Cohomological obstructions to cleft extensions over cocommutative Hopf algebras

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Abstract. We extend the cohomological treatment of cleft extensions over cocommutative Hopf algebras by giving an interpretation of degree three Sweedler cohomology classes as obstructions to extensions:

We show that every twisted action of a cocommutative Hopf algebra on an algebra R gives rise to an obstruction, a degree three Sweedler cohomology class of H with values in the center of R. The obstruction vanishes if and only if the twisted action belongs to a crossed product extension. We also show that every Sweedler three-cocycle can be realized as an obstruction.

Keywords: Sweedler cohomology, cleft extension, obstruction

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1. Introduction

The cohomology of a cocommutative Hopf algebra H with coefficients in a commutative algebra R on which H acts was introduced by Sweedler in [11]; we will refer to it as Sweedler cohomology $\mathcal{H}^\bullet(H,R)$. In the special cases where the Hopf algebra H is a group algebra or the universal enveloping algebra of a Lie algebra it is closely related to group cohomology or Lie algebra cohomology. The second cohomology group $\mathcal{H}^2(H,R)$ has an interpretation in terms of extensions: It classifies all algebra extensions $R \subset A$ in which A is an H-comodule algebra with coinvariant subalgebra R that is cleft (i. e. splits in a certain sense) and 'induces' the given action on R. This is modelled on a classical result in group cohomology: The second cohomology group $\mathcal{H}^2(G,M)$ of a group G with coefficients in a G-module M describes extensions of G with kernel M.

In a sense the cohomological description of cleft extensions has been extended completely to the general, non-commutative non-cocommutative case by Doi and Takeuchi [5], Blattner, Cohen and Montgomery [1], and Blattner and Montgomery [2]: A cleft H-exension A of an algebra R can be described equivalently as a crossed product $A \cong R\#_\sigma H$ given by an 'action' of H on R and a 'two-cocycle' $\sigma\colon H\otimes H \to R$. The action is not quite a module structure, but only up to conjugation by σ. By two-cocycle we mean that σ fulfills an equation that looks formally just like the condition to be a two-cocycle in Sweedler cohomology. Doi [4] shows that two crossed products are isomorphic if and only if their cocycles are related in a way that looks formally like homology of Sweedler cocycles.

In another sense, this is not at all a cohomological description of extensions, even when H is cocommutative. If R is noncommutative, the 'cocycles' are simply not cocycles in a complex defining cohomology groups — the analogy is purely formal. Let us compare this with the group case: Here one has a description of all extensions of a group G with nonabelian kernel K which is similar to (and of course precedes) the crossed product construction for Hopf algebras. Each such extension is isomorphic to $K \times G$ with a certain group structure built from maps $G \times K \to K$ and $G \times G \to K$. The former deviates somewhat from being an action of G on K by automorphisms, and the latter has to fulfill an equation that looks like a group cocycle condition — except that there is no cohomology theory in the usual sense of G with coefficients in the nonabelian group K. This description of extensions with nonabelian kernel, see e. g. Lemma 8.1 and Lemma 8.2 of [7], is just the starting point of an honestly cohomological analysis of the extension problem, which we can outline as follows: An abstract kernel is by definition a group K with a group homomorphism from G to the outer automorphism group of K. Any extension of G with kernel K gives rise to such an abstract kernel. Conversely, given an abstract kernel, it may or may not arise in this fashion from an extension. To any kernel one can assign a cohomology class of degree three with coefficients in the center of K, called the kernel's obstruction. This obstruction vanishes if and only if the kernel comes from an extension. Not only is the third cohomology group the place where obstructions live, but conversely every group cocycle arises as the obstruction to some kernel. Finally, if a kernel K admits an extension, then the second cohomology group of G with coefficients in the center of K acts simply transitively on the isomorphism classes of extensions giving rise to the kernel.

The last part of this programme was carried out for cleft extensions over a cocommutative Hopf algebra H by Doi [4]: Cleft extensions of an algebra R with the same action can be classified by the second Sweedler cohomology group with coefficients in the center of R.

The purpose of this paper is to extend the homological treatment of extensions over cocommutative Hopf algebras to include a theory of abstract kernels and their obstructions.

We define an abstract kernel to be a twisted action of H on a (noncommutative) algebra R in the sense of Schneider [10] and Montgomery

[8]. (More precisely, an equivalence class of such twisted actions). Aided by the key fact that twisted actions of cocommutative Hopf algebras are invertible by a result of Montgomery, we show that the center C of R is an H-module algebra, and we assign to an abstract kernel an obstruction, which is a class in the third Sweedler cohomology group with coefficients in the center of R. The obstruction vanishes if and only if the abstract kernel arises from an H-cleft extension of R. If this is the case, Doi's result shows that the second Sweedler cohomology group acts simply transitively on the isomorphism classes of cleft extensions inducing the given kernel. Finally we show that all Sweedler three-cocycles can be realized as obstructions: For each three-cocycle κ of H with coefficients in a commutative H-module algebra we construct an abstract kernel whose obstruction is κ.

2. Preliminaries

Throughout the paper we let k denote a commutative base ring. All algebras, coalgebras, modules etc. are meant to be over k if nothing else is specified. Unadorned $\otimes$ means tensor product over k.

Our general references on Hopf algebra theory are [9, 12]. We use the variant $\Delta(h) =: h_{(1)} \otimes h_{(2)}$ of Sweedler's notation for the comultiplication $\Delta\colon H \to H \otimes H$ of a coalgebra H. The comodule structure $\rho\colon V \to V \otimes H$ of a right H-comodule will be written $\rho(v) =: v_{(0)} \otimes v_{(1)}$.

If H is a coalgebra and R is an algebra we write $\mathrm{Reg}(H, R)$ for the group of convolution invertible maps from H to R. If H is a bialgebra and $n \in \mathbb{N}$ we write $\mathrm{Reg}^n_+(H, R) = \mathrm{Reg}_+(H \otimes H \otimes \ldots \otimes H, R)$ for the subgroup of those $\phi \in \mathrm{Reg}(H \otimes H \otimes \ldots \otimes H, R)$ that are normalized in the sense that $\phi(h^1 \otimes h^2 \otimes \ldots \otimes h^n) = \varepsilon(h^1 h^2 \ldots h^n)$ whenever one of the $h^i = 1$. We let $\mathrm{Reg}^0_+(H, R) = \mathrm{Reg}(k, R)$.

If H is a cocommutative Hopf algebra and R is a commutative H-module algebra, the elements of the abelian group $\mathrm{Reg}^n_+(H, R)$ are by definition the normalized Sweedler n-cochains of H with coefficients in R. We will not review the definition of the Sweedler coboundary operators $D^q\colon \mathrm{Reg}^q_+(H, R) \to \mathrm{Reg}^{q+1}_+(H, R)$ except for the following special cases:

$$\begin{aligned} D^2\sigma(f|g|h) = (f_{(1)} \rightharpoonup \sigma(g_{(1)}|h_{(1)}))\sigma^{-1}(f_{(2)}g_{(2)}|h_{(2)}) \\ \sigma(f_{(3)}|g_{(3)}h_{(3)})\sigma^{-1}(f_{(4)}|h_{(4)}) \end{aligned}$$

$$\begin{aligned} D^3\kappa(f|g|h|j) = (f_{(1)} \rightharpoonup \kappa(g_{(1)}|h_{(1)}|j_{(1)}))\kappa^{-1}(f_{(2)}g_{(2)}|h_{(2)}|j_{(2)}) \\ \kappa(f_{(3)}|g_{(3)}h_{(3)}|j_{(3)})\kappa^{-1}(f_{(4)}|g_{(4)}|h_{(4)}j_{(4)})\kappa(f_{(5)}|g_{(5)}|h_{(5)}) \end{aligned}$$

for $\sigma \in \mathrm{Reg}^2_+(H,R)$ and $\kappa \in \mathrm{Reg}^3_+(H,R)$. Sweedler cohomology can be defined as $\mathcal{H}^q(H,R) := \mathrm{Ker}(D^q)/\mathrm{Im}(D^{q-1})$.

3. Generalities on twisted actions and crossed products

Let H be a coalgebra and A, B algebras. Recall that a measuring from A to B by H is a map $\rightharpoonup\colon H \otimes A \ni h \otimes x \mapsto h \rightharpoonup x \in B$ satisfying $h \rightharpoonup (xy) = (h_{(1)} \rightharpoonup x)(h_{(2)} \rightharpoonup y)$ and $h \rightharpoonup 1 = \varepsilon(h)1$ for all $h \in H$ and $x, y \in R$. We are most interested in measurings from an algebra R to itself by H, which we will call measurings of R by H.

DEFINITION 3.1. *Let H be a Hopf algebra, R an algebra, and $\rightharpoonup\colon H \otimes R \to R$ a measuring of R by H.*

An **associator** *for $\rightharpoonup$ is a map $\sigma \in \mathrm{Reg}_+(H \otimes H, R)$ satisfying*

$$\sigma(g_{(1)}|h_{(1)})(g_{(2)}h_{(2)} \rightharpoonup r) = (g_{(1)} \rightharpoonup h_{(1)} \rightharpoonup r)\sigma(g_{(2)}|h_{(2)}) \tag{1}$$

for all $g, h \in H$ and $r \in R$.

The measuring $\rightharpoonup$ is a **twisted action** *of H on R if an associator for $\rightharpoonup$ exists, and $1 \rightharpoonup r = r$ holds for all $r \in R$.*

An associator for $\rightharpoonup$ is called a **cocycle** *if it satisfies*

$$(f_{(1)} \rightharpoonup \sigma(g_{(1)}|h_{(1)}))\sigma(f_{(2)}|g_{(2)}h_{(2)}) = \sigma(f_{(1)}|g_{(1)})\sigma(f_{(2)}g_{(2)}|h)$$

for all $f, g, h \in H$.

A **crossed product action** *of H on R is a twisted action one of whose associators is a cocycle.*

REMARK 3.2. *In [8, 10] a twisted action is a measuring for which $1 \rightharpoonup r$ holds for all $r \in R$, and there is $\sigma \in \mathrm{Reg}(H \otimes H, R)$ satisfying (1). For an associator, we have required an additional normalization condition. However, if σ is not normalized, then we can replace it by*

$$\sigma'(g|h) = \sigma(g|h)\sigma^{-1}(1|1)$$

This will be an associator, since $\sigma(1|1)$ is central by the calculation

$$\sigma(1|1)r = \sigma(1|1)(1 \rightharpoonup r) = (1 \rightharpoonup 1 \rightharpoonup r)\sigma(1|1) = r\sigma(1|1),$$

so that our definition is equivalent to that in [8, 10].

A measuring of R by a coalgebra H is called invertible, if the associated map $\phi\colon H \to \mathrm{End}(R)$, mapping h to $(r \mapsto h \rightharpoonup r)$, is convolution invertible. A key fact for our calculations is [8, Lemma 2.1], which shows in particular that every twisted action of a cocommutative Hopf algebra is invertible. We present a proof for completeness, and because we can offer a very short one: It is trivial to check that for any $u \in \operatorname{Reg}(H,R)$ the inner measuring induced by u, which corresponds to

$$\operatorname{inn}_u\colon H \ni h \mapsto (r \mapsto u(h_{(1)})ru^{-1}(h_{(2)})) \in \operatorname{End}(R),$$

is invertible with $(\operatorname{inn}_u)^{-1} = \operatorname{inn}_{u^{-1}}$. Now for cocommutative H the formula

$$S(h_{(1)}) \rightharpoonup h_{(2)} \rightharpoonup r = \sigma^{-1}(S(h_{(1)})|h_{(2)})r\sigma(S(h_{(3)})|h_{(4)})$$

shows $(\phi S) * \phi = \operatorname{inn}_u$ for $u(h) = \sigma^{-1}(S(h_{(1)})|h_{(2)})$, and

$$h_{(1)} \rightharpoonup S(h_{(2)}) \rightharpoonup r = \sigma^{-1}(h_{(1)}|S(h_{(2)}))r\sigma(h_{(3)}|S(h_{(4)}))$$

shows $\phi * (\phi S) = \operatorname{inn}_v$ for $v(h) = \sigma^{-1}(h_{(1)}|S(h_{(2)}))$. In particular ϕ has a right as well as a left convolution inverse. We note that the argument carries over immediately to prove the more general statement of [8, Lemma 2.1]. We will apply the invertibility of twisted actions several times, using the following Lemma:

LEMMA 3.3. *Let H be a cocommutative coalgebra, R an algebra, and $\phi\colon H \to \operatorname{End}_k(R)$ convolution invertible. Write $\phi(h)(r) =: h \rightharpoonup r$. Assume $u\colon H \to R$ satisfies $u(h_{(1)})(h_{(2)} \rightharpoonup r) = (h_{(1)} \rightharpoonup r)u(h_{(2)})$ for all $h \in H$ and $r \in R$. Then u takes values in the center of R.*

Proof. For $r \in R$ and $h \in H$ we have

$$\begin{aligned} u(h)r = u(h_{(1)})\phi(h_{(2)})\phi^{-1}(h_{(3)})(r) &= \phi(h_{(1)})\phi^{-1}(h_{(3)})(r)u(h_{(2)}) \\ &= \phi(h_{(1)})\phi^{-1}(h_{(2)})(r)u(h_{(3)}) = ru(h). \end{aligned}$$

The cocycle condition in Definition 3.1 is precisely what is needed to define an associative algebra, the crossed product $R\#_\sigma H$, by endowing $R \otimes H$ with the multiplication

$$(r\#g)(s\#h) = r(g_{(1)} \rightharpoonup s)\sigma(g_{(2)}|h_{(1)})\#g_{(3)}h_{(2)}.$$

The crossed product was introduced by Doi and Takeuchi [5] and Blattner, Cohen and Montgomery [1]. Every cleft H-comodule algebra is such a crossed product. Here, an H-comodule algebra A is called cleft, if there exists a cleaving map for A, that is, a convolution invertible right H-comodule map $\gamma\colon H \to A$ satisfying $\gamma(1) = 1$. (The last condition can always be met by modifying γ if necessary.) If A is H-cleft, and $R := A^{\operatorname{co} H} := \{a \in A|\rho(a) = a \otimes 1\}$, then $h \rightharpoonup r := \gamma(h_{(1)})r\gamma^{-1}(h_{(2)})$ and $\sigma(g|h) := \gamma(g_{(1)})\gamma(h_{(1)})\gamma^{-1}(g_{(2)}h_{(2)})$ are in R for $g, h \in H$ and

$r \in R$, and $\rightharpoonup$ is a crossed product action of H on R with cocycle σ. An isomorphism $R\#_\sigma H \to A$ of right H-comodule algebras is given by $r\#h \mapsto r\gamma(h)$. Conversely, if $\rightharpoonup$ is a crossed product action with cocycle σ, then the crossed product $R\#_\sigma H$ is cleft with cleaving map $\gamma(h) = 1\#h$ by [2]. Note that $h \rightharpoonup r = \gamma(h_{(1)})r\gamma^{-1}(h_{(2)})$ and $\sigma(g|h) = \gamma(g_{(1)})\gamma(h_{(1)})\gamma^{-1}(g_{(2)}h_{(2)})$ in this case.

The following observation is implicit in the proof of Lemma 2.1 of [4]:

LEMMA 3.4. *Let H be a Hopf algebra and A an H-cleft extension of R. Then by convolution $\mathrm{Reg}_+(H, R)$ acts simply transitively on the left on the set of all cleaving maps for A.*

Proof. Let γ be a cleaving map and $u \in \mathrm{Reg}_+(H, R)$. Then $u * \gamma$ is clearly a cleaving map as well, so that we have a left action as claimed. If $u' * \gamma = u * \gamma$ for $u, u' \in \mathrm{Reg}_+(H, R)$, the conclusion $u = u'$ is immediate. Let $\phi\colon H \to A$ be another cleaving map. Then $u(h) := \phi(h_{(1)})\gamma^{-1}(h_{(2)})$ gives a well-defined map $u\colon H \to R$, since $\rho\gamma^{-1}(h) = \gamma^{-1}(h_{(2)}) \otimes S(h_{(1)})$ by [3, Lemma 3.2], and hence

$$\rho u(h) = \phi(h_{(1)})\gamma^{-1}(h_{(4)}) \otimes h_{(2)}S(h_{(3)}) = u(h) \otimes 1.$$

The map $v\colon H \to R$ with $v(h) = \gamma(h_{(1)})\phi^{-1}(h_{(2)})$ is well-defined by the same reasoning, and clearly convolution inverse to u. Of course $\phi = u * \gamma$.

LEMMA 3.5. *Let H be a Hopf algebra and R an algebra.*

If $\rightharpoonup$ is a measuring of R by H and $u \in \mathrm{Reg}_+(H, R)$, then $\overset{u}{\rightharpoonup}$, defined by $h \overset{u}{\rightharpoonup} b := u(h_{(1)})(h_{(2)} \rightharpoonup b)u^{-1}(h_{(3)})$, is also a measuring of R by H. In this way the group $\mathrm{Reg}_+(H, R)$ acts on the left on the set of all measurings of R by H.

The subsets of twisted actions and of crossed product actions are stable under the action of $\mathrm{Reg}_+(H, R)$.

Proof. It is trivial to check that $\overset{u}{\rightharpoonup}$ is a measuring. If σ is an associator for $\rightharpoonup$, define ${}^u\sigma \in \mathrm{Reg}_+(H \otimes H, R)$ by

$${}^u\sigma(g|h) = u(g_{(1)})(g_{(2)} \rightharpoonup u(h_{(1)}))\sigma(g_{(3)}|h_{(2)})u^{-1}(g_{(4)}h_{(3)})$$

(cf. the definition of equivalent crossed systems by Doi [4]) and find

$$\begin{aligned}
&{}^u\sigma(g_{(1)}|h_{(1)})(g_{(2)}h_{(2)} \overset{u}{\rightharpoonup} r) \\
&= u(g_{(1)})(g_{(2)} \rightharpoonup u(h_{(1)}))\sigma(g_{(3)}|h_{(2)})u^{-1}(g_{(4)}h_{(3)})(g_{(5)}h_{(4)} \overset{u}{\rightharpoonup} r) \\
&= u(g_{(1)})(g_{(2)} \rightharpoonup u(h_{(1)}))\sigma(g_{(3)}|h_{(2)})(g_{(4)}h_{(3)} \rightharpoonup r)u^{-1}(g_{(5)}h_{(4)}) \\
&= u(g_{(1)})(g_{(2)} \rightharpoonup u(h_{(1)}))(g_{(3)} \rightharpoonup h_{(2)} \rightharpoonup r)\sigma(g_{(4)}|h_{(3)})u^{-1}(g_{(5)}h_{(4)}) \\
&= u(g_{(1)})(g_{(2)} \rightharpoonup (u(h_{(1)})(h_{(2)} \rightharpoonup r))\sigma(g_{(3)}|h_{(3)})u^{-1}(g_{(4)}h_{(4)}) \\
&= (g_{(1)} \overset{u}{\rightharpoonup} ((h_{(1)} \overset{u}{\rightharpoonup} r)u(h_{(2)}))u(g_{(2)})\sigma(g_{(3)}|h_{(3)})u^{-1}(g_{(4)}h_{(4)}) \\
&= (g_{(1)} \overset{u}{\rightharpoonup} h_{(1)} \overset{u}{\rightharpoonup} r){}^u\sigma(g_{(2)}|h_{(2)}).
\end{aligned}$$

If $\rightharpoonup$ is a crossed product action and σ a cocycle, then one can check by direct computation that ${}^u\sigma$ is a cocycle for $\overset{u}{\rightharpoonup}$ (see Remark 4.9 below), but there is an easier trick, again implicit in the proof of Lemma 2.1 of [4]: There is — by the crossed product construction — an H-cleft extension A of R with cleaving map $\gamma\colon H \to A$ such that $h \rightharpoonup r = \gamma(h_{(1)})r\gamma^{-1}(h_{(2)})$ for all $h \in H$ and $r \in R$. Now if $u \in \mathrm{Reg}_+(H, R)$, then $u * \gamma\colon H \to A$ is also a cleaving map, and $(u*\gamma)(h_{(1)})r(u*\gamma)^{-1}(h_{(2)}) = u(h_{(1)})\gamma(h_{(2)})r\gamma^{-1}(h_{(3)})u(h_{(4)}) = h \overset{u}{\rightharpoonup} r$, showing that $\overset{u}{\rightharpoonup}$ is also a crossed product action.

4. Kernels and their obstructions

DEFINITION 4.1. *Let H be a Hopf algebra. An abstract kernel $[R, \rightharpoonup]$ is the orbit of a twisted action $\rightharpoonup$ of H on an algebra R under the action of* $\mathrm{Reg}_+(H, R)$.

From the results on crossed products we recalled in the preceding section, one sees immediately:

COROLLARY 4.2. *Let H be a Hopf algebra, and A an H-cleft extension of R. For each cleaving map $\gamma\colon H \to A$, the equation $h \rightharpoonup r := \gamma(h_{(1)})r\gamma^{-1}(h_{(2)})$ defines a crossed product action of H on R.*

The abstract kernel $[R, \rightharpoonup]$ depends only on the isomorphism class of A.

Any twisted action $\rightharpoonup'$ in the $\mathrm{Reg}_+(H, R)$*-orbit of $\rightharpoonup$ can be obtained from a suitable choice of cleaving map for A.*

DEFINITION 4.3. *We say that an abstract kernel $[R, \rightharpoonup]$ admits an extension, if one (hence any) of its representatives can be obtained from a cleft extension as in Corollary 4.2.*

Clearly, $[R, \rightharpoonup]$ admits an extension if and only if $\rightharpoonup$ is a crossed product action.

THEOREM 4.4. *Let H be a cocommutative Hopf algebra and R an algebra with center $Z = Z(R)$.*

Let $\rightharpoonup : H \otimes R \to R$ be a twisted action and $\sigma \in \mathrm{Reg}_+(H \otimes H, R)$ an associator for $\rightharpoonup$. Then $\rightharpoonup$ defines an H-module algebra structure on Z, and

$$\kappa(f|g|h) := (f_{(1)} \rightharpoonup \sigma(g_{(1)}|h_{(1)}))\sigma(f_{(2)}|g_{(2)}h_{(2)}) \\ \sigma^{-1}(f_{(3)}g_{(3)}|h_{(3)})\sigma^{-1}(f_{(4)}|g_{(4)})$$

defines a Sweedler three-cocycle on H with values in Z.

The cohomology class $\mathrm{obs}[R, \rightharpoonup] := [\kappa] \in \mathcal{H}^3(H, Z)$ depends only on the abstract kernel $[R, \rightharpoonup]$, and it vanishes if and only if this kernel admits an extension; we will call it the obstruction to $[R, \rightharpoonup]$.

Note that the obstruction κ is defined from σ by the same formula as the Sweedler coboundary of a Sweedler two-cochains.

The proof of the theorem will be split into several lemmata.

LEMMA 4.5. *Let H be a cocommutative Hopf algebra and $\rightharpoonup : H \otimes R \to R$ a twisted action of H on an algebra R. Then $\rightharpoonup$ induces an H-module algebra structure on the center $Z(R)$, which depends only on the abstract kernel $[R, \rightharpoonup]$.*

Proof. It suffices to check that the center Z of R is stable under $\rightharpoonup$. For $z \in Z$, $r \in R$ and $h \in H$ we have

$$(h_{(1)} \rightharpoonup z)(h_{(2)} \rightharpoonup r) = h \rightharpoonup (zr) = h \rightharpoonup (rz) = (h_{(1)} \rightharpoonup r)(h_{(2)} \rightharpoonup z)$$

Since the twisted action $\rightharpoonup$ is invertible by [8, Lemma 2.1], Lemma 3.3 applies to show that $H \ni h \mapsto h \rightharpoonup z \in R$ takes values in Z.

LEMMA 4.6. *Let $\rightharpoonup : H \otimes R \to R$ be a twisted action of a cocommutative Hopf algebra H on an algebra R. Let $\sigma \in \mathrm{Reg}_+(H \otimes H, R)$ be an associator for $\rightharpoonup$. Then $\kappa \in \mathrm{Reg}_+(H \otimes H \otimes H, R)$, defined by*

$$\kappa(f|g|h) := (f_{(1)} \rightharpoonup \sigma(g_{(1)}|h_{(1)}))\sigma(f_{(2)}|g_{(2)}h_{(2)}) \\ \sigma^{-1}(f_{(3)}g_{(3)}|h_{(3)})\sigma^{-1}(f_{(4)}|g_{(4)}),$$

takes values in the center Z of R.

Proof. From

$$
\begin{aligned}
&(f_{(1)} \rightharpoonup g_{(1)} \rightharpoonup h_{(1)} \rightharpoonup r)(f_{(2)} \rightharpoonup \sigma(g_{(2)}|h_{(2)}))\sigma(f_{(3)}|g_{(3)}h_{(3)}) \\
&\quad = (f_{(1)} \rightharpoonup ((g_{(1)} \rightharpoonup h_{(1)} \rightharpoonup r)\sigma(g_{(2)}|h_{(2)})))\sigma(f_{(2)}|g_{(3)}h_{(3)}) \\
&\quad = (f_{(1)} \rightharpoonup (\sigma(g_{(1)}|h_{(1)})(g_{(2)}h_{(2)} \rightharpoonup r))\sigma(f_{(2)}|g_{(3)}h_{(3)}) \\
&\quad = (f_{(1)} \rightharpoonup \sigma(g_{(1)}|h_{(1)}))(f_{(2)} \rightharpoonup g_{(2)}h_{(2)} \rightharpoonup r)\sigma(f_{(3)}|g_{(3)}h_{(3)}) \\
&\quad = (f_{(1)} \rightharpoonup \sigma(g_{(1)}|h_{(1)}))\sigma(f_{(2)}|g_{(2)}h_{(2)})(f_{(3)}g_{(3)}h_{(3)} \rightharpoonup r)
\end{aligned}
$$

and

$$
\begin{aligned}
&(f_{(1)} \rightharpoonup g_{(1)} \rightharpoonup h_{(1)} \rightharpoonup r)\sigma(f_{(2)}|g_{(2)})\sigma(f_{(3)}g_{(3)}|h_{(2)}) \\
&\quad = \sigma(f_{(1)}|g_{(1)})(f_{(2)}g_{(2)} \rightharpoonup h_{(1)} \rightharpoonup r)\sigma(f_{(3)}g_{(3)}|h_{(2)}) \\
&\quad = \sigma(f_{(1)}|g_{(1)})\sigma(f_{(2)}g_{(2)}|h_{(1)})(f_{(3)}g_{(3)}h_{(2)} \rightharpoonup r)
\end{aligned}
$$

we conclude

$$
\begin{aligned}
&(f_{(1)} \rightharpoonup g_{(1)} \rightharpoonup h_{(1)} \rightharpoonup r)\kappa(f_{(2)}|g_{(2)}|h_{(2)}) \\
&\quad = \kappa(f_{(1)}|g_{(1)}|h_{(1)})(f_{(2)} \rightharpoonup g_{(2)} \rightharpoonup h_{(2)} \rightharpoonup r)
\end{aligned}
$$

for all $f, g, h \in H$ and $r \in R$. Now since $\rightharpoonup$ is invertible by [8, Lemma 2.1], so is

$$
\begin{aligned}
H \otimes H \otimes H &\to \mathrm{End}_k(R) \\
f \otimes g \otimes h &\mapsto (r \mapsto f \rightharpoonup g \rightharpoonup h \rightharpoonup r)
\end{aligned}
$$

and Lemma 3.3 applies to prove the claim.

We will call the induced map $\kappa\colon H \otimes H \otimes H \to Z$ the obstruction to $(\rightharpoonup, \sigma)$. Clearly, it is convolution invertible and a normalized three-cochain in $\mathrm{Reg}_+(H \otimes H \otimes H, Z)$.

LEMMA 4.7. *The obstruction to $(\rightharpoonup, \sigma)$ is a Sweedler three-cocycle.*

Proof. We have to show that

$$
\begin{aligned}
&(f_{(1)} \rightharpoonup \kappa(g_{(1)}|h_{(1)}|j_{(1)}))\kappa(f_{(2)}|g_{(2)}h_{(2)}|j_{(2)})\kappa(f_{(3)}|g_{(3)}|h_{(3)}) \\
&\quad = \kappa(f_{(1)}|g_{(1)}|h_{(1)}j_{(1)})\kappa(f_{(2)}g_{(2)}|h_{(2)}|j_{(2)})
\end{aligned}
$$

holds for all $f, g, h, j \in H$. Now

$$
\begin{aligned}
&(f_{(1)} \rightharpoonup ((g_{(1)} \rightharpoonup \sigma(h_{(1)}|j_{(1)}))\sigma(g_{(2)}|h_{(2)}j_{(2)})))\sigma(f_{(2)}|g_{(3)}h_{(3)}j_{(3)}) \\
&\quad = (f_{(1)} \rightharpoonup (\kappa(g_{(1)}|h_{(1)}|j_{(1)})\sigma(g_{(2)}|h_{(2)})\sigma(g_{(3)}h_{(3)}|j_{(2)}))) \\
&\qquad\quad \sigma(f_{(2)}|g_{(4)}h_{(4)}j_{(3)}) \\
&\quad = (f_{(1)} \rightharpoonup \kappa(g_{(1)}|h_{(1)}|j_{(1)}))(f_{(2)} \rightharpoonup \sigma(g_{(2)}|h_{(2)})) \\
&\qquad\quad (f_{(3)} \rightharpoonup \sigma(g_{(3)}h_{(3)}|j_{(2)}))\sigma(f_{(4)}|g_{(4)}h_{(4)}j_{(3)}) \\
&\quad = (f_{(1)} \rightharpoonup \kappa(g_{(1)}|h_{(1)}|j_{(1)}))(f_{(2)} \rightharpoonup \sigma(g_{(2)}|h_{(2)})) \\
&\qquad\quad \kappa(f_{(3)}|g_{(3)}h_{(3)}|j_{(2)})\sigma(f_{(4)}|g_{(4)}h_{(4)})\sigma(f_{(5)}g_{(5)}h_{(5)}|j_{(3)}) \\
&\quad = (f_{(1)} \rightharpoonup \kappa(g_{(1)}|h_{(1)}|j_{(1)}))\kappa(f_{(2)}|g_{(2)}h_{(2)}|j_{(2)}) \\
&\qquad\quad (f_{(3)} \rightharpoonup \sigma(g_{(3)}|h_{(3)}))\sigma(f_{(4)}|g_{(4)}h_{(4)})\sigma(f_{(5)}g_{(5)}h_{(5)}|j_{(3)}) \\
&\quad = (f_{(1)} \rightharpoonup \kappa(g_{(1)}|h_{(1)}|j_{(1)}))\kappa(f_{(2)}|g_{(2)}h_{(2)}|j_{(2)})\kappa(f_{(3)}|g_{(3)}|h_{(3)}) \\
&\qquad\quad \sigma(f_{(4)}|g_{(4)})\sigma(f_{(5)}g_{(5)}|h_{(4)})\sigma(f_{(6)}g_{(5)}h_{(5)}|j_{(3)})
\end{aligned}
$$

and

$$
\begin{aligned}
&(f_{(1)} \rightharpoonup ((g_{(1)} \rightharpoonup \sigma(h_{(1)}|j_{(1)}))\sigma(g_{(2)}|h_{(2)}j_{(2)})))\sigma(f_{(2)}|g_{(3)}h_{(3)}j_{(3)}) \\
&\quad = (f_{(1)} \rightharpoonup g_{(1)} \rightharpoonup \sigma(h_{(1)}|j_{(1)}))(f_{(2)} \rightharpoonup \sigma(g_{(2)}|h_{(2)}j_{(2)})) \\
&\qquad\quad \sigma(f_{(3)}|g_{(3)}h_{(3)}j_{(3)}) \\
&\quad = (f_{(1)} \rightharpoonup g_{(1)} \rightharpoonup \sigma(h_{(1)}|j_{(1)})) \\
&\qquad\quad \kappa(f_{(2)}|g_{(2)}|h_{(2)}j_{(2)})\sigma(f_{(3)}|g_{(3)})\sigma(f_{(4)}g_{(4)}|h_{(3)}j_{(3)}) \\
&\quad = \kappa(f_{(1)}|g_{(1)}|h_{(1)}j_{(1)})\sigma(f_{(2)}|g_{(2)})(f_{(3)}g_{(3)} \rightharpoonup \sigma(h_{(2)}|j_{(2)})) \\
&\qquad\quad \sigma(f_{(4)}g_{(4)}|h_{(3)}j_{(3)}) \\
&\quad = \kappa(f_{(1)}|g_{(1)}|h_{(1)}j_{(1)})\sigma(f_{(2)}|g_{(2)}) \\
&\qquad\quad \kappa(f_{(3)}g_{(3)}|h_{(2)}|j_{(2)})\sigma(f_{(4)}g_{(4)}|h_{(3)})\sigma(f_{(5)}g_{(5)}h_{(4)}|j_{(3)}) \\
&\quad = \kappa(f_{(1)}|g_{(1)}|h_{(1)}j_{(1)})\kappa(f_{(2)}g_{(2)}|h_{(2)}|j_{(2)}) \\
&\qquad\quad \sigma(f_{(3)}|g_{(3)})\sigma(f_{(4)}g_{(4)}|h_{(3)})\sigma(f_{(5)}g_{(5)}h_{(4)}|j_{(3)})
\end{aligned}
$$

and the claim follows by cancellation.

To complete the proof of Theorem 4.4 we have to show that the cohomology class of the obstruction to $(\rightharpoonup, \sigma)$ does not depend on the choice of an associator, nor on the choice of a representative for the kernel:

LEMMA 4.8. *Let H be a cocommutative Hopf algebra, R an algebra, $\rightharpoonup: H \otimes R \to R$ a twisted action, σ an associator for $\rightharpoonup$, and κ the obstruction to $(\rightharpoonup, \sigma)$.*

1. *If σ' is another associator for $\rightharpoonup$ and κ' is the obstruction to $(\rightharpoonup, \sigma')$, then κ and κ' are homologous.*

2. *If κ' is a three-cocycle homologous to κ, then there is an associator σ' for $\rightharpoonup$ such that κ' is the obstruction to $(\rightharpoonup, \sigma')$.*

3. *If $\rightharpoonup'\colon H \otimes R \to R$ is in the orbit of $\rightharpoonup$ under the action of $\mathrm{Reg}_+(H, R)$, then there is an associator σ' for $\rightharpoonup'$ such that the obstruction to $(\rightharpoonup', \sigma')$ is κ.*

Proof. We observe first that the group $\mathrm{Reg}_+(H \otimes H, Z)$ of two-cochains acts transitively on the set of associators by convolution. For let $\tau \in \mathrm{Reg}_+(H \otimes H, R)$ and $\sigma' := \tau * \sigma$. Then

$$\begin{aligned}
\sigma'(g_{(1)}|h_{(1)})&(g_{(2)}h_{(2)} \rightharpoonup r) \\
&= \tau(g_{(1)}|h_{(1)})\sigma(g_{(2)}|h_{(2)})(g_{(3)}h_{(3)} \rightharpoonup r) \\
&= \tau(g_{(1)}|h_{(1)})(g_{(2)} \rightharpoonup h_{(2)} \rightharpoonup r)\sigma(g_{(3)}|h_{(3)}) \\
&= \tau(g_{(1)}|h_{(1)})(g_{(2)} \rightharpoonup h_{(2)} \rightharpoonup r)\tau^{-1}(g_{(3)}|h_{(3)})\sigma'(g_{(4)}|h_{(4)})
\end{aligned}$$

for $r \in R$ and $g, h \in H$, so that σ' is an associator for $\rightharpoonup$ if and only if

$$\tau(g_{(1)}|h_{(1)})(g_{(2)} \rightharpoonup h_{(2)} \rightharpoonup r)\tau^{-1}(g_{(3)}|h_{(3)}) = g \rightharpoonup h \rightharpoonup r \tag{2}$$

holds for all $g, h \in H$ and $r \in R$. Now since $\rightharpoonup$ is invertible by [8, Lemma 2.1], so is $H \otimes H \ni g \otimes h \mapsto (r \mapsto g \rightharpoonup h \rightharpoonup r) \in \mathrm{End}_k(R)$, so that Lemma 3.3 applies to prove that (2) holds if and only if τ takes values in Z.

To prove the first two claims one now has to check that the obstruction to $\tau * \sigma$ is $(D^2\tau) * \kappa$ for $\tau \in \mathrm{Reg}_+(H \otimes H, Z)$. This is straightforward: κ is defined from σ by the same formula as that for the Sweedler coboundary operator, and all the terms containing τ in the formula for the obstruction to $\tau * \sigma$ can be moved to the left since H is cocommutative and τ takes values in the center.

Now let $u \in \mathrm{Reg}_+(H, R)$. Using that

$${}^u\sigma(g_{(1)}|h_{(1)})u(g_{(2)}h_{(2)}) = u(g_{(1)})(g_{(2)} \rightharpoonup u(h_{(1)}))\sigma(g_{(3)}|h_{(2)}),$$

we find

$$
\begin{aligned}
&(f_{(1)} \overset{u}{\rightharpoonup} {}^u\sigma(g_{(1)}|h_{(1)}))\,{}^u\sigma(f_{(2)}|g_{(2)}|h_{(2)})u(f_{(3)}g_{(3)}h_{(3)})\\
&\quad = (f_{(1)} \overset{u}{\rightharpoonup} {}^u\sigma(g_{(1)}|h_{(1)}))u(f_{(1)})(f_{(3)} \rightharpoonup u(g_{(2)}h_{(2)}))\sigma(f_{(4)}|g_{(3)}h_{(3)})\\
&\quad = (f_{(1)} \overset{u}{\rightharpoonup} {}^u\sigma(g_{(1)}|h_{(1)}))(f_{(2)} \overset{u}{\rightharpoonup} u(g_{(2)}h_{(2)}))u(f_{(3)})\sigma(f_{(4)}|g_{(3)}h_{(3)})\\
&\quad = (f_{(1)} \overset{u}{\rightharpoonup} ({}^u\sigma(g_{(1)}|h_{(1)})u(g_{(2)}h_{(2)})))u(f_{(2)})\sigma(f_{(3)}|g_{(3)}h_{(3)})\\
&\quad = u(f_{(1)})(f_{(2)} \rightharpoonup (({}^u\sigma(g_{(1)}|h_{(1)})u(g_{(2)}h_{(2)})))\sigma(f_{(3)}|g_{(3)}h_{(3)})\\
&\quad = \ u(f_{(1)})(f_{(2)} \rightharpoonup (u(g_{(1)})(g_{(2)} \rightharpoonup u(h_{(1)}))\sigma(g_{(3)}|h_{(2)})))\\
&\qquad\quad \sigma(f_{(3)}|g_{(4)}h_{(3)})\\
&\quad = \ u(f_{(1)})(f_{(2)} \rightharpoonup u(g_{(1)}))(f_{(3)} \rightharpoonup g_{(2)} \rightharpoonup u(h_{(1)}))\\
&\qquad\quad (f_{(4)} \rightharpoonup \sigma(g_{(3)}|h_{(2)}))\sigma(f_{(5)}|g_{(4)}h_{(3)})\\
&\quad = \ u(f_{(1)})(f_{(2)} \rightharpoonup u(g_{(1)}))(f_{(3)} \rightharpoonup g_{(2)} \rightharpoonup u(h_{(1)}))\\
&\qquad\quad \kappa(f_{(4)}|g_{(3)}|h_{(2)})\sigma(f_{(5)}|g_{(4)})\sigma(f_{(6)}g_{(5)}|h_{(3)})\\
&\quad = \ \kappa(f_{(1)}|g_{(1)}|h_{(1)})u(f_{(2)})(f_{(3)} \rightharpoonup u(g_{(2)}))\sigma(f_{(4)}|g_{(3)})\\
&\qquad\quad (f_{(5)}g_{(4)} \rightharpoonup u(h_{(2)}))\sigma(f_{(6)}g_{(5)}|h_{(3)})\\
&\quad = \ \kappa(f_{(1)}|g_{(1)}|h_{(1)})\,{}^u\sigma(f_{(2)}|g_{(2)})\\
&\qquad\quad u(f_{(3)}g_{(3)})(f_{(4)}g_{(4)} \rightharpoonup u(h_{(2)}))\sigma(f_{(5)}g_{(5)}|h_{(3)})\\
&\quad = \kappa(f_{(1)}|g_{(1)}|h_{(1)})\,{}^u\sigma(f_{(2)}|g_{(2)})\,{}^u\sigma(f_{(3)}g_{(3)}|h_{(2)})
\end{aligned}
$$

proving κ is the obstruction to $(\overset{u}{\rightharpoonup}, {}^u\sigma)$.

REMARK 4.9. *In the proof of part (3) of the preceding Lemma, cocommutativity is only used when κ is not trivial. Thus it contains a direct proof of the fact that $\overset{u}{\rightharpoonup}$ is a crossed product action if $\rightharpoonup$ is.*

5. Realization of obstructions

To complete the interpretation of degree three Sweedler cohomology in terms of the theory of cleft extensions begun in the last section, we show that every Sweedler three-cocycle arises as the obstruction to a suitable kernel.

THEOREM 5.1. *Let H be a cocommutative Hopf algebra, C a commutative H-module algebra, and $\kappa\colon H \otimes H \otimes H \to C$ a normalized Sweedler three-cocycle.*

Then there exists an algebra R with center C and a twisted action of H on R such that κ is one of its obstructions.

The proof will occupy most of the section.

DEFINITION 5.2. *Let H be a Hopf algebra and V a k-module.*

We let $(S(H,V),\sigma,t)$ denote a solution of the following universal problem: $S(H,V)$ is a k-algebra, $t\colon H\otimes V\to S(H,V)$ is a k-linear map, and $\sigma\in\operatorname{Reg}_+(H\otimes H,S(H,V))$. For every k-algebra S', k-linear map $t'\colon H\otimes V\to S'$, and $\sigma'\in\operatorname{Reg}_+(H\otimes H,S)$ there is a unique k-algebra map $F\colon S(H,V)\to S'$ with $t'=Ft$ and $\sigma'=F\sigma$. We shall write $B(H)=S(H,0)$; for a k-algebra C we put $R(C,H,V)=C\otimes S(H,V)$.

Of course, a solution to this universal problem exists. For example, we may first take $B_0(H):=T(H\otimes H\oplus H\otimes H)$ the tensor algebra over two copies of $H\otimes H$. We let $\sigma(g|h)=g\otimes h+0$ and $\overline{\sigma}(g|h):=0+g\otimes h$ in $H\otimes H\oplus H\otimes H$. Next, we let $B(H)=B_0(H)/I$, where the ideal I is generated by the elements

$$\begin{gathered}\sigma(g_{(1)}|h_{(1)})\overline{\sigma}(g_{(2)}|h_{(2)})-\varepsilon(gh)1\\ \overline{\sigma}(g_{(1)}|h_{(1)})\sigma(g_{(2)}|h_{(2)})-\varepsilon(gh)1\\ \sigma(g|1)-\varepsilon(g)1\\ \sigma(1|h)-\varepsilon(h)1\end{gathered}$$

for all $g,h\in H$. Finally, we let $S(H,V)$ be the free $B(H)$-ring generated by the k-module $H\otimes V$. This can be obtained by a tensor algebra-like construction:

$$S(H,V):=T_B(B\otimes H\otimes V\otimes B):=\bigoplus_{n\in\mathbb{N}_0}(B\otimes H\otimes V\otimes B)^{\otimes^n_B}.$$

(where $B=B(H)$).

By the universal property of $B(H)$ there exists an algebra map $F\colon B(H)\to k$, determined by $F\sigma=\varepsilon$. In particular, k is a direct summand of the k-module $B(H)$. By the construction above, or also by the universal property, there exists an algebra map $S(H,V)\to B(H)$, so that $B(H)$ and also k are direct summands of $S(H,V)$. In particular, we can consider C as a subalgebra of $R(C,H,V)$. We denote the composition of σ and t with the canonical algebra map $S(H,V)\to R(C,H,V)$ by the same symbols. We write $t(1,v)=v$ whenever convenient.

LEMMA 5.3. *Let H be a cocommutative Hopf algebra, C a commutative H-module algebra, $\kappa\colon H\otimes H\otimes H\to C$ a normalized Sweedler three-cocycle, and V a k-module.*

Put $R:=R(C,H,V)$. Then there is a unique measuring $\rightharpoonup\colon H\otimes R\to R$ such that

1. *$\rightharpoonup$ extends the given module algebra structure of C,*
2. *For all $f,g,h\in H$ we have*
$$\kappa(f_{(1)}|g_{(1)}|h_{(1)})\sigma(f_{(2)}|g_{(2)})\sigma(f_{(3)}g_{(3)}|h_{(2)}) = (f_{(1)}\rightharpoonup\sigma(g_{(1)}|h_{(1)}))\sigma(f_{(2)}|g_{(2)}h_{(2)}).$$

3. $t(h,v) = h \rightharpoonup v$ *for all* $h \in H$ *and* $v \in V$.

The measuring $\rightharpoonup$ *is a twisted action with associator* σ.

Proof. A measuring $\rightharpoonup\colon H \otimes R \to R$ is described equivalently as an algebra map $F\colon R \to \mathrm{Hom}(H,R)$ where the codomain is an algebra by convolution. Hence we can use the universal properties of $S(H,V)$ and $R(C,H,V)$.

Assume given a measuring satisfying (1), (2), and (3), and consider the corresponding $F\colon R \to \mathrm{Hom}(H,R)$. Condition (1) fixes $F(c)(h) = h \rightharpoonup c$ for $c \in C$ and $h \in H$. Condition (2) can be solved to say

$$\begin{aligned} F\sigma(g|h)(f) &= f \rightharpoonup \sigma(g|h) \\ &= \kappa(f_{(1)}|g_{(1)}|h_{(1)})\sigma(f_{(2)}|g_{(2)})\sigma(f_{(3)}g_{(3)}|h_{(2)})\sigma^{-1}(f_{(4)}|g_{(4)}h_{(3)}). \end{aligned}$$

Condition (3) implies that

$$\begin{aligned} Ft(h,v)(g) &= g \rightharpoonup t(h,v) = g \rightharpoonup h \rightharpoonup v \\ &= \sigma(g_{(1)}|h_{(1)})(g_{(2)}h_{(2)} \rightharpoonup v)\sigma^{-1}(g_{(3)}|h_{(3)}) \\ &= \sigma(g_{(1)}|h_{(1)})t(g_{(2)}h_{(2)},v)\sigma^{-1}(g_{(3)}|h_{(3)}). \end{aligned}$$

Now define $\alpha\colon C \to \mathrm{Hom}(H,R)$, $\hat\sigma\colon H \otimes H \to \mathrm{Hom}(H,R)$, and $\hat t\colon H \otimes V \to \mathrm{Hom}(H,R)$ by $\alpha(c)(h) = h \rightharpoonup c$,

$$\begin{aligned} &\hat\sigma(g|h)(f) \\ &\quad := \kappa(f_{(1)}|g_{(1)}|h_{(1)})\sigma(f_{(2)}|g_{(2)})\sigma(f_{(3)}g_{(3)}|h_{(2)})\sigma^{-1}(f_{(4)}|g_{(4)}h_{(3)}), \end{aligned}$$

and

$$\hat t(h \otimes v)(g) = \sigma(g_{(1)}|h_{(1)})t(g_{(2)}h_{(2)},v)\sigma^{-1}(g_{(3)}|h_{(3)}).$$

To define our measuring, we have to check that $\hat\sigma$ is convolution invertible and satisfies $\hat\sigma(1|h) = \hat\sigma(h|1) = \varepsilon(h)1$ for all $h \in H$, and that the image of α lies in the center of $\mathrm{Hom}(H,R)$. The latter is immediate by the definition, the inverse of $\hat\sigma$ is given by

$$\begin{aligned} \hat\sigma^{-1}(g|h)(f) = \sigma(f_{(1)}|g_{(1)}h_{(1)})&\sigma^{-1}(f_{(2)}g_{(2)}|h_{(2)}) \\ &\sigma^{-1}(f_{(3)}|g_{(3)})\kappa^{-1}(f_{(4)}|g_{(4)}|h_{(3)}), \end{aligned}$$

and the normalization condition on $\hat\sigma$ is obvious.

It remains to show that σ is an associator. Since both $(g \otimes h) \triangleright r = g \rightharpoonup h \rightharpoonup r$ and $(g \otimes h) \bullet r = \sigma(g_{(1)}|h_{(1)})(g_{(2)}h_{(2)} \rightharpoonup r)\sigma^{-1}(g_{(3)}|h_{(3)})$ define measurings of R by $H \otimes H$, it suffices by the universal property of R to check that

$$\begin{aligned} \sigma(f_{(1)}|g_{(1)})(f_{(2)}g_{(2)} \rightharpoonup c) &= (f_{(1)} \rightharpoonup g_{(1)} \rightharpoonup c)\sigma(f_{(2)}|g_{(2)}) \\ \sigma(f_{(1)}|g_{(1)})(f_{(2)}g_{(2)} \rightharpoonup \sigma(h|j)) &= (f_{(1)} \rightharpoonup g_{(1)} \rightharpoonup \sigma(h|j))\sigma(f_{(2)}|g_{(2)}) \\ \sigma(f_{(1)}|g_{(1)})(f_{(2)}g_{(2)} \rightharpoonup t(h,v)) &= (f_{(1)} \rightharpoonup g_{(1)} \rightharpoonup t(h,v))\sigma(f_{(2)}|g_{(2)}) \end{aligned}$$

hold for all $f, g, h, j \in H$, $c \in C$, and $v \in V$. The first formula holds since H is cocommutative, and C is an H-module algebra in the center of R. For the second formula we compute

$$\begin{aligned}
&\kappa(f_{(1)}|g_{(1)}|h_{(1)}j_{(1)})\ (f_{(2)} \rightharpoonup g_{(2)} \rightharpoonup \sigma(h_{(2)}|j_{(2)})) \\
&\qquad\qquad \sigma(f_{(3)}|g_{(3)})\sigma(f_{(4)}g_{(4)}|h_{(3)}j_{(3)}) \\
&= (f_{(1)} \rightharpoonup g_{(1)} \rightharpoonup \sigma(h_{(1)}|j_{(1)}))\kappa(f_{(2)}|g_{(2)}|h_{(2)}j_{(2)})\sigma(f_{(3)}|g_{(3)}) \\
&\quad \sigma(f_{(4)}g_{(4)}|h_{(3)}j_{(3)}) \\
&= (f_{(1)} \rightharpoonup g_{(1)} \rightharpoonup \sigma(h_{(1)}|j_{(1)}))(f_{(2)} \rightharpoonup \sigma(g_{(2)}|h_{(2)}j_{(2)})) \\
&\quad \sigma(f_{(3)}|g_{(3)}h_{(3)}j_{(3)}) \\
&= (f_{(1)} \rightharpoonup ((g_{(1)} \rightharpoonup \sigma(h_{(1)}|j_{(1)}))\sigma(g_{(2)}|h_{(2)}j_{(2)}))) \\
&\quad \sigma(f_{(2)}|g_{(3)}h_{(3)}j_{(2)}) \\
&= (f_{(1)} \rightharpoonup (\kappa(g_{(1)}|h_{(1)}|j_{(1)})\sigma(g_{(2)}|h_{(2)})\sigma(g_{(3)}h_{(3)}|j_{(2)}))) \\
&\quad \sigma(f_{(2)}|g_{(4)}h_{(4)}j_{(3)}) \\
&= (f_{(1)} \rightharpoonup \kappa(g_{(1)}|h_{(1)}|j_{(1)}))(f_{(2)} \rightharpoonup \sigma(g_{(2)}|h_{(2)})) \\
&\quad (f_{(3)} \rightharpoonup \sigma(g_{(3)}h_{(3)}|j_{(2)}))\sigma(f_{(4)}|g_{(4)}h_{(4)}j_{(3)}) \\
&= (f_{(1)} \rightharpoonup \kappa(g_{(1)}|h_{(1)}|j_{(1)}))(f_{(2)} \rightharpoonup \sigma(g_{(2)}|h_{(2)})) \\
&\quad \kappa(f_{(3)}|g_{(3)}h_{(3)}|j_{(2)})\sigma(f_{(4)}|g_{(4)}h_{(4)})\sigma(f_{(5)}g_{(5)}h_{(5)}|j_{(3)}) \\
&= (f_{(1)} \rightharpoonup \kappa(g_{(1)}|h_{(1)}|j_{(1)}))\kappa(f_{(2)}|g_{(2)}h_{(2)}|j_{(2)}) \\
&\quad (f_{(3)} \rightharpoonup \sigma(g_{(3)}|h_{(3)}))\sigma(f_{(4)}|g_{(4)}h_{(4)})\sigma(f_{(5)}g_{(5)}h_{(5)}|j_{(3)}) \\
&= (f_{(1)} \rightharpoonup \kappa(g_{(1)}|h_{(1)}|j_{(1)}))\kappa(f_{(2)}|g_{(2)}h_{(2)}|j_{(2)})\kappa(f_{(3)}|g_{(3)}|h_{(3)}) \\
&\quad \sigma(f_{(4)}|g_{(4)})\sigma(f_{(5)}g_{(5)}|h_{(4)})\sigma(f_{(6)}g_{(6)}h_{(5)}|j_{(3)}) \\
&= \kappa(f_{(1)}|g_{(1)}|h_{(1)}j_{(1)})\kappa(f_{(2)}g_{(2)}|h_{(2)}|j_{(2)}) \\
&\quad \sigma(f_{(4)}|g_{(4)})\sigma(f_{(5)}g_{(5)}|h_{(3)})\sigma(f_{(6)}g_{(6)}h_{(4)}|j_{(3)}) \\
&= \kappa(f_{(1)}|g_{(1)}|h_{(1)}j_{(1)})\sigma(f_{(2)}|g_{(2)}) \\
&\quad \kappa(f_{(3)}g_{(3)}|h_{(2)}|j_{(2)})\sigma(f_{(4)}g_{(4)}|h_{(3)})\sigma(f_{(5)}g_{(5)}h_{(4)}|j_{(3)}) \\
&= \kappa(f_{(1)}|g_{(1)}|h_{(1)}j_{(1)})\sigma(f_{(2)}|g_{(2)})(f_{(3)}g_{(3)} \rightharpoonup \sigma(h_{(2)}|j_{(2)})) \\
&\quad \sigma(f_{(4)}g_{(4)}|h_{(3)}j_{(3)})
\end{aligned}$$

and for the third

$$\begin{aligned}
&\kappa(f_{(1)}|g_{(1)}|h_{(1)})\sigma(f_{(2)}|g_{(2)})(f_{(3)}g_{(3)} \rightharpoonup t(h_{(2)}, v))\sigma(f_{(4)}g_{(4)}|h_{(3)}) \\
&\quad = \kappa(f_{(1)}|g_{(1)}|h_{(1)})\sigma(f_{(2)}|g_{(2)})\sigma(f_{(3)}g_{(3)}|h_{(2)})t(f_{(4)}g_{(4)}h_{(3)}, v) \\
&\quad = (f_{(1)} \rightharpoonup \sigma(g_{(1)}|h_{(1)}))\sigma(f_{(2)}|g_{(2)}h_{(2)})t(f_{(3)}g_{(3)}h_{(3)}, v) \\
&\quad = (f_{(1)} \rightharpoonup \sigma(g_{(1)}|h_{(1)}))(f_{(2)} \rightharpoonup t(g_{(2)}h_{(2)}, v))\sigma(f_{(3)}|g_{(3)}h_{(3)}) \\
&\quad = (f_{(1)} \rightharpoonup (\sigma(g_{(1)}|h_{(1)})t(g_{(2)}h_{(2)}, v)))\sigma(f_{(2)}|g_{(3)}h_{(3)}) \\
&\quad = (f_{(1)} \rightharpoonup ((g_{(1)} \rightharpoonup t(h_{(1)}, v))\sigma(g_{(2)}|h_{(2)})))\sigma(f_{(2)}|g_{(3)}h_{(3)}) \\
&\quad = (f_{(1)} \rightharpoonup g_{(1)} \rightharpoonup t(h_{(1)}, v))(f_{(2)} \rightharpoonup \sigma(g_{(2)}|h_{(2)}))\sigma(f_{(3)}|g_{(3)}h_{(3)})
\end{aligned}$$

$$\begin{aligned}
&= (f_{(1)} \rightharpoonup g_{(1)} \rightharpoonup t(h_{(1)}, v))\kappa(f_{(2)}|g_{(2)}|h_{(2)})\sigma(f_{(3)}|g_{(3)}) \\
&\qquad \sigma(f_{(4)}g_{(4)}|h_{(3)}) \\
&= \kappa(f_{(1)}|g_{(1)}|h_{(1)})(f_{(2)} \rightharpoonup g_{(2)} \rightharpoonup t(h_{(2)}, v)) \; . \\
&\qquad \sigma(f_{(3)}|g_{(3)})\sigma(f_{(4)}g_{(4)}|h_{(3)})
\end{aligned}$$

That the measuring satisfies $1 \rightharpoonup r = r$ for all $r \in R$ is straightforward to check.

REMARK 5.4. *In the proof of the preceding Lemma cocommutativity of H was only needed when κ is nontrivial. In particular $B := B(H)$ has a unique crossed product action of H with cocycle σ for arbitrary H. The crossed product $B\#_\sigma H$ is a 'versal' H-cleft extension, and thus B is somewhat analogous to a 'classifying space' for H-cleft extensions: For all H-cleft extensions A there exists at least one algebra map $B \to A^{\mathrm{co}\,H}$ that extends to an H-comodule algebra map $B\#_\sigma H \to A$. With a suitable definition of this notion, the map $B \to A^{\mathrm{co}\,H}$ is in fact a morphism of kernels.*

Lemma 5.3 almost completes the proof of Theorem 5.1; in fact, it looks as if we could have avoided some complications by just using $R(C, H, 0)$. However, it may happen that the center of this is bigger than C.

A similar situation arises in the classical case of group cohomology, where an extra construction is necessary to realize an obstruction over the cyclic group of order two, precisely because the construction that works for other groups has too big a center in this special case. One can check that if H is the group algebra of a group of order two, then $R(C, H, 0)$ is the (commutative) Laurent polynomial ring in one variable over C.

The following result finishes the job:

PROPOSITION 5.5. *Let H be a coalgebra, C a commutative algebra, and V a free k-module of rank at least two.*

Then the center of $R(C, H, V)$ is precisely C.

Proof. We let $R = R(C, H, V)$, $S := S(H, V)$, and $B := B(H)$. We realize S as a tensor algebra over B as in the remarks following Definition 5.2. Thus $S \cong \bigoplus_{n \in \mathbb{N}_0} S_n$ with $S_n = B \otimes (H \otimes V \otimes B)^{\otimes n}$.

Let $(v_i)_{i \in I}$ be a basis of V. Then R is naturally graded by the free monoid generated by the set I. We note that the only elements of S whose degree is the empty word are the elements of $C \otimes S_0$.

Since k is a direct summand in the k-modules B and H, right multiplication by v_i is a split monomorphism from S to itself: Its component $S_n \to S_{n+1}$ is given by tensoring on the right with $1 \otimes v_i \otimes 1 \in H \otimes V \otimes B$.

Similarly left multiplication by v_i is split monic as an endomorphism of S.

We show first that an element of the center $Z(R)$ must be homogeneous of empty degree, hence in $C \otimes S_0$. Let ξ be in the center, and assume that ξ has homogeneous components of nonempty degree. If all the degrees of nonzero homogeneous components of ξ involve only the letter i, let $j \in I \setminus \{i\}$. Then $v_j\xi$ has nonzero homogeneous components with degree a word of at least two letters starting with j, but all the degrees of length at least two of homogeneous components of ξv_j start with i, a contradiction. Thus we can assume that there is a homogeneous component of ξ whose degree contains the letter i and another letter. Let $n(\xi)$ denote the maximal number of leading i's in the degree of a homogeneous component of ξ that does not only contain the letter i. Then $n(v_i\xi) = n(\xi)+1$, but $n(\xi v_i) = n(\xi)$, a contradiction.

Now let $\xi = \sum c_i \otimes x_i \in C \otimes S_0 = C \otimes B$ be contained in the center of R. Then for $j \in I$ we have $v_j\xi = \xi v_j$, that is,

$$\sum c_i \otimes 1 \otimes 1 \otimes v_j \otimes x_i = \sum c_i \otimes x_i \otimes 1 \otimes v_j \otimes 1 \in C \otimes B \otimes H \otimes V \otimes H,$$

hence

$$\sum c_i \otimes 1 \otimes x_i = \sum c_i \otimes x_i \otimes 1 \in C \otimes B \otimes B.$$

Since k is a direct summand in B, it follows that $\xi \in C \otimes k \subset C \otimes B$.

In their cohomological treatment of group extensions with nonabelian kernel Eilenberg and MacLane [6] also define an intrinsic product of abstract kernels corresponding to the product of their obstructions. This carries over partly to our situation: Assume H is a cocommutative Hopf algebra and $[R, \rightharpoonup]$ and $[S, \rightharpoonup]$ are abstract kernels with the same center $(Z, \rightharpoonup)$. Then $R \underset{Z}{\otimes} S$ is a k-algebra, $h \rightharpoonup (r \otimes s) = h \rightharpoonup r \otimes h \rightharpoonup s$ defines a measuring of $R \underset{Z}{\otimes} S$ by H, and if σ and τ are associators for $(R, \rightharpoonup)$ and $(S, \rightharpoonup)$ respectively, then

$$\theta(g|h) = \sigma(g_{(1)}|h_{(1)}) \otimes \tau(g_{(2)}|h_{(2)})$$

defines an associator θ for $(R \underset{Z}{\otimes} S, \rightharpoonup)$. One has a natural algebra map $j\colon Z \to R \underset{Z}{\otimes} S$, and it is not hard to check that the obstruction to $(\rightharpoonup, \theta)$ is the composition with j of the product of the obstructions to $(\rightharpoonup, \sigma)$ and $(\rightharpoonup, \tau)$. However, it may happen that the center of $R \underset{Z}{\otimes} S$ is larger than Z, or that j is not injective.

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