overline{R}-}(L, R)$$

with the R^e-bimodule structure defined by

$$((r \otimes \overline{s})\xi(t \otimes \overline{u}))(\ell) = r\xi(t\ell u\overline{s})$$

for $r, s, t, u \in R$, $\xi \in L^\vee$ and $\ell \in L$.

Furthermore, we define the right dual R^e-bimodule of Λ to be

$${}^\vee\Lambda := \mathrm{Hom}_{R-}(\Lambda, R)$$

with the R^e-bimodule structure defined by

$$((r \otimes \overline{s})\ell(t \otimes \overline{u}))(\xi) = \ell(\overline{u}\xi r\overline{t})s$$

for $r, s, t, u \in R$, $\xi \in \Lambda$ and $\ell \in L$.

If no confusion is likely, we use the notation $\langle\xi|\ell\rangle := \xi(\ell)$ for $\ell \in L$ and $\xi \in L^\vee$ as well as $\langle\xi|\ell\rangle := \ell(\xi)$ for $\ell \in {}^\vee\Lambda$ and $\xi \in \Lambda$.

Next, we endow the left and right dual R^e-bimodules of a $\times_R$-coalgebra with R^e-ring structures. Again, we are guided by the definition of skew pairings.

Lemma 5.5. *Let L be a $\times_R$-coalgebra. Then an R^e-ring structure on $L^\vee$ is defined by*

$$\langle\xi\zeta|\ell\rangle := \langle\xi|\langle\zeta|\ell_{(1)}\rangle\ell_{(2)}\rangle = \langle\xi\langle\zeta|\ell_{(1)}\rangle|\ell_{(2)}\rangle$$

for $\xi, \zeta \in L^\vee$, and

$$i : R^e \ni r \otimes \overline{s} \mapsto (\ell \mapsto r\varepsilon(\ell)(s)) \in L^\vee.$$

Proof. We have to show that $L^\vee$ is really an associative R^e-ring. One checks immediately that multliplication is well-defined. For $\xi, \zeta, \theta \in L^\vee$ and $\ell \in L$ we have

$$\begin{aligned}\langle(\xi\zeta)\theta|\ell\rangle &= \langle\xi\zeta|\langle\theta|\ell_{(1)}\rangle\ell_{(2)}\rangle = \langle\xi|\langle\zeta|\langle\theta|\ell_{(1)}\rangle\ell_{(2)(1)}\rangle\ell_{(2)(2)}\rangle \\ &\overset{*}{=} \langle\xi|\langle\zeta|\langle\theta|\ell_{(1)}\rangle\ell_{(2)}\rangle\ell_{(3)}\rangle = \langle\xi|\langle\zeta|\langle\theta|\ell_{(1)(1)}\rangle\ell_{(1)(2)}\rangle\ell_{(2)}\rangle \\ &= \langle\xi|\langle\zeta\theta|\ell_{(1)}\rangle\ell_{(2)}\rangle = \langle\xi(\zeta\theta)|\ell\rangle\end{aligned}$$

using crucially that the expression on the right of the equality marked $*$ is well-defined, which is straightforward to check. For $\xi \in L^\vee, \ell \in L$ and $r, s \in R$ we have

$$\begin{aligned}\langle i(r \otimes \overline{s})\xi|\ell\rangle &= \langle i(r \otimes \overline{s})|\langle\xi|\ell_{(1)}\rangle\ell_{(2)}\rangle = r\varepsilon(\langle\xi|\ell_{(1)}\rangle\ell_{(2)})(s) \\ &= r\langle\xi|\ell_{(1)}\rangle\varepsilon(\ell_{(2)}\overline{s})(1) = r\langle\xi|\overline{\varepsilon(\ell_{(2)}\overline{s})(1)}\ell_{(1)}\rangle = r\langle\xi|\ell\overline{s}\rangle\end{aligned}$$

and

$$\begin{aligned}\langle\xi i(r \otimes \overline{s})|\ell\rangle &= \langle\xi|\langle i(r \otimes \overline{s})|\ell_{(1)}\rangle\ell_{(2)}\rangle = \langle\xi|r\varepsilon(\ell_{(1)})(s)\ell_{(2)}\rangle \\ &= \langle\xi|r\varepsilon(\ell_{(1)}s)(1)\ell_{(2)}\rangle = \langle\xi|r\ell s\rangle.\end{aligned}$$

This enables us to show that i is an algebra map:

$$\langle i(r \otimes \overline{s})i(t \otimes \overline{u})|\ell\rangle = r\langle i(t \otimes \overline{u})|\ell\overline{s}\rangle = rt\varepsilon(\ell\overline{s})(u) = rt\varepsilon(\ell)(us) = \langle i(rt \otimes \overline{us})|\ell\rangle$$

and verifies that the underlying R^e-bimodule structure on $L^\vee$ is the same as in Definition 5.4. $\square$

Similarly one proves

LEMMA 5.6. *Let Λ be a $\times_R$-coalgebra. Then an R^e-ring structure on ${}^\vee\Lambda$ is defined by*

$$\langle\xi|\ell m\rangle = \langle\xi_{(1)}|\ell\overline{\langle\xi_{(2)}|m\rangle}\rangle = \langle\overline{\langle\xi_{(2)}|m\rangle}\xi_{(1)}|\ell\rangle$$

and

$$i\colon R^e \ni r\otimes\overline{s} \mapsto (\xi \mapsto \varepsilon(\xi)(r)s) \in {}^\vee\Lambda$$

REMARK 5.7. If $R = k$, so that a $\times_R$-coalgebra is just a k-coalgebra, then $L^\vee = (L^*)^{\mathrm{op}}$ and ${}^\vee\Lambda = \Lambda^*$ as algebras.

For any R, we have $(R^e)^\vee = \mathrm{Hom}_{\overline{R}-}(R\otimes\overline{R}, R) \cong \mathrm{End}(R)$. One checks that the isomorphism

$$\varphi_{(5.8)}\colon \mathrm{End}(R) \ni f \mapsto (r\otimes\overline{s} \mapsto f(r)s) \in (R^e)^\vee \tag{5.8}$$

is in fact an isomorphism of R^e-rings. Similarly ${}^\vee(R^e) \cong \mathrm{End}(\overline{R})$ as algebras.

It will require suitable finiteness conditions to also dualize an R^e-ring structure to give a $\times_R$-coalgebra structure, and to dualize $\times_R$-bialgebra structures.

DEFINITION 5.8. Let M be an R^e-bimodule. We call M left finite if it is finitely generated projective as a left $\overline{R}$-module. We call M right finite if it is finitely generated projective as a left R-module.

REMARK 5.9. As a first application of the definition, the categories of left finite and of right finite R^e-bimodules are dual to each other. In more detail:

If L and Λ are R^e-bimodules, then R^e-bimodule maps

$$\varphi_{(5.9)}\colon L \to {}^\vee(L^\vee) \tag{5.9}$$

$$\varphi_{(5.10)}\colon \Lambda \to ({}^\vee\Lambda)^\vee \tag{5.10}$$

are defined by $\langle\xi|\varphi_{(5.9)}(\ell)\rangle = \langle\xi|\ell\rangle$ for $\xi\in L^\vee$ and $\ell\in L$, and $\langle\varphi_{(5.10)}(\xi)|\ell\rangle = \langle\xi|\ell\rangle$ for $\xi\in\Lambda$ and $\ell\in{}^\vee\Lambda$.

If L is left finite, then $L^\vee$ is right finite, and $\varphi_{(5.9)}$ is an isomorphism. If Λ is right finite, then ${}^\vee\Lambda$ is left finite, and $\varphi_{(5.10)}$ is an isomorphism.

LEMMA 5.10. *Let L be a $\times_R$-coalgebra. Then a functor ${}^L\mathcal{M} \to {}_{L^\vee}\mathcal{M}$ is given by endowing a left L-comodule V with the module structure defined by*

$$\xi \bullet v = \langle\xi|v_{(-1)}\rangle v_{(0)} \tag{5.11}$$

for $v\in V$ and $\xi\in L^\vee$.

If L is left finite, the functor is an equivalence.

PROOF. The functor is well-defined by the same reasoning as in the proof of Lemma 5.3. Now assume that L is left finite. We have to show that any left $L^\vee$-module V arises from a left L-comodule structure on V in the way described in (5.11). We start by endowing V with the R-bimodule structure induced canonically by the underlying R^e-module structure of the $L^\vee$-module V. Since L is a finite projective left $\overline{R}$-module, we have, for any left R-module W, an isomorphism

$$\varphi_{(5.12)}\colon L\diamond W \to \mathrm{Hom}_R(L^\vee, W) \tag{5.12}$$

given by $\varphi_{(5.12)}(\ell\otimes w)(\xi) = \langle\xi|\ell\rangle w$. Note $\varphi_{(5.12)}$ is left R-linear when we endow $\mathrm{Hom}_R(L^\vee, W)$ with the left R-module structure induced by the right R-module structure of $L^\vee$. Now, using $\varphi_{(5.12)}$ for $W = V$, we define $\lambda_0\colon V \to L\diamond V$ by $\lambda_0(v) = v_{(-1)}\otimes v_{(0)}$, where $\langle\xi|v_{(-1)}\rangle v_{(0)} = \xi \triangleright v$ for all $\xi\in L^\vee$. One checks that λ_0

takes values in $L\times_R V$, and the induced map $\lambda\colon V\to L\times_R V$is an R^e-module map. To see coassociativity of λ_0, we apply $\varphi_{(5.12)}$ twice to obtain an isomorphism

$$\varphi_{(5.13)}\colon L\diamond L\diamond V\to \mathrm{Hom}_R(L^\vee,\mathrm{Hom}_R(L^\vee,V)) \tag{5.13}$$

with $\varphi_{(5.13)}(\ell\otimes m\otimes v)(\xi)(\zeta)=\langle\zeta|\langle\xi|\ell\rangle m\rangle v$. The calculations (5.6) and (5.7) for $\tau=\langle -|-\rangle$ show

$$\varphi_{(5.13)}(v_{(-1)(1)}\otimes v_{(-1)(2)}\otimes v_{(0)})(\xi)(\zeta)=(\xi\zeta)\triangleright v$$

and

$$\varphi_{(5.13)}(v_{(-1)}\otimes v_{(0)(-1)}\otimes v_{(0)(0)})(\xi)(\zeta)=\xi\triangleright(\zeta\triangleright v)$$

which shows coassociativity. □

Before dualizing R^e-ring structures to give $\times_R$-coalgebra structures, we recall a standard fact from ring and module theory: Let A and B be two rings, X a left A-module, and Y a B-A-bimodule. Then we have a map

$$\varphi_{(5.14)}\colon \mathrm{Hom}_A(X,A)\underset{A}{\otimes}\mathrm{Hom}_B(Y,B)\to\mathrm{Hom}_B(Y\underset{A}{\otimes}X,B), \tag{5.14}$$

with $\varphi_{(5.14)}(\xi\otimes\zeta)(y\otimes x)=\zeta(y\xi(x))$. If X is finitely generated projective, then $\varphi_{(5.14)}$ is an isomorphism; its inverse is given by ${\varphi_{(5.14)}}^{-1}(T)=\sum x^i\otimes T(-\otimes x_i)$ when $x_1,\dots,x_n\in X$ and $x^1,\dots,x^n\in\mathrm{Hom}_A(X,A)$ satisfy $\sum x^i(x)x_i=x$ for all $x\in X$.

Now let L and M be two R^e-bimodules. Then a version of $\varphi_{(5.14)}$ gives a map

$$\varphi_{(5.15)}\colon L^\vee\diamond M^\vee\to\mathrm{Hom}_{\overline{R}}(L\underset{\overline{R}}{\otimes}M,R) \tag{5.15}$$

which is an isomorphism if M is left finite. The calculation

$$\begin{aligned}\varphi_{(5.15)}(\xi\overline{r}\otimes\zeta s)(\ell\otimes m)=\langle\xi\overline{r}|\ell\overline{\langle\zeta s|m\rangle}\rangle=\langle\xi|\ell r\overline{\langle\zeta|sm\rangle}\rangle\\ =\varphi_{(5.15)}(\xi\otimes\zeta)(\ell r\otimes sm)\end{aligned} \tag{5.16}$$

for $\xi\in L^\vee$, $\zeta\in M^\vee$, $\ell\in L$, $m\in M$, and $r,s\in R$ shows that if $\xi\otimes\zeta\in L^\vee\times_R M^\vee$, then $\varphi_{(5.15)}(\xi\otimes\zeta)$ factors over $L\otimes_{R^e}M$, and thus

$$\varphi_{(5.17)}\colon L^\vee\underset{R}{\times}M^\vee\to(L\underset{R^e}{\otimes}M)^\vee \tag{5.17}$$

with $\langle\varphi_{(5.17)}(\xi\otimes\zeta)|\ell\otimes m\rangle=\langle\xi|\ell\overline{\langle\zeta|m\rangle}\rangle=\langle\overline{\langle\zeta|m\rangle}\xi|\ell\rangle$ is well-defined. If M is left finite, so that $\varphi_{(5.17)}$ is an isomorphism, then (5.16) shows more precisely that $\varphi_{(5.15)}(\xi\otimes\zeta)$ factors over $L\otimes_{R^e}M$ if and only if $\xi\otimes\zeta\in L^\vee\times_R M^\vee$, so that $\varphi_{(5.17)}$ is an isomorphism. $\varphi_{(5.17)}$ is a morphism of R^e-bimodules, since for $r,s,t,u\in R$

$$\begin{aligned}\langle\varphi_{(5.17)}(r\overline{s}(\xi\otimes\zeta)t\overline{u})|\ell\otimes m\rangle&=\langle\varphi_{(5.17)}(r\xi t\otimes\overline{s}\zeta\overline{u})|\ell\otimes m\rangle=\langle r\xi t|\ell\overline{\langle\overline{s}\zeta\overline{u}|m\rangle}\rangle\\&=r\langle\xi|t\ell\overline{\langle\zeta|mu\overline{s}\rangle}\rangle=r\langle\varphi_{(5.17)}(\xi\otimes\zeta)|t\ell\otimes mu\overline{s}\rangle=\langle r\overline{s}\varphi_{(5.17)}(\xi\otimes\zeta)t\overline{u}|\ell\otimes m\rangle.\end{aligned}$$

Theorem 5.11. *Let L be a left finite R^e-ring. Then there is a unique $\times_R$-coalgebra structure on $L^\vee$ satisfying*

$$\langle\xi|\ell m\rangle=\langle\xi_{(1)}|\ell\overline{\langle\xi_{(2)}|m\rangle}\rangle=\langle\overline{\langle\xi_{(2)}|m\rangle}\xi_{(1)}|\ell\rangle \tag{5.18}$$

for all $\xi\in L^\vee$ and $\ell,m\in L$. $\varphi_{(5.9)}\colon L\to{}^\vee(L^\vee)$ is an isomorphism of R^e-rings. The categories of left finite R^e-rings and right finite $\times_R$-coalgebras are dual to each other.

PROOF. Clearly (5.18) is equivalent to $\varphi_{(5.17)}(\xi_{(1)} \otimes \xi_{(2)}) = \nabla^\vee(\xi)$, where $\nabla\colon L \otimes_{R^e} L \to L$ denotes multiplication. This leaves us with the only possibility to define $\Delta\colon L^\vee \to L^\vee \times_R L^\vee$ by $\Delta = {\varphi_{(5.17)}}^{-1}\nabla^\vee$. With this definition, (5.18) is satisfied. To check that comultiplication is coassociative, we consider $\Delta_0\colon L^\vee \ni \xi \mapsto \Delta(\xi) \in L^\vee \diamond L^\vee$ and verify $(\Delta_0 \diamond L)\Delta_0 = (L \diamond \Delta_0)\Delta_0$. To this end we use (5.14) twice to obtain the isomorphism

$$\varphi_{(5.19)}\colon L^\vee \diamond L^\vee \diamond L^\vee \to \mathrm{Hom}_{\overline{R}}(L \underset{\overline{R}}{\otimes} L \underset{\overline{R}}{\otimes} L, R) \tag{5.19}$$

with $\varphi_{(5.19)}(\xi \otimes \zeta \otimes \theta)(\ell \otimes m \otimes n) = \langle \xi | \ell \overline{\langle \zeta | m \overline{\langle \theta | n \rangle} \rangle} \rangle$. Now

$$\begin{aligned} \varphi_{(5.19)}(\Delta_0 \diamond L)\Delta_0(\xi)(\ell \otimes m \otimes n) &= \varphi_{(5.19)}(\xi_{(1)(1)} \otimes \xi_{(1)(2)} \otimes \xi_{(2)})(\ell \otimes m \otimes n) \\ &= \langle \xi_{(1)(1)} | \ell \overline{\langle \xi_{(1)(2)} | m \overline{\langle \xi_{(2)} | n \rangle} \rangle} \rangle = \langle \xi_{(1)} | \ell m \overline{\langle \xi_{(2)} | n \rangle} \rangle = \langle \xi | \ell m n \rangle \\ &= \langle \xi_{(1)} | \ell \overline{\langle \xi_{(2)} | mn \rangle} \rangle = \langle \xi_{(1)} | \ell \overline{\langle \xi_{(2)(1)} | m \overline{\langle \xi_{(2)(2)} | n \rangle} \rangle} \rangle \\ &= \varphi_{(5.19)}(\xi_{(1)} \otimes \xi_{(2)(1)} \otimes \xi_{(2)(2)})(\ell \otimes m \otimes n) = \varphi_{(5.19)}(L \diamond \Delta_0)\Delta_0(\xi)(\ell \otimes m \otimes n) \end{aligned}$$

proves coassociativity. It is easy to check that $\varepsilon(\xi)(r) = \langle \xi \overline{r} | 1 \rangle = \langle \xi r | 1 \rangle$ defines a counit $\varepsilon\colon L^\vee \to \mathrm{E}$. The remaining assertions follow. □

Similarly, one proves:

THEOREM 5.12. *Let Λ be a right finite R^e-ring. Then there is a unique $\times_R$-coalgebra structure on ${}^\vee\Lambda$ satisfying*

$$\langle \xi\zeta | \ell \rangle = \langle \xi \langle \zeta | \ell_{(1)} \rangle | \ell_{(2)} \rangle = \langle \xi | \langle \zeta | \ell_{(1)} \rangle \ell_{(2)} \rangle \tag{5.20}$$

for all $\xi, \zeta \in \Lambda$ and $\ell \in L$. $\varphi_{(5.10)}\colon \Lambda \to ({}^\vee\Lambda)^\vee$ is an isomorphism of R^e-rings. The categories of right finite R^e-rings and left finite $\times_R$-coalgebras are dual to each other.

THEOREM 5.13. *Let L (resp. Λ) be a left (resp. right) finite $\times_R$-bialgebra. Then $L^\vee$ is a right finite $\times_R$-bialgebra, ${}^\vee\Lambda$ is a left finite $\times_R$-bialgebra, and $\varphi_{(5.9)}\colon L \to {}^\vee(L^\vee)$ and $\varphi_{(5.10)}\colon \Lambda \to ({}^\vee\Lambda)^\vee$ are isomorphisms of $\times_R$-bialgebras.*

The categories of left finite and right finite $\times_R$-bialgebras are dual to each other.

PROOF. We only show that $L^\vee$ is a $\times_R$-bialgebra. To do so, we apply $\varphi_{(5.17)}$ to $(\xi\zeta)_{(1)} \otimes (\xi\zeta)_{(2)} \in L^\vee \times_R L^\vee$ and $\xi_{(1)}\zeta_{(1)} \otimes \xi_{(2)}\zeta_{(2)} \in L^\vee \times_R L^\vee$ for $\xi, \zeta \in L^\vee$ and get

$$\begin{aligned} \langle \varphi_{(5.17)}((\xi\zeta)_{(1)} \otimes (\xi\zeta)_{(2)}) | \ell \otimes m \rangle &= \langle \xi\zeta | \ell m \rangle = \langle \xi | \langle \zeta | (\ell m)_{(1)} \rangle (\ell m)_{(2)} \rangle \\ &= \langle \xi | \langle \zeta | \ell_{(1)} m_{(1)} \rangle \ell_{(2)} m_{(2)} \rangle = \langle \xi_{(1)} | \langle \zeta_{(1)} | \ell_{(1)} \overline{\langle \zeta_{(2)} | m_{(1)} \rangle} \rangle \ell_{(2)} \overline{\langle \xi_{(2)} | m_{(2)} \rangle} \rangle \\ &= \langle \xi_{(1)} | \langle \zeta_{(1)} | \ell_{(1)} \rangle \ell_{(2)} \langle \zeta_{(2)} | m_{(1)} \rangle \overline{\langle \xi_{(2)} | m_{(2)} \rangle} \rangle \\ &= \langle \xi_{(1)} \overline{\langle \zeta_{(2)} | m_{(1)} \rangle} | \langle \zeta_{(1)} | \ell_{(1)} \rangle \ell_{(2)} \overline{\langle \xi_{(2)} | m_{(2)} \rangle} \rangle \\ &= \langle \xi_{(1)} | \langle \zeta_{(1)} | \ell_{(1)} \rangle \ell_{(2)} \overline{\langle \xi_{(2)} \langle \zeta_{(2)} | m_{(1)} \rangle | m_{(2)} \rangle} \rangle \\ &= \langle \xi_{(1)}\zeta_{(1)} | \ell \, \overline{\langle \xi_{(2)}\zeta_{(2)} | m \rangle} \rangle = \langle \varphi_{(5.17)}(\xi_{(1)}\zeta_{(1)} \otimes \xi_{(2)}\zeta_{(2)}) | \ell \otimes m \rangle \end{aligned}$$

for $\ell, m \in L$. □

EXAMPLE 5.14. If R is a finitely generated projective k-module, the duality theory above applies to the $\times_R$-bialgebra R^e to give a $\times_R$-bialgebra structure on $E = \mathrm{End}(R)$. For $f \in E$, write $\hat{f} := \varphi_{(5.8)}(f) \in (R^e)^\vee$. Then $\Delta_0(f) \in E \diamond E$ is determined by $\varphi_{(5.15)}(\hat{f}_{(1)} \otimes \hat{f}_{(2)})(r \otimes \overline{s} \otimes u \otimes \overline{v}) = \hat{f}(ru \otimes \overline{vs})$. With the canonical isomorphism

$$\varphi_{(5.21)} \colon \operatorname{Hom}_{\overline{R}-}(R^e \underset{\overline{R}}{\otimes} R^e, R) \to \operatorname{Hom}(R \otimes R, R) \tag{5.21}$$

defined by $\varphi_{(5.21)}(F)(r \otimes \overline{s} \otimes u \otimes \overline{v}) = F(ru)vs$, we have $\varphi_{(5.21)}\varphi_{(5.15)}(\hat{f} \otimes \hat{g})(r \otimes s) = f(r)g(s)$, so that $\Delta_0(f)$ is characterized by the equation $f_{(1)}(r)f_{(2)}(s) = f(rs)$ for all $r, s \in R$. This $\times_R$-bialgebra structure on $\operatorname{End}(R)$ was given by Lu in [**4**].

Corollary 5.15. *Let L be a left finite $\times_R$-bialgebra. Then an equivalence of monoidal categories between ${}^L\mathcal{M}$ and ${}_{L^\vee}\mathcal{M}$ is given by endowing an L-comodule V with the $L^\vee$-module structure defined by $\xi \bullet v = \langle \xi | v_{(-1)} \rangle v_{(0)}$ for $\xi \in L^\vee$ and $v \in V$.*

Remark 5.16. If $R = k$, so that a $\times_R$-bialgebra is just a bialgebra, then $L^\vee = (L^*)^{\mathrm{op}}$ and ${}^\vee\Lambda = (\Lambda^{\mathrm{op}})^*$ as bialgebras. Note that a direct generalization to $\times_R$-bialgebras of L^* is not defined, since the opposite of a $\times_R$-bialgebra is not a $\times_R$-bialgebra.

6. A generalized Drinfeld double

In this final section we will construct a $\times_R$-bialgebra analog of Drinfeld's quantum double, and, more generally, a version of the twist of a tensor product bialgebra by a skew pairing, due to Doi and Takeuchi [**2**]. Up to trivial modifications in our conventions, the outcome will be the same as in [**2**]. There, given two k-bialgebras L and Λ and a convolution invertible skew pairing $\tau \colon \Lambda \otimes L \to k$, one can define a twisted tensor product $L \bowtie_\tau \Lambda$ to be $L \otimes \Lambda$ as a coalgebra, with multiplication governed by the formula

$$(1 \bowtie \xi)(\ell \bowtie 1) = \tau(\xi_{(1)}|\ell_{(1)})\ell_{(2)} \bowtie \xi_{(2)}\tau^{-1}(\xi_{(3)}|\ell_{(3)}).$$

When L has an antipode, a skew pairing is automatically invertible with $\tau^{-1}(\xi|\ell) = \tau(\xi|S(\ell))$; if we substitute this above and further write $\ell_{(1)} \otimes S(\ell_{(2)}) = \ell_+ \otimes \ell_-$ as suggested by our discussion of the notion of $\times_R$-Hopf algebra in section 3, we obtain

$$(1 \bowtie \xi)(\ell \bowtie 1) = \tau(\xi_{(1)}|\ell_{(1)})\ell_{(2)+} \bowtie \xi_{(2)}\tau(\xi_{(3)}|\ell_{(2)-}).$$

It will be possible to generalize this formula almost directly to the $\times_R$-bialgebra case. However, it will take a major amount of work to show that multiplication, as we shall define it, is a well-defined map. Also, one should note that a naive generalization of [**2**] meets a major obstacle: There is no tensor product bialgebra of two $\times_R$-bialgebras that one could twist to obtain $L \bowtie \Lambda$: "componentwise" multiplication on the tensor product $L \otimes_{R^e} \Lambda$ is not well-defined.

Lemma 6.1. *Let L be a $\times_R$-Hopf algebra, Λ a $\times_R$-bialgebra, and $\tau \colon \Lambda \otimes L \to R$ a skew pairing. Then the map*

$$\begin{aligned}\varphi_{(6.1)} \colon \int_{uv} {}_{\overline{u}}\Lambda \otimes {}_{u\overline{v}}\Lambda \otimes {}_{v}\Lambda \otimes \int^t \int_{rs} {}_{\overline{s}}L_{\overline{t}} \otimes {}_{s}L_{t\overline{r}} \otimes {}_{\overline{r}}L &\to L \underset{R^e}{\otimes} \Lambda \\ \xi \otimes \xi' \otimes \xi'' \otimes m \otimes m' \otimes m'' &\mapsto \tau(\xi|m)m' \otimes \overline{\tau(\xi''|m'')}\xi'\end{aligned} \tag{6.1}$$

is well-defined.

PROOF. First define

$$\varphi_{(6.2)}\colon \Lambda^{\otimes 3}\otimes L^{\otimes 3}\to L\underset{R^e}{\otimes}\Lambda \tag{6.2}$$
$$\xi\otimes\xi'\otimes\xi''\otimes m\otimes m'\otimes m''\mapsto \tau(\xi|m)m'\otimes\overline{\tau(\xi''|m'')}\xi'.$$

Then $\varphi_{(6.2)}$ factors through

$$\varphi_{(6.3)}\colon \Lambda^{\otimes 3}\otimes\int_{rs} {}_{\overline{s}}L\otimes {}_sL_{\overline{r}}\otimes {}_{\overline{r}}L\to L\underset{R^e}{\otimes}\Lambda \tag{6.3}$$

since

$$\begin{aligned}\tau(\xi|\overline{s}m)m'\overline{r}\otimes\overline{\tau(\xi''|m'')}\xi' &= \tau(\xi|m)sm'\otimes\overline{\tau(\xi''|m'')r}\xi'\\ &= \tau(\xi|m)sm'\otimes\overline{\tau(\xi''|\overline{r}m'')}\xi'.\end{aligned}$$

We restrict $\varphi_{(6.3)}$ to give

$$\varphi_{(6.4)}\colon \Lambda^{\otimes 3}\otimes\int^t\int_{rs} {}_{\overline{s}}L_{\overline{t}}\otimes {}_sL_{t\overline{r}}\otimes {}_{\overline{r}}L\to L\underset{R^e}{\otimes}\Lambda \tag{6.4}$$

and claim that $\varphi_{(6.4)}$ factors to give the desired map $\varphi_{(6.1)}$. In fact

$$\begin{aligned}&\varphi_{(6.4)}(\overline{u}\xi\otimes\overline{v}\xi'\otimes\xi''\otimes m\otimes m'\otimes m'')\\ &\qquad=\tau(\overline{u}\xi|m)m'\otimes\overline{v\tau(\xi''|m'')}\xi'=\tau(\xi|m\overline{u})m'\otimes\overline{\tau(v\xi''|m'')}\xi'\\ &\qquad=\tau(\xi|m)m'u\otimes\overline{\tau(v\xi''|m'')}\xi'=\varphi_{(6.4)}(\xi\otimes u\xi'\otimes v\xi''\otimes m\otimes m'\otimes m'')\qquad\square\end{aligned}$$

DEFINITION 6.2. Let L be a $\times_R$-Hopf algebra, Λ a $\times_R$-bialgebra and $\tau\colon\Lambda\otimes L\to R$ a skew pairing. Then $L\bowtie_\tau\Lambda$ is $L\otimes_{R^e}\Lambda$ as a $\times_R$-coalgebra, endowed with the multiplication defined by

$$\begin{aligned}(\ell\otimes\xi)(m\otimes\zeta)&=\ell\tau(\xi_{(1)}|m_{(1)})m_{(2)+}\otimes\overline{\tau(\xi_{(3)}|m_{(2)-})}\xi_{(2)}\zeta\\ &=\ell\tau(\xi_{(1)}|m_{+(1)})m_{+(2)}\otimes\overline{\tau(\xi_{(3)}|m_-)}\xi_{(2)}\zeta\end{aligned}$$

for $\ell,m\in L$ and $\xi,\zeta\in\Lambda$.

LEMMA 6.3. *Multiplication in $L\bowtie_\tau\Lambda$ is a well-defined R^e-bimodule map*

$$(L\bowtie_\tau\Lambda)\underset{R^e}{\otimes}(L\bowtie_\tau\Lambda)\to L\bowtie_\tau\Lambda.$$

In particular, $i\colon R^e\to L\bowtie_\tau\Lambda$ is multiplicative and $1\otimes 1\in L\bowtie_\tau\Lambda$ is a unit.

PROOF. Using $\varphi_{(6.1)}$ from Lemma 6.1 and (3.6), we can define a map

$$\varphi_{(6.5)}\colon L\otimes\Lambda\otimes L\otimes\Lambda\to L\bowtie_\tau\Lambda \tag{6.5}$$
$$\begin{aligned}\ell\otimes\xi\otimes m\otimes\zeta\mapsto{}&\ell\tau(\xi_{(1)}|m_{(1)})m_{(2)+}\otimes\overline{\tau(\xi_{(3)}|m_{(2)-})}\xi_{(2)}\zeta\\ &=\ell\tau(\xi_{(1)}|m_{+(1)})m_{+(2)}\otimes\overline{\tau(\xi_{(3)}|m_-)}\xi_{(2)}\zeta\end{aligned}$$

which factors through

$$\varphi_{(6.6)}\colon L\bowtie_\tau\Lambda\otimes L\bowtie_\tau\Lambda\to L\bowtie_\tau\Lambda \tag{6.6}$$

because

$$\begin{aligned}
&\varphi_{(6.5)}(\ell(r\otimes\overline{s})\otimes\xi\otimes m\otimes\zeta)\\
&\qquad= \ell r\tau(\xi_{(1)}|m_{(1)})\overline{s}m_{(2)+}\otimes\overline{\tau(\xi_{(3)}|m_{(2)-})}\xi_{(2)}\zeta\\
&\qquad= \ell\tau(r\xi_{(1)}|m_{(1)})m_{(2)+}\otimes\overline{\tau(\xi_{(3)}|m_{(2)-}\overline{s})}\xi_{(2)}\zeta\\
&\qquad= \ell\tau(r\xi_{(1)}|m_{(1)})m_{(2)+}\otimes\overline{\tau(\overline{s}\xi_{(3)}|m_{(2)-})}\xi_{(2)}\zeta\\
&\qquad\qquad= \varphi_{(6.5)}(\ell\otimes(r\otimes\overline{s})\xi\otimes m\otimes\zeta)
\end{aligned}$$

and

$$\begin{aligned}
&\varphi_{(6.5)}(\ell\otimes\xi\otimes m(r\otimes\overline{s})\otimes\zeta)\\
&\qquad= \ell\tau(\xi_{(1)}|m_{(1)}r)m_{(2)+}\otimes\overline{\tau(\xi_{(3)}|sm_{(2)-})}\xi_{(2)}\zeta\\
&\qquad= \ell\tau(\xi_{(1)}\overline{r}|m_{(1)})m_{(2)+}\otimes\overline{\tau(\xi_{(3)}s|m_{(2)-})}\xi_{(2)}\zeta\\
&\qquad= \ell\tau(\xi_{(1)}|m_{(1)})m_{(2)+}\otimes\overline{\tau(\xi_{(3)}|m_{(2)-})}\xi_{(2)}(r\otimes\overline{s})\zeta\\
&\qquad\qquad= \varphi_{(6.5)}(\ell\otimes\xi\otimes m\otimes(r\otimes\overline{s})\zeta)
\end{aligned}$$

Now $\varphi_{(6.6)}$ in turn factors through $(L\bowtie_\tau\Lambda)\otimes_{R^e}(L\bowtie_\tau\Lambda)$, because

$$\begin{aligned}
&\varphi_{(6.6)}(\ell\otimes\xi(r\otimes\overline{s})\otimes m\otimes\zeta) = \ell\tau(\xi_{(1)}r|m_{(1)})m_{(2)+}\otimes\overline{\tau(\xi_{(3)}\overline{s}|m_{(2)-})}\xi_{(2)}\zeta\\
&\quad= \ell\tau(\xi_{(1)}|rm_{(1)})m_{(2)+}\otimes\overline{\tau(\xi_{(3)}|m_{(2)-}s)}\xi_{(2)}\zeta = \varphi_{(6.6)}(\ell\otimes\xi\otimes(r\otimes\overline{s})m\otimes\zeta).
\end{aligned}$$

It is clear that multiplication in $L\bowtie_\tau\Lambda$ is left and right R^e-linear. $\square$

PROPOSITION 6.4. *Let L be a $\times_R$-Hopf algebra, Λ a $\times_R$-bialgebra, and $\tau\colon\Lambda\otimes L\to R$ a skew pairing. Then $L\bowtie_\tau\Lambda$ is an R^e-ring.*

PROOF. We have to show that multiplication is associative. Note first that we have $(\ell\otimes 1)(m\otimes\zeta)=\ell m\otimes\zeta$ and $(\ell\otimes\xi)(1\otimes\zeta)=\ell\otimes\xi\zeta$ for $\ell,m\in L$ and $\xi,\zeta\in\Lambda$. Further $(1\otimes\xi)((1\otimes\zeta)(\ell\otimes 1))=(1\otimes\xi\zeta)(\ell\otimes 1)$ by the following calculation:

$$\begin{aligned}
&(1\otimes\xi)((1\otimes\zeta)(\ell\otimes 1))\\
&\quad= (1\otimes\xi)(\tau(\zeta_{(1)}|\ell_{(1)})\ell_{(2)+}\otimes\overline{\tau(\zeta_{(3)}|\ell_{(2)-})}\zeta_{(2)})\\
&\quad= \tau(\xi_{(1)}|\tau(\zeta_{(1)}|\ell_{(1)})\ell_{(2)+(1)})\ell_{(2)+(2)+}\otimes\overline{\tau(\xi_{(3)}|\ell_{(2)+(2)-})}\xi_{(2)}\overline{\tau(\zeta_{(3)}|\ell_{(2)-})}\zeta_{(2)}\\
&\quad= \tau(\xi_{(1)}|\tau(\zeta_{(1)}|\ell_{(1)})\ell_{(2)+(1)})\ell_{(2)+(2)+}\otimes\overline{\tau(\xi_{(3)}\tau(\zeta_{(3)}|\ell_{(2)-})|\ell_{(2)+(2)-})}\xi_{(2)}\zeta_{(2)}\\
&\quad= \tau(\xi_{(1)}|\tau(\zeta_{(1)}|\ell_{(1)})\ell_{(2)})\ell_{(3)+}\otimes\overline{\tau(\xi_{(3)}\tau(\zeta_{(3)}|\ell_{(3)-(1)})|\ell_{(3)-(2)})}\xi_{(2)}\zeta_{(2)}\\
&\quad= \tau(\xi_{(1)}\zeta_{(1)}|\ell_{(1)})\ell_{(2)+}\otimes\overline{\tau(\xi_{(3)}\zeta_{(3)}|\ell_{(2)-})}\xi_{(2)}\zeta_{(2)} = (1\otimes\xi\zeta)(\ell\otimes 1)
\end{aligned}$$

in which the fourth equality needs extra justification. We have

$$\begin{aligned}
&\ell_{(1)}\otimes\ell_{(2)+(1)}\otimes\ell_{(2)+(2)+}\otimes\ell_{(2)-}\otimes\ell_{(2)+(2)-}\\
&\qquad\qquad= \ell_{(1)}\otimes\ell_{(2)}\otimes\ell_{(3)+}\otimes\ell_{(3)-(1)}\otimes\ell_{(3)-(2)}
\end{aligned}$$

as elements of

$$\mathcal{X} := \int^{vw}\int_{rstu} {}_{\overline{r}}L_{\overline{v}}\otimes{}_{r\overline{s}}L_{v\overline{w}}\otimes{}_{s}L_{w\overline{t}}\otimes{}_{\overline{u}}L\otimes{}_{u\overline{t}}L$$

by several applications of Proposition 3.7. We have to check that this can be substituted in the above calculation, that is, we have to check that

$$\varphi_{(6.7)}\colon \Lambda\otimes\Lambda\otimes\mathcal{X}\to L\bowtie_\tau\Lambda \tag{6.7}$$

with

$$\begin{aligned}\varphi_{(6.7)}&(\xi\otimes\zeta\otimes\ell\otimes\ell'\otimes\ldots\otimes\ell'''')\\&=\tau(\xi_{(1)}|\tau(\zeta_{(1)}|\ell)\ell')\ell''\otimes\overline{\tau(\xi_{(3)}\tau(\zeta_{(3)}|\ell''')|\ell'''')}\xi_{(2)}\zeta_{(2)}\end{aligned}$$

is well-defined. Now

$$\varphi_{(6.8)}\colon \Lambda^{\otimes 6}\otimes\mathcal{X}\to L\bowtie_\tau\Lambda \tag{6.8}$$

$$\begin{aligned}\varphi_{(6.8)}&(\xi\otimes\xi'\otimes\xi''\otimes\zeta\otimes\zeta'\otimes\zeta''\otimes\ell\otimes\ldots\otimes\ell'''')\\&=\tau(\xi|\tau(\zeta|\ell)\ell')\ell''\otimes\overline{\tau(\xi''\tau(\zeta''|\ell''')|\ell'''')}\xi'\zeta'\end{aligned}$$

is well-defined since for $r,s,t,u\in R$ we have

$$\begin{aligned}\tau(\xi|\tau(\zeta|\overline{r}\ell)\overline{s}\ell')\ell''\overline{t}&\otimes\overline{\tau(\xi''\tau(\zeta''|\overline{u}\ell''')|\ell'''')}\xi'\zeta'\\&=\tau(\xi|\tau(\zeta|\ell)r\ell')s\ell''\otimes\overline{\tau(\xi''\tau(\zeta''|\ell''')u|\ell'''')t}\xi''\zeta'\\&=\tau(\xi|\tau(\zeta|\ell)r\ell')s\ell''\otimes\overline{\tau(\xi''\tau(\zeta''|\ell''')|u\overline{t}\ell'''')}\xi'\zeta'\end{aligned}$$

by definition of a skew pairing. One checks that

$$\begin{aligned}\varphi_{(6.8)}&(\overline{x}\xi\otimes\overline{y}\xi'\otimes\xi''\otimes\zeta\otimes\zeta'\otimes\zeta''\otimes\ell\otimes\ldots\otimes\ell'''')\\&=\varphi_{(6.8)}(\xi\otimes x\xi'\otimes y\xi''\otimes\zeta\otimes\zeta'\otimes\zeta''\otimes\ell\otimes\ldots\otimes\ell'''')\end{aligned}$$

for $x,y\in R$, so that

$$\varphi_{(6.9)}\colon L^{\otimes 4}\otimes\mathcal{X}\to L\bowtie_\tau\Lambda \tag{6.9}$$

$$\begin{aligned}\varphi_{(6.9)}&(\xi\otimes\zeta\otimes\zeta'\otimes\zeta''\otimes\ell\otimes\ldots\otimes\ell'''')\\&=\varphi_{(6.8)}(\xi_{(1)}\otimes\xi_{(2)}\otimes\xi_{(3)}\otimes\zeta\otimes\zeta'\otimes\zeta''\otimes\ell\otimes\ldots\otimes\ell'''')\end{aligned}$$

is well-defined. Next, one proves that

$$\varphi_{(6.9)}(\xi\otimes\overline{x}\zeta\otimes\overline{y}\zeta'\otimes\zeta''\otimes\ell\otimes\ldots\otimes\ell'''')=\varphi_{(6.9)}(\xi\otimes\zeta\otimes x\zeta'\otimes y\zeta''\otimes\ell\otimes\ldots\otimes\ell'''')$$

for all $x,y\in R$, so that $\varphi_{(6.7)}$ is well-defined.

Next, we have $((1\otimes\xi)(\ell\otimes 1))(m\otimes 1)=(1\otimes\xi)(\ell m\otimes 1)$ by the following calculation:

$$\begin{aligned}((1&\otimes\xi)(\ell\otimes 1))(m\otimes 1)\\&=(\tau(\xi_{(1)}|\ell_{(1)})\ell_{(2)+}\otimes\overline{\tau(\xi_{(3)}|\ell_{(2)-})}\xi_{(2)})(m\otimes 1)\\&=\tau(\xi_{(1)}|\ell_{(1)})\ell_{(2)+}\tau(\xi_{(2)(1)}|m_{(1)})m_{(2)+}\otimes\overline{\tau(\overline{\tau(\xi_{(3)}|\ell_{(2)-})}\xi_{(2)(3)}|m_{(2)-})}\xi_{(2)(2)}\\&=\tau(\xi_{(1)}|\ell_{(1)}\overline{\tau(\xi_{(2)(1)}|m_{(1)})})\ell_{(2)+}m_{(2)+}\otimes\overline{\tau(\overline{\tau(\xi_{(3)}|\ell_{(2)-})}\xi_{(2)(3)}|m_{(2)-})}\xi_{(2)(2)}\\&=\tau(\xi_{(1)}|\ell_{(1)}\overline{\tau(\xi_{(2)}|m_{(1)})})\ell_{(2)+}m_{(2)+}\otimes\overline{\tau(\overline{\tau(\xi_{(5)}|\ell_{(2)-})}\xi_{(4)}|m_{(2)-})}\xi_{(3)}\\&=\tau(\xi_{(1)}|\ell_{(1)}m_{(1)})\ell_{(2)+}m_{(2)+}\otimes\overline{\tau(\xi_{(3)}|m_{(2)-}\ell_{(2)-})}\xi_{(2)}\end{aligned}$$

where, to be able to substitute coassociativity of Λ in the fourth equality, we have to show that the map

$$\varphi_{(6.10)}\colon \Lambda\otimes L\otimes L\to L\bowtie_\tau \Lambda \tag{6.10}$$

$$\begin{aligned}&\varphi_{(6.10)}(\xi\otimes\ell\otimes m)\\ &\qquad= \tau(\xi_{(1)}|\ell_{(1)}\overline{\tau(\xi_{(2)}|m_{(1)})})\ell_{(2)+}m_{(2)+}\otimes\overline{\tau(\overline{\tau(\xi_{(5)}|\ell_{(2)-})}\xi_{(4)}|m_{(2)-})}\xi_{(3)}\end{aligned}$$

is well-defined. This proceeds in the following steps: One defines

$$\varphi_{(6.11)}\colon \Lambda^{\otimes 5}\otimes L^{\otimes 6}\to L\bowtie_\tau \Lambda \tag{6.11}$$

$$\begin{aligned}&\varphi_{(6.11)}(\xi\otimes\ldots\otimes\xi''''\otimes\ell\otimes\ell'\otimes\ell''\otimes m\otimes m'\otimes m'')\\ &\qquad= \tau(\xi|\ell\overline{\tau(\xi'|m)})\ell'm'\otimes\overline{\tau(\overline{\tau(\xi''''|\ell'')}\xi'''|m'')}\xi'',\end{aligned}$$

then checks

$$\begin{aligned}&\varphi_{(6.11)}(\xi\otimes\ldots\otimes\xi''''\otimes\ell\otimes\ell'\otimes\ell''\otimes m\otimes m'\overline{r}\otimes m'')\\ &\qquad= \varphi_{(6.11)}(\xi\otimes\ldots\otimes\xi''''\otimes\ell\otimes\ell'\otimes\ell''\otimes m\otimes m'\otimes\overline{r}m'')\end{aligned}$$

to define

$$\varphi_{(6.12)}\colon \Lambda^{\otimes 5}\otimes L^{\otimes 5}\to L\bowtie_\tau \Lambda \tag{6.12}$$

$$\begin{aligned}&\xi\otimes\ldots\otimes\xi''''\otimes\ell\otimes\ell'\otimes\ell''\otimes m\otimes m'\\ &\qquad\mapsto \varphi_{(6.11)}(\xi\otimes\ldots\xi''''\otimes\ell\otimes\ell'\otimes\ell''\otimes m\otimes m'_+\otimes m'_-).\end{aligned}$$

Now

$$\begin{aligned}&\varphi_{(6.12)}(\xi\otimes\ldots\xi''''\otimes\overline{r}\ell\otimes\ell'\overline{s}\otimes\ell''\otimes m\otimes m')\\ &\qquad= \varphi_{(6.12)}(\xi\otimes\ldots\xi''''\otimes\ell\otimes r\ell'\otimes\overline{s}\ell''\otimes m\otimes m'),\end{aligned}$$

and we can proceed to define

$$\varphi_{(6.13)}\colon \Lambda^{\otimes 5}\otimes L^{\otimes 3}\to L\bowtie_\tau \Lambda \tag{6.13}$$

$$\begin{aligned}&\xi\otimes\ldots\otimes\xi''''\otimes\ell\otimes m\otimes m'\\ &\qquad\mapsto \varphi_{(6.12)}(\xi\otimes\ldots\otimes\xi''''\otimes\ell_{(1)}\otimes\ell_{(2)+}\otimes\ell_{(2)-}\otimes m\otimes m').\end{aligned}$$

Now

$$\varphi_{(6.13)}(\xi\otimes\ldots\otimes\xi''''\otimes\ell\otimes\overline{r}m\otimes m') = \varphi_{(6.13)}(\xi\otimes\ldots\otimes\xi''''\otimes\ell\otimes m\otimes rm'),$$

and we can define

$$\varphi_{(6.14)}\colon \Lambda^{\otimes 5}\otimes L\otimes L\to L\bowtie_\tau \Lambda \tag{6.14}$$

$$\xi\otimes\ldots\otimes\xi''''\otimes\ell\otimes m\mapsto \varphi_{(6.13)}(\xi\otimes\ldots\xi''''\otimes\ell\otimes m_{(1)}\otimes m_{(2)}).$$

Finally, one can now verify

$$\begin{aligned}&\varphi_{(6.14)}(\overline{r}\xi\otimes\overline{s}\xi'\otimes\overline{t}\xi''\otimes\overline{u}\xi'''\otimes\xi''''\otimes\ell\otimes m)\\ &\qquad= \varphi_{(6.14)}(\xi\otimes r\xi'\otimes s\xi''\otimes t\xi'''\otimes u\xi''''\otimes\ell\otimes m),\end{aligned}$$

and thus $\varphi_{(6.10)}$ is well-defined.

From the special cases of associativity collected above, it is no problem to deduce associativity of multiplication in $L\bowtie_\tau\Lambda$. □

PROPOSITION 6.5. *Let L be a $\times_R$-Hopf algebra, Λ a $\times_R$-bialgebra, and $\tau\colon \Lambda \otimes L \to R$ a skew pairing.*

Let T be an R^e-ring. The following are equivalent for two maps $f\colon L \to T$ and $g\colon \Lambda \to T$ of R^e-rings:

1. *There is a (unique) R^e-ring map $h\colon L \bowtie_\tau \Lambda \to T$ with $h(\ell \bowtie 1) = f(\ell)$ and $h(1 \bowtie \xi) = g(\xi)$ for all $\ell \in L$ and $\xi \in \Lambda$.*
2. *We have*

$$\begin{aligned} g(\xi)f(\ell) &= \tau(\xi_{(1)}|\ell_{+(1)})f(\ell_{+(2)})\overline{\tau(\xi_{(3)}|\ell_{-})}g(\xi_{(2)}) \\ &= \tau(\xi_{(1)}|\ell_{(1)})f(\ell_{(2)+})\overline{\tau(\xi_{(3)}|\ell_{(2)-})}g(\xi_{(2)}) \end{aligned} \tag{6.15}$$

for all $\ell \in L$ and $\xi \in \Lambda$.

3. *We have*

$$\tau(\xi_{(1)}|\ell_{(1)})f(\ell_{(2)})g(\xi_{(2)}) = \overline{\tau(\xi_{(2)}|\ell_{(2)})}g(\xi_{(1)})f(\ell_{(1)}) \tag{6.16}$$

for all $\ell \in L$ and $\xi \in \Lambda$.

PROOF. Note that the second and third expression in (6.15) are well-defined and equal by the same reasoning as in the proof of Lemma 6.3.

The equivalence of (1) and (2) is easy (the unique map h in (1) is, of course, given by $h(\ell \bowtie \xi) = f(\ell)g(\xi)$.

Assume (2). Then we have

$$\begin{aligned} &\overline{\tau(\xi_{(2)}|\ell_{(2)})}g(\xi_{(1)})f(\ell_{(1)}) \\ &\quad= \overline{\tau(\xi_{(2)}|\ell_{(2)})}\tau(\xi_{(1)(1)}|\ell_{(1)+(1)})f(\ell_{(1)+(2)})\overline{\tau(\xi_{(1)(3)}|\ell_{(1)-})}g(\xi_{(1)(2)}) \\ &\quad= \tau(\xi_{(1)(1)}|\ell_{(1)+(1)})f(\overline{\tau(\xi_{(2)}|\ell_{(2)})}\ell_{(1)+(2)})\overline{\tau(\xi_{(1)(3)}|\ell_{(1)-})}g(\xi_{(1)(2)}) \\ &\quad= \tau(\xi_{(1)(1)}|\ell_{(1)+(1)})f(\ell_{(1)+(2)})\overline{\tau(\xi_{(1)(3)}|\ell_{(1)-}\overline{\tau(\xi_{(2)}|\ell_{(2)})})}g(\xi_{(1)(2)}) \\ &\quad\overset{*}{=} \tau(\xi_{(1)}|\ell_{(1)+(1)})f(\ell_{(1)+(2)})\overline{\tau(\xi_{(3)}|\ell_{(1)-}\overline{\tau(\xi_{(4)}|\ell_{(2)})})}g(\xi_{(2)}) \\ &\quad= \tau(\xi_{(1)}|\ell_{(1)+(1)})f(\ell_{(1)+(2)})\overline{\tau(\xi_{(2)}|\ell_{(1)-}\ell_{(2)})}g(_{(2)}) \\ &\quad= \tau(\xi_{(1)}|\ell_{(1)})f(\ell_{(2)})\overline{\tau(\xi_{(3)}|1)}g(\xi_{(2)}) \\ &\quad= \tau(\xi_{(1)}|\ell_{(1)})f(\ell_{(2)})g(\overline{\varepsilon(\xi_{(3)})(1)}\xi_{(2)}) \\ &\quad= \tau(\xi_{(1)}|\ell_{(1)})f(\ell_{(2)})g(\xi_{(2)}) \end{aligned}$$

for $\ell \in L$ and $\xi \in \Lambda$, provided we can show that the expression on the right hand side of the equality marked "$*$" is well-defined. One checks immediately that

$$\varphi_{(6.17)}\colon \Lambda^{\otimes 4} \otimes L^{\otimes 3} \to T \tag{6.17}$$

$$\xi \otimes \xi' \otimes \xi'' \otimes \xi''' \otimes \ell \otimes \ell' \otimes \ell'' \mapsto \tau(\xi|\ell_{(1)})f(\ell_{(2)})\overline{\tau(\xi''|\ell'\overline{\tau(\xi'''|\ell'')})}g(\xi')$$

is well-defined. Now one verifies

$$\varphi_{(6.17)}(\overline{r}\xi \otimes \overline{s}\xi' \otimes \overline{t}\xi'' \otimes \xi''' \otimes \ell \otimes \ell' \otimes \ell'') = \varphi_{(6.18)}(\xi \otimes r\xi' \otimes s\xi'' \otimes t\xi''' \otimes \ell \otimes \ell' \otimes \ell'')$$

for $r, s, t \in R$, so that

$$\varphi_{(6.18)}\colon \Lambda \otimes L^{\otimes 3} \to T \tag{6.18}$$

$$\xi \otimes \ell \otimes \ell' \otimes \ell'' \mapsto \varphi_{(6.17)}(\xi_{(1)} \otimes \xi_{(2)} \otimes \xi_{(3)} \otimes \xi_{(4)} \otimes \ell \otimes \ell' \otimes \ell'')$$

is well-defined. Next, since

$$\varphi_{(6.18)}(\xi \otimes \ell\overline{r} \otimes \ell's \otimes \ell'') = \varphi_{(6.18)}(\xi \otimes \ell \otimes \overline{r}\ell' \otimes s\ell''),$$

the expression

$$\varphi_{(6.18)}(\xi \otimes \ell_{(1)+} \otimes \ell_{(1)-} \otimes \ell_{(2)}) \\ = \tau(\xi_{(1)}|\ell_{(1)+(1)})f(\ell_{(1)+(2)})\overline{\tau(\xi_{(3)}|\ell_{(1)-}\overline{\tau(\xi_{(4)}|\ell_{(2)})})}g(\xi_{(2)})$$

is well-defined.

Now assume (3). Then we have

$$\begin{aligned}\tau(\xi_{(1)}|\ell_{+(1)})f(\ell_{+(2)})\overline{\tau(\xi_{(3)}|\ell_{-})}g(\xi_{(2)}) &= \tau(\xi_{(1)}|\ell_{+(1)})f(\ell_{+(2)}\overline{\tau(\xi_{(3)}|\ell_{-})})g(\xi_{(2)}) \\ = \overline{\tau(\xi_{(2)}|\ell_{+(2)}\overline{\tau(\xi_{(3)}|\ell_{-})})}g(\xi_{(1)})f(\ell_{+(1)}) &= \overline{\tau(\xi_{(2)}|\ell_{+(2)}\ell_{-})}g(\xi_{(1)})f(\ell_{+(1)}) \\ &= \overline{\tau(\xi_{(2)}|1)}g(\xi_{(1)})f(\ell_{+(1)}) = g(\xi)f(\ell) \quad \square\end{aligned}$$

Remark 6.6. Notice that by (3) above we have, in particular,

$$\tau(\xi_{(1)}|\ell_{(1)})(\ell_{(2)} \bowtie \xi_{(2)}) = \overline{\tau(\xi_{(2)}|\ell_{(2)})}(1 \bowtie \xi_{(1)})(\ell_{(1)} \bowtie 1)$$

in $L \bowtie_\tau \Lambda$.

Theorem 6.7. *Let L be a $\times_R$-Hopf algebra, Λ a $\times_R$-bialgebra, and $\tau\colon \Lambda \otimes L \to R$ a skew pairing. Then $L \bowtie_\tau \Lambda$ is a $\times_R$-bialgebra.*

Proof. Consider the R^e-ring maps

$$f\colon L \ni \ell \mapsto (\ell_{(1)} \bowtie 1) \otimes (\ell_{(2)} \bowtie 1) \in (L \bowtie_\tau \Lambda) \underset{R}{\times} (L \bowtie_\tau \Lambda)$$
$$g\colon \Lambda \ni \xi \mapsto (1 \bowtie \xi_{(1)}) \otimes (1 \bowtie \xi_{(2)}) \in (L \bowtie_\tau \Lambda) \underset{R}{\times} (L \bowtie_\tau \Lambda).$$

We have

$$\begin{aligned}\tau(\xi_{(1)}|\ell_{(1)})f(\ell_{(2)})g(\xi_{(2)}) &= \tau(\xi_{(1)}|\ell_{(1)})(\ell_{(2)(1)} \bowtie \xi_{(2)(1)}) \otimes (\ell_{(2)(2)} \otimes \xi_{(2)(2)}) \\ &= \tau(\xi_{(1)}|\ell_{(1)})(\ell_{(2)} \bowtie \xi_{(2)}) \otimes (\ell_{(3)} \bowtie \xi_{(3)}) \\ &= \overline{\tau(\xi_{(2)}|\ell_{(2)})}(1 \bowtie \xi_{(1)})(\ell_{(1)} \bowtie 1) \otimes (\ell_{(3)} \bowtie \xi_{(3)}) \\ &= (1 \bowtie \xi_{(1)})(\ell_{(1)} \bowtie 1) \otimes \tau(\xi_{(2)}|\ell_{(2)})(\ell_{(3)} \bowtie \xi_{(3)}) \\ &= (1 \bowtie \xi_{(1)})(\ell_{(1)} \bowtie 1) \otimes \overline{\tau(\xi_{(3)}|\ell_{(3)})}(1 \bowtie \xi_{(2)})(\ell_{(2)} \bowtie 1) \\ &= \overline{\tau(\xi_{(3)}|\ell_{(3)})}((1 \bowtie \xi_{(1)})(\ell_{(1)} \bowtie 1) \otimes (1 \bowtie \xi_{(2)})(\ell_{(2)} \bowtie 1) \\ &= \overline{\tau(\xi_{(2)}|\ell_{(2)})}g(\xi_{(1)})f(\xi_{(1)}),\end{aligned}$$

using that the expression on the right of the second equality is indeed a well-defined element of $(L \bowtie_\tau \Lambda) \diamond (L \bowtie_\tau \Lambda)$, which is easy to check. Now Proposition 6.5 shows that the standard comultiplication of $L \bowtie_\tau \Lambda$ is indeed an algebra map. $\square$

Corollary 6.8. *Let L be a $\times_R$-Hopf algebra, Λ a $\times_R$-bialgebra, and $\tau\colon \Lambda \otimes L \to R$ a skew pairing.*

If V is a Yetter-Drinfeld L-module, then V is a left $L \bowtie_\tau \Lambda$-module via

$$(\ell \otimes \xi)v := \ell \triangleright (\tau(\xi|v_{(-1)})v_{(0)}).$$

Proof. Let V be a left L-module and L-comodule such that the underlying left R-module structures agree, and $vr = \overline{r}v$ holds for all $r \in R$ and $v \in V$. We make V a left Λ-module by

$$\xi \bullet v := \tau(\xi|v_{(-1)})v_{(0)}. \tag{6.19}$$

For $\xi\in\Lambda$, $\ell\in L$ and $v\in V$ we calculate

$$\begin{aligned}(6.20)\quad \tau(\xi|\ell_{(1)}v_{(-1)})(\ell_{(2)}\triangleright v_{(0)}) &= \tau(\xi_{(1)}|\ell_{(1)}\overline{\tau(\xi_{(2)}|m_{(-1)})})(\ell_{(2)}\triangleright v_{(0)})\\ &= \tau(\xi_{(1)}|\ell_{(1)})\ell_{(2)}\tau(\xi_{(2)}|v_{(-1)})\triangleright v_{(0)} = \tau(\xi_{(1)}|\ell_{(1)})\ell_{(2)}\triangleright(\xi\bullet v)\end{aligned}$$

and

$$\begin{aligned}(6.21)\quad &\tau(\xi|(\ell_{(1)}\triangleright v)_{(-1)}\ell_{(2)})(\ell_{(1)}\triangleright v)_{(0)}\\ &= \tau(\xi_{(1)}|(\ell_{(1)}\triangleright v)_{(-1)}\overline{\tau(\xi_{(2)}|\ell_{(2)})})(\ell_{(1)}\triangleright v)_{(0)}\\ &= \tau(\xi_{(1)}|(\ell_{(1)}\triangleright v)_{(-1)})(\ell_{(1)}\triangleright v)_{(0)}\tau(\xi_{(2)}|\ell_{(2)})\\ &= (\xi_{(1)}\bullet(\ell_{(1)}\triangleright v))\tau(\xi_{(2)}|\ell_{(2)}) = \overline{\tau(\xi_{(2)}|\ell_{(2)})}\xi_{(1)}\bullet(\ell_{(1)}\triangleright v)\end{aligned}$$

Hence, if V is a Yetter-Drinfeld module, then, by Proposition 6.5, V is a module over $L\bowtie_\tau\Lambda$ in the claimed way. □

Definition 6.9. Let L be a left finite $\times_R$-Hopf algebra. Let $L^\vee$ be the left dual of L, and $\tau\colon L^\vee\otimes L\to R$ the canonical skew pairing. Then

$$D(L):=L\bowtie_\tau L^\vee$$

is called the Drinfeld double of L.

Theorem 6.10. *Let L be a left finite $\times_R$-Hopf algebra. Then the functor*

$${}^L_L\mathcal{YD}\to{}_{D(L)}\mathcal{M}$$

defined by endowing a Yetter-Drinfeld module V with the $D(L)$-module structure $(\ell\otimes\xi)v=\ell\triangleright\langle\xi|v_{(-1)}\rangle v_{(0)}$ is a monoidal category equivalence.

Proof. After Lemma 5.10 and Proposition 6.5, it remains to show that, for a left L-comodule and left L-module V whose underlying R-R-bimodule and R^e-module structures agree up to natural identification, the following are equivalent:

(1) V is a Yetter-Drinfeld module.
(2) The L-module structure and $L^\vee$-module structure of V satisfy

$$(6.22)\qquad \langle\xi_{(1)}|\ell_{(1)}\rangle\ell_{(2)}\triangleright(\xi_{(2)}\bullet v)=\overline{\langle\xi_{(2)}|\ell_{(2)}\rangle}\xi_{(1)}\bullet(\ell_{(1)}\triangleright v)$$

for $\xi\in L^\vee$, $\ell\in L$ and $v\in V$.

Note that the defining equation of a Yetter-Drinfeld module is an equation on elements of $L\diamond V$. When we apply $\varphi_{(5.12)}$ to it, then, by the calculations (6.20) and (6.21) (with, of course, $\tau=\langle-|-\rangle$), we get precisely (6.22). □

Example 6.11. If R is a finitely generated projective k-module, then $D(R^e)=R^e\bowtie\mathrm{End}(R)\cong\mathrm{End}(R)$ as $\times_R$-bialgebras, so that we have a category equivalence $\mathcal{Z}_0({}_R\mathcal{M}_R)\cong\mathcal{Z}_0({}_{R^e}\mathcal{M})\cong{}_{\mathrm{End}(R)}\mathcal{M}$. Assume further that R is a faithfully flat k-module. Then it is easy to see that k is a direct summand of R, and R is a progenerator. Hence ${}_{\mathrm{End}(R)}\mathcal{M}\cong{}_k\mathcal{M}$ by Morita theory. It follows that every object $M\in\mathcal{Z}_0({}_R\mathcal{M}_R)$ has the form $M\cong R\otimes V$ for a k-module V, with the R-bimodule structure induced by that of R. This can be proved directly by using faithfully flat descent, cf. **[16]**.

References

[1] Böhm, G., Nill, F., and Szlachányi, K. Weak Hopf algebras I: Integral theory and C^*-structure. *J. Algebra 221* (1999), 385–438.

[2] Doi, Y., and Takeuchi, M. Multiplication alteration by two-cocycles — the quantum version. *Comm. in Alg. 22* (1994), 5715–5732.

[3] Hayashi, T. Face algebras I–A generalization of quantum group theory. *J. Math. Soc. Japan 50* (1998), 293–315.

[4] Jiang-Hua Lu. Hopf algebroids and quantum groupoids. *Int. J. Math. 7* (1996), 47–70.

[5] Joyal, A., and Street, R. Tortile Yang-Baxter operators in tensor categories. *J. Pure Appl. Algebra 71* (1991), 43–51.

[6] Kassel, C. *Quantum Groups*, vol. 155 of *GTM*. Springer, 1995.

[7] Maltsiniotis, G. Groupoïdes quantiques. *C. R. Acad. Sci., Paris, Ser I, Math. 314* (1992), 249–252.

[8] Nikshych, D., and Vainerman, L. Algebraic versions of a finite-dimensional quantum groupoid. *preprint* (math.QA/9808054).

[9] Nill, F. Axioms for weak bialgebras. *preprint* (q-alg/9708023).

[10] Ping Xu. Quantum groupoids. *preprint* (math.QA/9905192).

[11] Ping Xu. Quantum groupoids and deformation quantization. *C. R. Acad. Sci., Paris, Ser. I, Math. 326* (1998), 289–294.

[12] Rumynin, D. Duality for Hopf algebroids. *J. Algebra 223* (2000), 237–255.

[13] Saavedra Rivano, N. *Catégories tannakiennes*, vol. 265 of *Lecture Notes in Mathematics*. Springer, 1972.

[14] Schauenburg, P. Bialgebras over noncommutative rings and a structure theorem for Hopf bimodules. *Appl. Categorical Structures 6* (1998), 193–222.

[15] Schauenburg, P. Face algebras are $\times_R$-bialgebras. In *Rings, Hopf algebras, and Brauer groups; Proceedings of the fourth week on algebra and algebraic geometry* (1998), S. Caenepeel and A. Verschoren, Eds., vol. 197 of *Lecture Notes in Pure and Applied Mathematics*, Marcel Dekker, Inc., pp. 275–285.

[16] Schauenburg, P. The monoidal center construction and bimodules. *J. Pure Appl. Algebra* (to appear)

[17] Schneider, H.-J. Representation theory of Hopf-Galois extensions. *Israel J. of Math. 72* (1990), 196–231.

[18] Sweedler, M. E. Groups of simple algebras. *Publ. Math. I.H.E.S. 44* (1974), 79–189.

[19] Takeuchi, M. Groups of algebras over $A \otimes \overline{A}$. *J. Math. Soc. Japan 29* (1977), 459–492.

[20] Takeuchi, M. $\sqrt{\text{Morita}}$ theory. *J. Math. Soc. Japan 39* (1987), 301–336.

[21] Ulbrich, K.-H. On Hopf algebras and rigid monoidal categories. *Israel J. Math. 72* (1990), 252–256.

[22] Yetter, D. N. Quantum groups and representations of monoidal categories. *Math. Proc. Camb. Phil. Soc. 108* (1990), 261–290.

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