On Coquasitriangular Hopf Algebras and the Quantum Yang-Baxter Equation

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Introduction

The present paper investigates the close connections between the quantum Yang-Baxter equation, quasisymmetric monoidal categories and quasitriangular Hopf algebras. The most part of these connections has already been touched upon in various articles ([4], [10], [12], [13]). I think, however, that some aspects of the theory will be made more precise by this paper.

Symmetry groups and quantum groups

Classical mechanics takes place in a (say, differentiable) manifold M, e.g. in $\mathbb{R}^6$, the first three coordinates representing the location, the other three the momentum of a particle. Physical quantities are (say, smooth) functions on this manifold, e. g. the coordinate functions or angular momentum with respect to the origin. In this setting physical laws can be described in two different ways: Either as equations governing the evolution in time of a point in phase space, or as equations describing the evolution of functions on phase space, that is, of observables.

It does not matter if, describing geometry or mechanics, we refer to the manifold itself or to the algebra of functions defined on it. Suitably restricting the classes of algebras, "spaces" and functions, we obtain an equivalence between the category of spaces and the category of commutative algebras or, rather, a duality: Mappings between the spaces correspond to mappings between the algebras going in the opposite direction. One suitable set of restrictions is to C^* algebras, locally compact Hausdorff spaces and continuous complex valued functions vanishing at infinity (Gelfand-Naimark theorem). The simplest possibility from the point of view of the algebras is (almost by definition) the restriction of the spaces to affine schemes: The corresponding class of algebras allows all commutative algebras.

An affine scheme is roughly speaking the algebraic version of a manifold: the set of zeros in k^n of a set of polynomials in n indeterminates, where k is a fixed commutative ground ring, preferably a field. However, an affine scheme contains more information. It takes account not only of the points with coordinates in k but also of the zeros the polynomials in consideration may have in any ring extension R of k. This can be reformulated saying that an affine scheme is a representable functor from the category of commutative k-algebras to the category of sets. For an algebra R, the set of R-points of the scheme has the form $\underline{Alg}(A, R)$ with A a commutative k-algebra. The algebra A is the algebra of "functions" defined on the scheme. The correspondence between affine schemes and commutative algebras is a duality. (Actually affine schemes are almost defined as commutative algebras, more precisely, as their spectra.)

A symmetry group for a physical system is a group G (preferably a Lie group) acting on phase space M by a mapping $m : G \times M \to M$ such that (in a sense that has to be made precise) "physics is invariant" under the action of the group. In classical mechanics, each symmetry of a system implies that a certain physical quantity is preserved during the motion of the system, e.g. translation symmetry of usual Newtonian mechanics implies preservation of momentum. One may say, if we know the symmetry groups of a physical system, then we know the "good", meaning the preserved quantities.

In quantum mechanics and recent particle physics the importance of symmetries of physical systems is even greater than in classical mechanics. Here, however, also the idea arose that symmetry groups may not describe all the reasonable symmetries of a quantum mechanical system. This is one of the motivations for recent research on so called "quantum groups", objects that are (as the name indicates?) *not* groups, but non-commutative non-cocommutative Hopf algebras.

The process of producing a quantum mechanical system out of a classical mechanical system crudely corresponds to making a commutative algebra non-commutative: Location and momentum of a quantum mechanical particle do not commute. The transition to a non-commutative algebra of observables forces us to abandon the classical phase space — for both a physical and a mathematical reason. The physical reason is that, once it is impossible to simultaneously measure location and momentum of a particle, its location in phase space cannot be a reasonable notion any more. Mathematically it is obvious that a non-commutative algebra can never be the algebra of functions on any space. The basic idea of noncommutative geometry is to deal with certain classes of non-commutative algebras still picturing them as being the algebra of functions on a "non-commutative space". Thus, non-commutative spaces are (reasonably restricted) non-commutative algebras, and typical examples are the phase spaces of quantum mechanical systems. There is a quite developed theory of noncommutative geometrical spaces roughly following the scheme: Take a notion or fact referring to manifolds, translate it to the language of the algebra of functions on the manifold, transfer it in a reasonable way to (a suitable class of) non-commutative algebras and investigate it there.

What corresponds to the action of a group manifold on another manifold in the language of the algebras of functions? This can be answered most easily in the case of affine schemes and affine groups.

An affine group scheme is an affine scheme which is a group at the same time, the group operation being described by polynomial mappings between the manifolds. Stated in a different way, an affine group scheme is a representable functor from the category of commutative algebras to the category of groups, that is, a functor $\underline{Alg}(H, —)$ equipped with a functorial group multiplication $\underline{Alg}(H, —) \times \underline{Alg}(H, —) \to \underline{Alg}(H, —)$. Taking the isomorphism $\underline{Alg}(H, —) \times \underline{Alg}(H, —) \cong \underline{Alg}(H \otimes H, —)$ into account, Yoneda's lemma allows to equivalently describe the group multiplication by a "comultiplication" $H \to H \otimes H$ going in the opposite direction. All the group axioms can be translated to commutative diagrams (coassociativity, etc.) involving the comultiplication and certain other structure maps representing the unit and inverses. An algebra equipped with such a set of structure maps (all of them being algebra maps) is called a Hopf algebra. Thus an affine group scheme is represented by, or, more brutally, *is* a commutative Hopf algebra.

An action of an affine group on an affine scheme (described by polynomial mappings) is a functorial action of a representable group functor $\underline{Alg}(H, —)$ on a representable functor $\underline{Alg}(A, —)$. Such an action translates to a comodule structure $A \to A \otimes H$ on the algebra A, more precisely, an H-comodule algebra structure. Thus, the non-commutative-geometric structure corresponding to a group manifold is a non-commutative Hopf algebra (a non-cocommutative one anyway, if the "group" is not meant to be commutative). Consequently, comodule structures over non-commutative Hopf algebras may be hoped to be the non-commutative generalized notion of the action of a group on a space.

Also, one can hope to realize a kind of "Erlangen program" for non-commutative geometry with non-commutative non-cocommutative Hopf algebras, characterizing special types of geometries (say, affine or euclidean) as those theories on more general manifolds whose notions (like parallelism or distance) are invariant under the action of a "symmetry group" of non-commutative geometry.

Representations of braid groups

The n-strand braid group consists of the "essentially different" possibilities to plait a braid from n ropes. Group multiplication is performed by fitting the top of one braid to the bottom of another. I will not try too hard to make this intuitive picture a topological one; a few hints shall suffice: The "top ends" of the "ropes" are "fitted" to a plane in $\mathbb{R}^3$, the "bottom ends" are fitted in the same manner to a plane parallel to the first one[1]. Between the two planes, the ropes are not allowed to "turn back", that is, their axes are to have a non-zero component perpendicular to the two planes everywhere. (Thus, the ropes cannot be knotted in themselves.) Two braids are considered to be the same if they can be skilfully manipulated to look the same; the "manipulations" are encoded mathematically as isotopies of $\mathbb{R}^3$ fixing the two planes pointwise and transforming the two patterns into each other. Apparently the group operation, that is, concatenating braids[2] is compatible with the equivalence relation induced by skilful manipulations. The fact that it is possible to disentangle however complicated braids shows that we have indeed a group.

Generators and relations for the n-strand braid groups have been derived by Artin in 1925. The group B_n is generated by $n-1$ elements $b_1, \ldots, b_{n-1}$ subject to the relations

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \qquad i = 1, \ldots, n-1 \tag{1}$$

$$b_i b_j = b_j b_i \qquad |i-j| > 1. \tag{2}$$

[1] imagine the fitting points arranged in a straight line; however, this does not actually matter.

[2] Formally, of course, we have to rescale to take care that concatenation yields a braid situated between the same two planes.

The generator b_i corresponds to the braid

$$\underbrace{\qquad\qquad\qquad\qquad}_{i-1} \qquad\qquad \underbrace{\qquad\qquad\qquad\qquad}_{n-i-1}$$

The second relation just states that the order in which we plait braids into two disjoint sets of strands does not matter. The first relation gives the following picture (with the unused strands omitted):

We see that these two patterns can indeed easily be transformed into each other. The statement that the two relations given form a complete set of relations for the braid group means that any possible manipulation on a braid can be decomposed as a finite sequence of "moves", each of which replaces one side of the (depicted) relations, found at any position within the braid, with the other side. Strictly speaking, we have to list another two relations, hidden above in the requirement that B_n be a group: The element $b_i^{-1} \in B_n$ corresponds to the braid

and the additional manipulations needed are

In the more general setting of knot theory, such a set of "moves" is known as the Reidemeister moves.

Now let V be a finite dimensional vector space. The symmetric group S_n acts naturally on the n-fold tensor product $V^{\otimes n}$ by "permuting tensorands". This representation is characterized by the action of the transpositions $(i, i+1)$ on $V^{\otimes n}$, which is

$$v_1 \otimes \cdots \otimes v_n \mapsto v_1 \otimes \cdots \otimes v_{i-1} \otimes v_{i+1} \otimes v_i \otimes v_{i+2} \otimes \cdots \otimes v_n.$$

Let $\sigma : V \otimes V \to V \otimes V$ be an isomorphism. On what conditions can we define a representation of the n-strand braid group B_n on $V^{\otimes n}$ such that $b_i \in B_n$ acts on $V^{\otimes n}$ by

$$v_1 \otimes \cdots \otimes v_n \mapsto v_1 \otimes \cdots \otimes v_{i-1} \otimes \sigma(v_i \otimes v_{i+1}) \otimes v_{i+2} \otimes \cdots \otimes v_n? \tag{3}$$

Of course, prescribing (3) for the operation of b_i defines a unique representation of the free group on the generators b_i on $V^{\otimes n}$. We seek a condition on σ ensuring that the isomorphism assigned to a word in the free group depends only on the class of this word in the braid group, that is, ensuring that the operation be compatible with the relations defining the braid group. Relation (2) does not give any condition. The first relation requires commutativity of the hexagonal diagram

$$\begin{tikzcd}
 & V \otimes V \otimes V \arrow[dl, "\sigma \otimes 1"'] \arrow[dr, "1 \otimes \sigma"] & \\
V \otimes V \otimes V \arrow[d, "1 \otimes \sigma"'] & & V \otimes V \otimes V \arrow[d, "\sigma \otimes 1"] \\
V \otimes V \otimes V \arrow[dr, "\sigma \otimes 1"'] & & V \otimes V \otimes V \arrow[dl, "1 \otimes \sigma"] \\
 & V \otimes V \otimes V &
\end{tikzcd}$$

that is, $(\sigma \otimes 1)(1 \otimes \sigma)(\sigma \otimes 1) = (1 \otimes \sigma)(\sigma \otimes 1)(1 \otimes \sigma)$. This condition is called the quantum Yang-Baxter equation for the isomorphism σ.

Let's have another look at the topological meaning of such representations of the braid groups. To give a representation of the braid group as indicated by (3) means to assign to a given braid on n strands an isomorphism $V^{\otimes n} \to V^{\otimes n}$ according to the following directions: Start with the identity on $V^{\otimes n}$. "Put the braid on your desk" and read it from top to bottom, multiplying your result with the isomorphism σ on the corresponding tensorands whenever you meet an overcrossing from right to left, and with σ^{-1} when you meet one from left to right. The fact that this procedure actually gives a representation of B_n means that the resulting isomorphism does not depend on the way we arranged the braid on the desk; stated more sophisticatedly, the isomorphism we obtain is an isotopy invariant of the braid.

So far, everything is fairly dull. Recently, however, new classes of isotopy invariants of knots and more complicated "tangles" have been discovered that actually arise as vector space morphisms assigned to tangles built up in an analogous manner from morphisms assigned to the elementary parts of those tangles (like crossings, turning points, etc.). These knot invariants, the most illustrous one being the Jones polynomial, are still an area of research, and they have helped going a long way further towards the old problem of deciding by some algorithm wether or not two given knots are isotopic.

Representations of quantum groups

The approach to "quantum groups" central for the present paper is already touched upon in a brief remark in [4, p. 812]. To "understand" the structure of quasitriangular Hopf algebras we look at the structures of the category of their modules; the properties of these categories are (hopefully) easier to grasp than those of the Hopf algebra itself. This approach is also suggested by the first section of this introduction, where I tried to motivate quantum groups as the objects typically acting on the spaces of non-commutative geometry, that is, via their representations.

Contrary to the setting of actions of affine group schemes on affine schemes (and also to the setting in this paper) "representation" shall mean in the following a module rather than a comodule.

Let G be a group acting on vector spaces V and W. Then we also have a reasonable action of G on the tensor product $V \otimes W$, an element $g \in G$ just acting simultaneously on each tensorand: $g(v \otimes w) = gv \otimes gw$. Making the group algebra kG a Hopf algebra in the usual manner, comultiplying elements $g \in G$ as $\Delta(g) = g \otimes g \in kG \otimes kG$, we can also describe this action on tensor procucts in terms of the comultiplication. To let an element $x \in kG$ act on $V \otimes W$, comultiply it first, and then let the element $\Delta(x) \in kG \otimes kG$ obtained act on $V \otimes W$ componentwise.

If H is a k-bialgebra, the tensor product (over k) of two left H-modules can be made an H-module in an analogous manner. The well known "diagonal structure" on $V \otimes W$ is given by $h(v \otimes w) = \sum h_{(1)}v \otimes h_{(2)}w$. The nice properties of this structure can be summarized as follows: The category ${}_H\mathcal{M}$ of left H-modules over a bialgebra H is a monoidal category (with a monoidal underlying functor), that is, there is a tensor product (namely, the usual tensor product over k, equipped with the diagonal structure), the triple tensor products $V \otimes (W \otimes X)$ and $(V \otimes W) \otimes X$ are naturally isomorphic, and there is a neutral object I such that $V \otimes I \cong V \cong I \otimes V$.

If H is even a Hopf algebra, then, moreover, the hom set $\mathrm{Hom}_k(V, W)$, for V and W H-modules admits a canonical H-module structure. That this structure is "reasonable" can also be made more precise: The H-modules $\mathrm{Hom}_k(V, W)$ define inner hom-functors for the monoidal category ${}_H\mathcal{M}$, meaning that we have a functorial isomorphism

$$\mathrm{Hom}_H(X \otimes V, W) \cong \mathrm{Hom}_H(X, \mathrm{Hom}_k(V, W))$$

(as for k-modules).

If the bialgebra or Hopf algebra in consideration is cocommutative, the category ${}_H\mathcal{M}$ inherits another property of the category of vector spaces: The canonical isomorphism $\tau : V \otimes W \to W \otimes V, \quad v \otimes w \mapsto w \otimes v, \quad$ is H-linear for any two H-modules V

and W. In this way the monoidal category of H-modules is made a symmetric one. If the bialgebra is not cocommutative, the usual twist on twofold tensor products is not generally H-linear any more. Neither can we expect to find any other reasonable H-linear isomorphism between the tensor products $V \otimes W$ and $W \otimes V$ with different order of tensorands.

Quasitriangular Hopf algebras are "almost cocommutative". More precisely, in a quasitriangular Hopf algebra the comultiplication $\Delta : H \to H \otimes H$ and the twisted comultiplication $\Delta^{\mathrm{cop}} : H \ni h \mapsto \sum h_{(2)} \otimes h_{(1)} \in H \otimes H$ agree up to an inner automorphism of the algebra $H \otimes H$. Thus there is an invertible $R \in H \otimes H$ such that for all $h \in H$ we have $\Delta^{\mathrm{cop}}(h) = R\Delta(h)R^{-1}$. (The element R is required to fulfill two more equations. For triangular Hopf algebras we also require $R^{-1} = \tau R$.) The definition goes back to Drinfeld [4].

Now if H is a *triangular* Hopf algebra, the element R allows to define an H-linear isomorphism between tensor products formed in different orders that makes the monoidal category ${}_H\mathcal{M}$ a symmetric one. Note, however, that (except for cocommutative Hopf algebras) we cannot expect the isomorphism defined to agree with the usual twist on the underlying vector spaces; the usual twist still fails to be H-linear. If the Hopf algebra in consideration is only quasitriangular we have to do with still less: the category of modules is now only "quasisymmetric", the twisting morphism σ does not necessarily satisfy $\sigma^2 = id$.

The isomorphism $\sigma_{VW} : V \otimes W \to W \otimes V$ defined for modules over a quasitriangular Hopf algebra satisfies the quantum Yang-Baxter equation, as we can easily deduce from the properties of quasisymmetric monoidal categories. In particular, the n-fold tensor product of a module over a quasitriangular Hopf algebra admits a natural representation of the n-strand braid group.

Until now, I have tried to sketch the following:

- The category of modules over a quasitriangular Hopf algebra is a quasisymmetric closed monoidal category.
- A quasisymmetric monoidal category together with an underlying functor to the category of vector spaces defines solutions of the quantum Yang-Baxter equation.

The present paper investigates the reverse implications. The starting point was [10]. Lyubashenko sketches the following construction: From each unitary solution of the quantum Yang-Baxter equation we can build a symmetric monoidal category $\mathcal{C}$ consisting of certain subobjects and quotients of direct sums of tensor products of the vector space on which the solution is given. The symmetry of this category is a suitable extension of the given Yang-Baxter operator. He then constructs a Hopf algebra A acting on all the vector spaces in the category, that is, a functor from $\mathcal{C}$ to the category of A-modules. The Hopf algebra A contains "all the information" contained in $\mathcal{C}$ in the sense that this functor is fully faithful. For the same Hopf algebra (up to a twisting of the multiplication or comultiplication) we have that $\mathcal{C}$ is equivalent to the category of finite dimensional A-comodules. Lyubashenko lists properties of A that correspond to the original solution of the quantum Yang-Baxter equation. In particular, he states a kind of self-duality of the Hopf algebra that corresponds to the fact that modules and comodules over A are about the same. (Lyubashenko does not mention the notion of quasitriangularity; however, [4], [14] and [12] cite his paper as a construction of quasitriangular Hopf algebras, but without explaining the connection.)

I have obtained Lyubashenkos construction with more general techniques. This allows, in particular, to state and prove the results of [10] in a coordinate-free form. Moreover, we see how Lyubashenko's results and techniques —that quite obviously make sense only for a special type of monoidal categories— are connected with much more generally applicable techniques. Some papers of Ulbrich ([26], [27]), motivated by [22] and [3], give a general construction of a coalgebra C for each category $\mathcal{C}$ equipped with an "underlying" functor to the category of finite dimensional vector spaces, such that the (underlying vector spaces of the) objects of $\mathcal{C}$ are comodules over C. If, moreover, the category $\mathcal{C}$ is k-linear abelian, and the underlying functor is k-linear, faithful and exact, then $\mathcal{C}$ is even equivalent to the category of comodules over the coalgebra constructed. Properties of the category considered can be translated to properties of the coalgebra: Ulbrich's most far-reaching result is that the coalgebra constructed from a rigid monoidal category is a Hopf algebra. This generalizes results from [22] und [3] that a symmetric monoidal k-linear abelian category with a symmetric monoidal k-linear faithful exact functor to the category of vector spaces is equivalent to the category of representations of an affine group scheme, that is, of a commutative Hopf algebra. In [24] I have reconsidered Ulbrich's constructions. In particular, I have shown that his construction of a coalgebra from a category defines a left adjoint to the functor assigning to a coalgebra the category of its right comodules. I have also redone Ulbrich's reconstruction of bialgebra and Hopf algebra structures from monoidal and rigid monoidal structures of the given category, respectively.

In the second chapter of this paper we extend the reconstruction theorems, which, up to this point, deal with reconstructing coalgebras and morphisms of coalgebras from categories and functors, to natural transformations between the functors involved. The correspondence between the natural transformations and "coelements" of the coalgebra can be formulated as an adjunction of 2-functors extending the adjunction obtained in [24]

It is an immediate consequence of this general adjunction result that the bialgebra constructed from a quasisymmetric monoidal category carries the dual analogue of the structure of a quasitriangular bialgebra. I simply called the type of Hopf algebra thus obtained "coquasitriangular". Thus, coquasitriangular Hopf algebras are "almost commutative", and one could say they are "affine quantum groups" not very different from affine group schemes.

The corresponding reconstruction of a coquasitriangular structure in Lyubashenko's paper depends on the particular form of the category considered and results in only a restricted class of cotriangular Hopf algebras being reconstructed. Thus Drinfeld is not quite right in saying that Lyubashenko shows all the symmetric monoidal categories with underlying functors to the category of vector spaces to arise from quasitriangular Hopf algebras. Majids reconstruction in [12] of a quasitriangular structure on a Hopf algebra whose finite dimensional modules form a quasisymmetric monoidal category is possible, strictly speaking, only if the Hopf algebra is finite dimensional. The same is true of the constructions in [13, p.5:"After some thought it is easy to see..."].

In the third chapter, starting from a not necessarily unitary solution of the quantum Yang-Baxter equation on finite dimensional vector spaces, I construct a quasisymmetric monoidal category with an underlying functor to the category of vector spaces. The construction, which does not agree with Lyubashenko's construction in [10] mentioned above, has a universal property. The techniques from the second chapter then give universal coquasitriangular bialgebras having given solutions of the quantum Yang-Baxter equation as comodules.

Another step in the construction, essentially furnishing the category constructed with as many morphisms, sub- and quotient objects as possible, or (which is actually the same!) making sure that certain actions of the coquasitriangular Hopf algebra on its *co*modules contain as much information as the comodule structures do, finally results in an "almost self-dual" coquasitriangular Hopf algebra, which, in the cotriangular case, agrees with the Hopf algebra constructed by Lyubashenko.

I hope these considerations have put Lyubashenko's work into the context of other results. In particular, the connections to Ulbrich's more general constructions and thus also to the old results on affine group schemes may be interesting, as well as the precise relation between Lyubashenko's Hopf algebras and Drinfeld's quasitriangular Hopf algebras.

We can have one more look at what the reconstruction of a coquasitriangular Hopf algebra from a solution of the quantum Yang-Baxter equation means in the context of the motivations given above. As every solution of the quantum Yang-Baxter equation "provides" a coquasitriangular Hopf algebra from which it arises, one might say that each knot invariant that can be written as a homomorphism between vector spaces "attached" to the ends of the knotted ropes which can be constructed step by step following directions in the sense explained above, can be derived as well from a coquasitriangular Hopf algebra (like in [21]). The result can also be interpreted in the way that the typical symmetry groups of a not too non-commutative geometry is a coquasitriangular Hopf algebra. Not too non-commutative in this sense are algebras whose generators obey commutation relations that can be expressed by saying that rearranging the monomials of a formal polynomial in the generators by a switching operator that fulfills the quantum Yang-Baxter equation does not change the value of the polynomial in the algebra. (Algebras of this type include the examples of quantum planes in [14]). This viewpoint corresponds to the idea from [14, p.82/83] to use a symmetric monoidal category with a (non-symmetric) underlying functor to the category of vector spaces as the "universe for a non-commutative algebraic geometry".

It is a pleasure to thank Prof. Dr. B. Pareigis for suggesting the subject of this paper, and for many a useful hint.

Chapter 1

Monoidal categories

This chapter is devoted to a brief exposition of the category theoretic notions suited to the study of representations of coquasitriangular Hopf algebras. In the first section we recall the results on dual objects in general monoidal categories from [24]

1.1 Dual objects

The following definition captures the nice properties of finitely generated projective modules and their duals over a commutative ring in the general context of monoidal categories.

1.1.1. Definition: *Let X be an object of a monoidal category $\mathcal{C}$.*
Let $X^* \in \operatorname{Ob} C, \operatorname{ev} : X^* \otimes X \to I$.

(X^*, ev) is called a **(left) dual** *for X if there is a morphism $\operatorname{db} : I \to X \otimes X^*$ in $\mathcal{C}$ such that*

$$(X \xrightarrow{\operatorname{db}\otimes 1} X \otimes X^* \otimes X \xrightarrow{1\otimes \operatorname{ev}} X) = 1_X$$

$$(X^* \xrightarrow{1\otimes \operatorname{db}} X^* \otimes X \otimes X^* \xrightarrow{\operatorname{ev}\otimes 1} X^*) = 1_{X^*}$$

$\mathcal{C}$ is said to be **(left) rigid**, *if each object of $\mathcal{C}$ admits a left dual.*

The notions of right dual, right rigid are defined in an analogous manner.

Sometimes we will also call an *object* rigid if it admits a left dual.

1.1.2. Example: *A k-module X is finitely generated projective iff there are elements $x_1, \dots, x_n \in X$ and $x^1, \dots, x^n \in X^* = \operatorname{Hom}_k(X, k)$ such that*

$$\forall x \in X : \quad \sum_i x^i(x) x_i = x.$$

Then also

$$\forall \varphi \in X^* : \quad \sum_i \varphi(x_i) x^i = \varphi.$$

Thus X^* is a left dual for X in the sense of 1.1.1, iff X is finitely generated projective. In this case, the mapping db *required by the definition is given by* $\mathrm{db}(1_k) = \sum_i x_i \otimes x^i$.

In calculations with this generalization of a "basis and dual basis" to finitely generated projective modules we shall mostly adopt the usual summation convention, writing $\mathrm{db}(1_k) = x_i \otimes x^i$, and we will denote the evaluation by brackets $<,>$, like, say, in $< x^i, x > x_i = x$.

1.1.3. Remark: *$(Y, \mathrm{ev} : Y \otimes X \to I)$ is a left dual for X iff (X, ev) is a right dual for Y.*

1.1.4. Corollary: *If (X^*, ev) is a left dual of X, then the morphism* db *occuring in the definition is unique.*

1.1.5. Corollary: *If X admits a left dual, this is unique in the sense that for any two left duals (X^*, ev) and (D, e) for X there is a unique isomorphism $j : D \to X^*$ making the diagrams*

$$\begin{tikzcd} D \otimes X \arrow[rr, "j \otimes 1"] \arrow[dr, "e"'] & & X^* \otimes X \arrow[dl, "\mathrm{ev}"] \\ & I & \end{tikzcd}$$

commute, namely

$$D \xrightarrow{1 \otimes \mathrm{db}} D \otimes X \otimes X^* \xrightarrow{e \otimes 1} X^*$$

1.1.6. Corollary: *Let $\mathcal{C}$ be rigid and choose a dual X^* for each object X in $\mathcal{C}$. Then $(\,—\,)^* : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ defines a functor.*

More explicitely, for $f : X \to X'$ the morphism f^* will be characterized by commutativity

$$\begin{tikzcd} (X')^* \otimes X \arrow[r, "1 \otimes f"] \arrow[d, "f^* \otimes 1"'] & (X')^* \otimes X' \arrow[d, "\mathrm{ev}"] \\ X^* \otimes X \arrow[r, "\mathrm{ev}"'] & I \end{tikzcd}$$

and given by

$$f^* = (\ (X')^* \xrightarrow{1 \otimes \mathrm{db}} (X')^* \otimes X \otimes X^* \xrightarrow{1 \otimes f \otimes 1} (X')^* \otimes X' \otimes X^* \xrightarrow{\mathrm{ev} \otimes 1} X^*)$$

1.1.7. Lemma: *Let* $(\mathcal{F},\xi) : \mathcal{C} \to \tilde{\mathcal{C}}$ *be a monoidal functor. Let* $X \in \mathrm{Ob}\, C$ *have* (X^*, ev) *as a left dual. Then* $(\mathcal{F}(X^*), \widetilde{\mathrm{ev}})$ *is a left dual for* $\mathcal{F}X$*, if we define*

$$\widetilde{\mathrm{ev}} = (\mathcal{F}X^* \otimes \mathcal{F}X \xrightarrow{\xi} \mathcal{F}(X^* \otimes X) \xrightarrow{\mathcal{F}(\mathrm{ev})} I)$$

1.1.8. Corollary: *Let* $(\mathcal{F},\xi)$ *be as in 1.1.7, and let both* $\mathcal{C}$ *and* $\tilde{\mathcal{C}}$ *be rigid with a chosen functor* $(\,—\,)^*$*. Then there is a unique functorial isomorphism* $u : \mathcal{F}(X^*) \to \mathcal{F}(X)^*$ *satisfying*

$$\begin{tikzcd}
\mathcal{F}(X^*) \otimes \mathcal{F}(X) \arrow[r, "u \otimes 1"] \arrow[d, "\xi"'] & \mathcal{F}(X)^* \otimes \mathcal{F}(X) \arrow[d, "\mathrm{ev}"] \\
\mathcal{F}(X^* \otimes X) \arrow[r, "\mathcal{F}(\mathrm{ev})"'] & I
\end{tikzcd}$$

namely

$$\mathcal{F}(X^*) \xrightarrow{1\otimes \mathrm{db}} \mathcal{F}(X^*) \otimes \mathcal{F}(X) \otimes \mathcal{F}(X)^* \xrightarrow{\xi\otimes 1} \mathcal{F}(X^* \otimes X) \otimes \mathcal{F}(X) \xrightarrow{\mathcal{F}(\mathrm{ev})\otimes 1} \mathcal{F}(X)$$

1.1.9. Remark: *Let* $\mathcal{C} = (\mathcal{C}_0, \otimes, \alpha, I, \lambda, \rho)$ *be a monoidal category. Let* X *and* Y *be objects of* $\mathcal{C}$ *having* (X^*, ev_X) *and* (Y^*, ev_Y) *as left duals, respectively.*

Then $(Y^* \otimes X^*, ev_{X\otimes Y})$ *is a left dual for* $X \otimes Y$*, with*

$$\mathrm{ev}_{X\otimes Y} : Y^* \otimes X^* \otimes X \otimes Y \xrightarrow{1\otimes\mathrm{ev}\otimes 1} Y^* \otimes Y \xrightarrow{\mathrm{ev}} I$$

$$\mathrm{db}_{X\otimes Y} : I \xrightarrow{\mathrm{db}} X \otimes X^* \xrightarrow{1\otimes\mathrm{db}\otimes 1} X \otimes Y \otimes Y^* \otimes X^*$$

Also, if $\mathcal{C}$ *is abelian monoidal and* X_1, X_2 *are objects of* $\mathcal{C}$ *with left duals* $(X_i^*, \mathrm{ev}_{X_i})$*, then* $(X_1^* \oplus X_2^*, \mathrm{ev}_{X_1\oplus X_2})$ *is a left dual for* $X_1 \oplus X_2$*.*

1.1.10. Corollary: *Let* $\mathcal{C} = (\mathcal{C}_0, \otimes, \alpha, I, \lambda, \rho)$ *be rigid with chosen left duals. Then there is a unique natural isomorphism* $\nu : (X \otimes Y)^* \cong Y^* \otimes X^*$ *satisfying*

$$\begin{tikzcd}
Y^* \otimes X^* \otimes X \otimes Y \arrow[r, "\nu \otimes id"] \arrow[d, "1 \otimes \mathrm{ev} \otimes 1"'] & (X \otimes Y)^* \otimes (X \otimes Y) \arrow[d, "\mathrm{ev}"] \\
Y^* \otimes Y \arrow[r, "\mathrm{ev}"'] & I
\end{tikzcd}$$

Moreover, if $\mathcal{C}$ *is abelian monoidal, then for* $X_1, X_2 \in \mathcal{C}$ *the object* $(X_1 \oplus X_2)^*$ *is a biproduct of* X_1 *and* X_2 *with projections and injections dual to the injections and projections of* $X_1 \oplus X_2$*, respectively.*

1.1.11. Corollary: *If $\mathcal{C}$ is rigid, then the functor $(\,—\,)^*$ is a monoidal functor from $\mathcal{C}$ to $\mathcal{C}^{\mathrm{sym\,op}}$. Moreover, if $\mathcal{C}$ is abelian monoidal, then $(\,—\,)^*$ is additive.*

The following definition generalizes the process $f \mapsto f^*$ of dualizing morphisms in a rigid monoidal category. A morphism $S : V \otimes W \to X \otimes Y$ gives a "halfway dualized" morphism $S^\flat : X^* \otimes V \to Y \otimes W^*$.

1.1.12. Lemma and Definition: *Let $\mathcal{C}$ be a monoidal category, let $V, W, X, Y \in \operatorname{Ob}\mathcal{C}$.*
Let (X^*, ev_X) and (Y^*, ev_Y) be left duals for X and W, respectively.
The following conditions on morphisms $F : V \otimes W \to X \otimes Y$ and $G : X^* \otimes V \to Y \otimes W^*$ are equivalent:

$$\begin{tikzcd}
X^* \otimes V \otimes W \arrow[r, "1 \otimes F"] \arrow[d, "G \otimes 1"'] & X^* \otimes X \otimes Y \arrow[d, "\mathrm{ev} \otimes 1"] \\
V \otimes W^* \otimes W \arrow[r, "1 \otimes \mathrm{ev}"'] & Y
\end{tikzcd} \tag{1}$$

$$\begin{tikzcd}
V \arrow[r, "1 \otimes \mathrm{db}"] \arrow[d, "\mathrm{db} \otimes 1"'] & V \otimes W \otimes W^* \arrow[d, "F \otimes 1"] \\
X \otimes X^* \otimes V \arrow[r, "1 \otimes G"'] & X \otimes V \otimes W^*
\end{tikzcd} \tag{2}$$

Furthermore, for each $F : V \otimes W \to X \otimes Y$ there is a unique $G : X^* \otimes V \to Y \otimes W^*$ fulfilling one (and thus both) of the two.

We define $F^\flat := G$ and $G^\sharp := F$.

We have the following formulas for the "operations" $(\,—\,)^\flat$ and $(\,—\,)^*$.

1.1.13. Remark: *We have seen that if X_1, X_2 and Y_1, Y_2 have X_i^* and Y_i^* as left duals, then we can choose $(X_i \otimes Y_i)^* := Y_i^* \otimes X_i^*$ as a left dual for $X_i \otimes Y_i$. With this choice, we get $(S^\flat)^\flat = S^*$ for a morphism $S : X_1 \otimes Y_1 \to X_2 \otimes Y_2$.*

1.1.14. Lemma: *Let $S : X \otimes Y \to V \otimes W$, $f : \tilde{X} \to X$, $g : \tilde{Y} \to Y$, $h : V \to \tilde{V}$ and $j : X \to \tilde{W}$ be morphisms in $\mathcal{C}$.*
Then (if all the dual objects occurring in the respective statements exist) we have:

$$\begin{aligned}
(S(f \otimes 1))^\flat &= S^\flat(1 \otimes f) && (1)\\
(S(1 \otimes g))^\flat &= (1 \otimes g^*)S^\flat && (2)\\
((h \otimes 1)S)^\flat &= S^\flat(h^* \otimes 1) && (3)\\
((1 \otimes j)S)^\flat &= (j \otimes 1)S^\flat && (4)
\end{aligned}$$

1.1.15. Lemma: *Let C, Z and V be objects of a monoidal category $\mathcal{C}$, that admit left duals. Let $S : A \otimes V \to Z \otimes X$ and $T : B \otimes C \to V \otimes Y$ be morphisms in $\mathcal{C}$. Then the morphisms $S^\flat : Z^* \otimes A \to X \otimes V^*$, $T^\flat : V^* \otimes B \to Y \otimes C^*$ and $((S \otimes 1)(1 \otimes T))^\flat : Z^* \otimes A \otimes B \to X \otimes Y \otimes C^*$ satisfy*

$$((S \otimes 1)(1 \otimes T))^\flat = (1 \otimes T^\flat)(S^\flat \otimes 1)$$

1.2 Quasisymmetric monoidal categories

For two objects X, Y in a general monoidal category the tensor products $X \otimes Y$ and $Y \otimes X$ are usually not isomorphic.

The definition of a quasisymmetric monoidal category requires the existence of a "reasonable" isomorphism between these two objects.

1.2.1. Definition: *Let $\mathcal{C} = (\mathcal{C}, \otimes, \alpha, I, \lambda, \rho)$ be a monoidal category.*
Let $\sigma : X \otimes Y \to Y \otimes X$ be a bifunctorial isomorphism.
$(\mathcal{C}, \sigma)$ is said to be a **quasisymmetric monoidal category***, σ is called a* **quasisymmetry***, if the diagram*

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "\sigma"] \arrow[d, "\alpha"'] & (Y \otimes Z) \otimes X & Y \otimes (Z \otimes X) \arrow[l, "\alpha"'] \\
(X \otimes Y) \otimes Z \arrow[r, "\sigma \otimes 1"'] & (Y \otimes X) \otimes Z & Y \otimes (X \otimes Z) \arrow[l, "\alpha"] \arrow[u, "1 \otimes \sigma"']
\end{tikzcd}$$

and the analogous diagram with $(\sigma_{BA})^{-1}$ in place of σ_{AB} commute.

σ is called a symmetry and $(\mathcal{C}, \sigma)$ is called symmetric, if in addition $\sigma^{-1}_{X,Y} = \sigma_{Y,X}$.

1.2.2. Remark: *Taking advantage of the coherence results for monoidal categories we can give the two diagrams the shorthand form*

$$\begin{tikzcd}
X \otimes Y \otimes Z \arrow[rr, "\sigma_{X,Y\otimes Z}"] \arrow[dr, "\sigma_{X,Y} \otimes 1"'] & & Y \otimes Z \otimes X \\
& Y \otimes X \otimes Z \arrow[ur, "1 \otimes \sigma_{X,Z}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes Y \otimes Z \arrow[rr, "\sigma_{X\otimes Y,Z}"] \arrow[dr, "1 \otimes \sigma_{Y,Z}"'] & & Z \otimes X \otimes Y \\
& X \otimes Z \otimes Y \arrow[ur, "\sigma_{X,Z} \otimes 1"'] &
\end{tikzcd}$$

Quasisymmetric monoidal categories are the generalization of symmetric ones designed to fit the study of quasitriangular (respectively, coquasitriangular) Hopf algebras, [12, p.71]. While for *symmetric* monoidal categories we have coherence theorems very similar to the ones valid for monoidal categories (all diagrams reasonably composed of the monoidal structure and the symmetry commute), coherence theorems for the quasisymmetric case are bound to be more restricted. Indeed, generally the morphisms $id_{X\otimes Y}$ and $\sigma_{YX}\sigma_{XY}$ fail to agree.

Restricted versions of quasisymmetric monoidal categories have been used, [6], to investigate knot and "tangle" invariants of Jones type.

1.2.3. Example: *Let H be a cocommutative k-bialgebra. Then the category of H-modules is symmetric. The symmetry τ for two H-modules M and N is given by*

$$\tau_{MN}(m \otimes n) = n \otimes m$$

on $m \in M, n \in N$.

The definition 1.2.1 imposes a certain compatibility condition on the quasisymmetry and the associativity constraint of the monoidal category. It is not necessary to seperately require compatibility with the unit object ([22, 2.4.1, p.39] in the symmetric case):

1.2.4. Lemma: *If $(\mathcal{C}, \sigma)$ is quasisymmetric, then σ is compatible with the morphisms λ, ρ associated with the unit object I of $\mathcal{C}$ in the sense that*

$$\begin{tikzcd}
X \otimes I \arrow[rr, "\sigma"] \arrow[dr, "\rho"'] & & I \otimes X \arrow[dl, "\lambda"] \\
& X &
\end{tikzcd}$$

and the same diagram for σ^{-1} commute.

Proof: In the following diagram

$$\begin{tikzcd}
& I \otimes (X \otimes I) \arrow[rr, "1 \otimes \sigma"] \arrow[dr, "1 \otimes \rho"] \arrow[ddl, "\alpha"'] & & I \otimes (I \otimes X) \arrow[dl, "1 \otimes \lambda"'] \arrow[ddr, "\alpha"] & \\
& & I \otimes X & & \\
(I \otimes X) \otimes I \arrow[urr, "\rho"] \arrow[ddr, "\sigma \otimes 1"'] & & & & (I \otimes I) \otimes X \arrow[ull, "\rho \otimes 1"'] \\
& & X \otimes I \arrow[uu, "\sigma"'] & & \\
& (X \otimes I) \otimes I \arrow[ur, "\rho"] & & X \otimes (I \otimes I) \arrow[ll, "\alpha"] \arrow[ul, "1 \otimes \rho"'] \arrow[uur, "\sigma"'] &
\end{tikzcd}$$

the uppermost triangle is our assertion, tensored with I on the left. All the other triangles in the diagram commute by compatibility of the unit object with the associativity constraint. The quadrangle on the left commutes by functoriality of ρ, the one on the right by functoriality of σ. The outer hexagon is a special case of the definition of quasisymmetry.

Since all the morphisms occuring are isomorphism, we are done. $\square$

Taken together with the compatibility of σ with tensor products expressed in the definition, the preceding lemma proves:

1.2.5. Lemma: *If σ is a quasisymmetry for $(\mathcal{C}, \otimes, \alpha)$, then the pair $(\mathcal{I}d_{\mathcal{C}}, \sigma)$ defines a monoidal equivalence $\mathcal{C}^{\mathrm{sym}} \to \mathcal{C}$.*

PROOF: The second part of the definition of a monoidal functor is a direct consequence of the lemma just proved. The first part requires

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "\sigma"] \arrow[d, "\alpha"'] & X \otimes^{\mathrm{sym}} (Y \otimes Z) \arrow[r, "1 \otimes^{\mathrm{sym}} \sigma"] & X \otimes^{\mathrm{sym}} (Y \otimes^{\mathrm{sym}} Z) \arrow[d, "\alpha^{\mathrm{sym}}"] \\
(X \otimes Y) \otimes Z \arrow[r, "\sigma"'] & (X \otimes Y) \otimes^{\mathrm{sym}} Z \arrow[r, "\sigma \otimes^{\mathrm{sym}} 1"'] & (X \otimes^{\mathrm{sym}} Y) \otimes^{\mathrm{sym}} Z
\end{tikzcd}$$

to commute. This in turn means commutativity of the outer rectangle of the following diagram:

$$\begin{tikzcd}
X \otimes Y \otimes Z \arrow[rr, "\sigma_{X,Y \otimes Z}"] \arrow[dr, "\sigma_{XY} \otimes 1"] \arrow[dd, "\sigma_{X \otimes Y, Z}"'] & & Y \otimes Z \otimes X \arrow[dd, "\sigma_{YZ} \otimes 1"] \\
& Y \otimes X \otimes Z \arrow[ur, "1 \otimes \sigma_{XZ}"'] \arrow[dr, "\sigma_{Y \otimes X, Z}"'] & \\
Z \otimes X \otimes Y \arrow[rr, "1 \otimes \sigma_{XY}"'] & & Z \otimes Y \otimes X
\end{tikzcd}$$

Now

- The "triangular quadrangle" bottom left commutes by functoriality of σ,
- The upper triangle and the one on the right commute by the definition of quasisymmetry (and the preceding remark).

$\square$

1.2.6. Definition: *A monoidal functor $(\mathcal{F}, \xi)$ between two quasisymmetric monoidal categories $(\mathcal{C}, \sigma)$ and $(\mathcal{D}, \gamma)$ is said to be compatible with the quasisymmetry or just quasisymmetric, if the diagrams*

$$\begin{tikzcd}
\mathcal{F}X \otimes \mathcal{F}Y \arrow[r, "\xi_{XY}"] \arrow[d, "\gamma_{\mathcal{F}X,\mathcal{F},Y}"'] & \mathcal{F}(X \otimes Y) \arrow[d, "\mathcal{F}\sigma_{XY}"] \\
\mathcal{F}Y \otimes \mathcal{F}X \arrow[r, "\xi_{YX}"'] & \mathcal{F}(Y \otimes X)
\end{tikzcd}$$

commute.

In the third chapter we will have to construct certain quasisymmetric monoidal categories. In particular we will have to make up a quasisymmetry for all of a category, given only the quasisymmetry for a "generating set" of objects. The definition of a quasisymmetry already leaves no choice for $\sigma_{X_1 \otimes X_2, Z}$ and $\sigma_{Z, X_1 \otimes X_2}$ when we already know $\sigma_{X_i, Z}$ and σ_{Z, X_i}. It is easy to see that fixing σ_{XZ} and σ_{ZX} also fixes $\sigma_{X^*, Z}$ and σ_{Z, X^*}. The morphisms σ that involve X^* can be expressed by those involving X via the operation $(—)^\flat$ from 1.1.12.

1.2.7. Lemma: *Let $(\mathcal{C}, \sigma)$ be quasisymmetric. Let $X \in \operatorname{Ob}\mathcal{C}$ have a left dual. Then for all $Y \in \operatorname{Ob}\mathcal{C}$ we have:*

$$\begin{aligned}
\sigma_{X^*, Y} &= (\sigma^{-1}_{X,Y})^\flat \\
\sigma_{Y, X^*} &= (\sigma^\flat_{Y,X})^{-1}
\end{aligned}$$

PROOF: From

$$\begin{tikzcd}
X^* \otimes X \otimes Y \arrow[r, "\sigma_{X^* \otimes X, Y}"] \arrow[d, "\mathrm{ev} \otimes 1"'] & Y \otimes X^* \otimes X \arrow[d, "1 \otimes \mathrm{ev}"] \\
I \otimes Y \arrow[r, "\sigma_{I,Y}"'] & Y \otimes I
\end{tikzcd}$$

(which commutes by functoriality of σ), the diagram

$$\begin{tikzcd}
X^* \otimes X \otimes Y \arrow[rr, "\sigma_{X^* \otimes X, Y}"] \arrow[dr, "1 \otimes \sigma_{XY}"'] & & Y \otimes X^* \otimes X \\
& X^* \otimes Y \otimes X \arrow[ur, "\sigma_{X^*, Y} \otimes 1"'] &
\end{tikzcd}$$

and lemma 1.2.4 we deduce commutativity of

$$
\begin{tikzcd}
X^* \otimes Y \otimes X \arrow[d, "\sigma_{X^*,Y} \otimes 1"'] & X^* \otimes X \otimes Y \arrow[l, "1 \otimes \sigma_{XY}"'] \arrow[d, "\mathrm{ev} \otimes 1"] \\
Y \otimes X^* \otimes X \arrow[r, "1 \otimes \mathrm{ev}"'] & Y
\end{tikzcd},
$$

which proves the first equation by definition of $(\text{—})^\flat$.

The other equation is obtained in a similar manner. $\square$

1.2.8. Corollary: *Let $\mathcal{C}$ and $\mathcal{D}$ be two quasisymmetric monoidal categories. Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a monoidal functor.*
To prove that $\mathcal{F}$ is quasisymmetric, it is sufficient to check commutativity of the diagram in definition 1.2.6 for all the objects X, Y taken from a set M such that all the objects of $\mathcal{C}$ are finite tensor products of elements of M and left duals of those.

In a general monoidal category we cannot expect that an object that has a left dual also has a right dual. If both a left and a right dual exist, they need not be isomorphic. In particular we do not know wether an object that has a left dual also has a bidual (a left dual of the left dual), and even if the category in consideration is rigid, an object need not be isomorphic to his bidual. A counterexample is provided by the known (though highly non-trivial) examples of Hopf algebras with non-injective antipodes, whose finite dimensional comodules form a left, but not right rigid category. (That this category is indeed not right rigid follows from the results of [24, sec. 2.4].)

By 1.2.5 a quasisymmetry gives a monoidal equivalence of a monoidal category to its symmetric category. Since this functor is compatible with left duals by 1.1.8, and since left duals in $\mathcal{C}^{\mathrm{sym}}$ are evidently right duals in $\mathcal{C}$, this equivalence provides us with an isomorphism of left duals with right duals.

More precisely, let the object X in a quasisymmetric monoidal category $\mathcal{C}$ have a left dual (X^*, ev). Then, by 1.1.8, $(X^*, \widetilde{\mathrm{ve}})$ with $\widetilde{\mathrm{ve}} : X \otimes X^* \xrightarrow{\sigma^{-1}} X^* \otimes X \xrightarrow{\mathrm{ev}} I$ is a left dual for X in $\mathcal{C}^{\mathrm{sym}}$, that is, a right dual for X in $\mathcal{C}$, and the diagram

$$
\begin{tikzcd}
X^* \otimes X \arrow[r, "1 \otimes 1"] \arrow[d, "\sigma"'] & X^* \otimes X \arrow[d, "\mathrm{ev}"] \\
X \otimes X^* \arrow[r, "\widetilde{\mathrm{ve}}"'] & I
\end{tikzcd}
$$

commutes trivially.

If $({}^*X, \mathrm{ve})$ is another right dual, then there is a unique isomorphism $j : {}^*X \to X^*$ (functorial in the case of a rigid category) such that

$$\begin{tikzcd}
X \otimes {}^*X \arrow[rr, "1 \otimes j"] \arrow[dr, "\mathrm{ve}"'] & & X \otimes X^* \arrow[dl, "\widetilde{\mathrm{ve}}"] \\
& I &
\end{tikzcd}$$

commutes, namely

$${}^*X \xrightarrow{1\otimes\widetilde{\mathrm{bd}}\otimes 1} X^* \otimes X \otimes {}^*X \xrightarrow{1\otimes \mathrm{ve}} X^*$$

where $\widetilde{\mathrm{bd}}$ is the "dual basis" morphism associated with $\widetilde{\mathrm{ve}}$.

Taking the symmetric reasoning into account, we have proved:

1.2.9. Lemma: *Let $(\mathcal{C}, \sigma)$ be a quasisymmetric monoidal category, let $X \in \mathrm{Ob}\,\mathcal{C}$.*

X has a left dual iff X has a right dual.

If (X^*, ev) and $({}^*X, \mathrm{ve})$ are a left and a right dual for X, respectively, then there is a unique isomorphism $j : {}^*X \to X^*$ satisfying

$$\begin{tikzcd}
{}^*X \otimes X \arrow[r, "j \otimes 1"] \arrow[d, "\sigma"'] & X^* \otimes X \arrow[d, "\mathrm{ev}"] \\
X \otimes {}^*X \arrow[r, "\mathrm{ve}"'] & I
\end{tikzcd}$$

namely (apply 1.1.5 to right instead of left duals):

$$j = ({}^*X \xrightarrow{\mathrm{db}\otimes 1} X \otimes X^* \otimes {}^*X \xrightarrow{\sigma\otimes 1} X^* \otimes X \otimes {}^*X \xrightarrow{1\otimes \mathrm{ve}} X^*)$$

If, moreover, $\mathcal{C}$ is left or (equivalently) right rigid, j is functorial.

1.2.10. Corollary: *Let $(\mathcal{C}, \sigma)$ be a quasisymmetric monoidal category, let $X \in \mathrm{Ob}\,\mathcal{C}$.*

If X has a left dual (X^*, ev_X), then X^* has a left dual $(X^{**}, \mathrm{ev}_{X^*})$ as well, and there is a unique isomorphism $j : X \to X^{**}$ satisfying

$$\begin{tikzcd}
X \otimes X^* \arrow[r, "j \otimes 1"] \arrow[d, "\sigma"'] & X^{**} \otimes X^* \arrow[d, "\mathrm{ev}_{X^*}"] \\
X^* \otimes X \arrow[r, "\mathrm{ev}_X"'] & I
\end{tikzcd}$$

namely

$$j = (X \xrightarrow{\mathrm{db}_{X^*}\otimes 1} X^* \otimes X^{**} \otimes X \xrightarrow{\sigma\otimes 1} X^{**} \otimes X^* \otimes X \xrightarrow{1\otimes \mathrm{ev}} X^{**})$$

If, moreover, $\mathcal{C}$ is rigid, then j is functorial.

PROOF: X is a right dual for X^*. □

1.3 2-categories

I will only present some definitions and basic facts from the theory of 2-categories, without going into the details.

A 2-category is a category enriched over the cartesian closed category <u>*Cat*</u> of all categories. Thus

1.3.1. Definition: *A 2-category $\mathcal{C}$ consists of*

- *a class* $\mathrm{Ob}\,\mathcal{C}$ *of objects.*
- *a family* $(\mathcal{M}or_{\mathcal{C}}(X,Y)|X,Y\in \mathrm{Ob}\,\mathcal{C})$ *of categories.*
- *a family* $(M_{X,Y,Z}:\mathcal{M}or(Y,Z)\times\mathcal{M}or(X,Y)\to\mathcal{M}or(X,Z)|X,Y,Z\in\mathrm{Ob}\,\mathcal{C})$ *of functors such that*

  $$M_{W,Y,Z}(\mathcal{I}d\times M_{W,X,Y})=M_{W,X,Z}(M_{X,Y,Z}\times\mathcal{I}d)$$

  holds and there is an object id_X *of* $\mathcal{M}or(X,X)$ *such that*

  $$M_{X,X,Z}(\,—\,,id_X)\qquad and\qquad M_{W,X,X}(id_X,\,—\,)$$

  are identity functors.

We fix some terminology: The objects of the $\mathcal{M}or$-categories are called 1-morphisms or 1-cells. The morphisms of these categories are called 2-morphisms or 2-cells. One may call the objects of $\mathcal{C}$ 0-cells.

We denote the set of 2-cells between two 1-cells f and g by $\mathfrak{mor}(f,g)$.

Composition within the $\mathcal{M}or$ categories is called *weak* composition and denoted $\varphi\cdot\psi$. Composition via the functors $M_{X,Y,Z}$ is called strong composition and denoted by a circle $f\circ g$, etc. Note that it is perfectly possible to compose (suitably chosen) 2-cells with 1-cells or other 2-cells by strong composition.

The fundamental role the category <u>*Set*</u> of all sets plays in category theory is played by <u>*Cat*</u> for the theory of 2-categories. Indeed <u>*Cat*</u> is a 2-category with

- categories as 0-cells,
- functors as 1-cells,
- natural transformations as 2-cells.

weak composition is the usual composition of natural transformations. Strong composition of 2-cells $\varphi:\mathcal{F}\to\mathcal{G}:\mathcal{B}\to\mathcal{C}$ and $\psi:\mathcal{H}\to\mathcal{J}:\mathcal{C}\to\mathcal{D}$ is given by

$$\psi\circ\varphi:=(\psi\mathcal{G})(\mathcal{H}\varphi)=(\mathcal{J}\varphi)(\psi\mathcal{F}):\mathcal{H}\mathcal{F}\to\mathcal{J}\mathcal{G}.$$

Strong composition of 2-cells and 1-cells is just

$$\psi\circ\mathcal{F}=\psi\mathcal{F}\qquad\mathcal{H}\circ\varphi=\mathcal{H}\varphi$$

The definition of a 2-functor is pretty obvious:

1.3.2. Definition: *A 2-functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ consists of a mapping $\mathcal{F} : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{D}$ and functors $\mathcal{F} : \mathcal{M}or(X,Y) \to \mathcal{M}or(\mathcal{F}X, \mathcal{F}Y)$ compatible with the composition functors and unit objects.*

Natural transformations of 2-functors (or natural 1-cells etc.) have to be compatible with both 1-cells and 2-cells.

1.3.3. Definition: *A natural 1-cell $f : \mathcal{F} \to \mathcal{G}$ is a family of 1-cells $f_X : \mathcal{F}(X) \to \mathcal{G}(X)$ such that*

$$f_Y \circ \mathcal{F}h = \mathcal{G}h \circ f_X$$

$$f_Y \circ \mathcal{F}\varphi = \mathcal{G}\varphi \circ f_X$$

holds for all 1-cells $h, h' : X \to Y$ and 2-cells $\varphi : h \to h'$. (That is, such that the functors

$$f_Y \circ \mathcal{F}(\,—\,),\ \mathcal{G}(\,—\,) \circ f_X : \mathcal{M}or(X,Y) \to \mathcal{M}or(\mathcal{F}X, \mathcal{G}Y)$$

are the same.)

Of course, there is also a notion of "natural 2-cells" between natural 1-cells, which are called modifications.

1.3.4. Definition: *A modification $\alpha : f \to g$ between natural 1-cells $f, g : \mathcal{F} \to \mathcal{G}$ is a family of 2-cells $\alpha_X : f_x \to g_X$ such that*

$$\alpha_Y \circ \mathcal{F}h = \mathcal{G}h \circ \alpha_X$$

$$\alpha_Y \circ \mathcal{F}\varphi = \mathcal{G}\varphi \circ \alpha_X$$

for all 1-cells and 2-cells as above.

With the natural transformations as 1-cells and the modifications as 2-cells $\mathcal{F}unct(\mathcal{C}, \mathcal{D})$ becomes a 2-category.

A 2-version of Yoneda's lemma gives isomorphisms of categories

$$\mathrm{Nat}(\mathcal{M}or(X, —), \mathcal{F}) \cong \mathcal{F}X \quad \text{for } \mathcal{F} : \mathcal{C} \to \underline{Cat}$$

$$\mathrm{Nat}(\mathcal{M}or(—, X), \mathcal{F}) \cong \mathcal{F}X \quad \text{for } \mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \underline{Cat}$$

(where $(\,—\,)^{\mathrm{op}}$ means reverting 1-cells, but not 2-cells.) As a consequence $X \mapsto \mathcal{M}or(X, —)$ and $X \mapsto \mathcal{M}or(—, X)$ define fully faithful (with respect to both 1-cells and 2-cells) functors

$$\mathcal{C} \to \mathcal{F}unct(\mathcal{C}^{\mathrm{op}}, \underline{Cat}) \quad \text{and} \quad \mathcal{C}^{\mathrm{op}} \to \mathcal{F}unct(\mathcal{C}, \underline{Cat})$$

A natural definition of a pair of adjoint functors is

1.3.5. Definition: *A 2-adjunction consists of 2-functors $\mathcal{F} : \mathcal{B} \to \mathcal{C}$ and $\mathcal{G} : \mathcal{C} \to \mathcal{B}$ and an isomorphism of categories, natural in X and Y:*

$$\mathcal{M}or(\mathcal{F}X, Y) \cong \mathcal{M}or(X, \mathcal{G}Y)$$

Of course, we call $\mathcal{F}$ a left adjoint for $\mathcal{G}$, etc. Once we have believed in Yoneda's lemma for 2-categories to hold, it is easy to believe or establish the usual facts on adjunctions in the 2-case.

1.3.6. Lemma: *Let $\mathcal{G} : \mathcal{C} \to \mathcal{B}$ be a 2-functor. The following sets of data are in bijective correspondence:*

(1) A 2-adjunction in which $\mathcal{G}$ is the right adjoint.

(2) A functor $\mathcal{F} : \mathcal{B} \to \mathcal{C}$ and natural 1-cells $U : \mathcal{I}d \to \mathcal{GF}$ and $C : \mathcal{FG} \to id$ such that the usual triangle equations hold.

(3) a mapping $\mathcal{F}_0 : \mathrm{Ob}\,\mathcal{B} \to \mathrm{Ob}\,\mathcal{C}$ and for each object X of $\mathcal{B}$ an isomorphism, natural in Y

$$\mathcal{M}or(\mathcal{F}_0 X, Y) \cong \mathcal{M}or(X, \mathcal{G}Y)$$

Note that, since our $\mathcal{M}or$-sets now become categories, we might want to weaken the adjointness conditions by replacing the isomorphisms between morphism categories with mere equivalences.

1.3.7. Definition: *A weak 2-adjunction between 2-functors $\mathcal{F}$ and $\mathcal{G}$ as above consists of an equivalence adjunction*

$$\mathcal{M}or(\mathcal{F}X, Y) \to \mathcal{M}or(X, \mathcal{G}Y),$$

natural in X and Y, such that the adjointness isomorphisms are modifications.

Even though we will actually encounter an example of such a weak adjointness, we shall not go into the fairly complicated and non-obvious theory of weakened adjunctions. A reference, wich is also a general reference on 2-category theory, is [7].

We define a monoidal 2-category to be the same as a monoidal category, except that we require $\otimes$ to be a 2-functor and α, λ and ρ to be natural transformations of 2-functors.

Chapter 2

Coquasitriangular Hopf algebras and other coquasi-notions

We assume given some symmetric monoidal category $\mathcal{A}$ (with unit object k) which is cocomplete, such that the tensor product commutes with arbitrary colimits. We denote by $\mathcal{A}_0$ the full subcategory of $\mathcal{A}$ consisting of the objects having left duals.

2.1 Results from a previous paper

We recall for reference some of the results obtained in [24].

The main results center on the possibility of reconstructing coalgebras (bialgebras, Hopf algebras) from the categories of their comodules (and the underlying functor of those categories to the category of, say, finite dimensional vector spaces.

2.1.1. Definition: *The category $\mathfrak{C} = \mathfrak{C}(\mathcal{A})$ has objects the pairs $(\mathcal{C}, \omega)$, consisting of a category $\mathcal{C}$ and a functor $\omega : \mathcal{C} \to \mathcal{A}_0$. Morphisms in $\mathfrak{C}$ from $(\mathcal{C}, \omega)$ to $(\mathcal{C}', \omega')$ are equivalence classes $[\mathcal{F}, \zeta]$ of pairs $(\mathcal{F}, \zeta)$ consisting of a functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ and a natural isomorphism $\zeta : \omega \cong \omega'\mathcal{F}$.*
Two such pairs $(\mathcal{F}, \zeta), (\mathcal{G}, \vartheta)$ shall be considered as equivalent iff there is a natural isomorphism $\varphi : \mathcal{F} \to \mathcal{G}$ satisfying

$$\begin{tikzcd} \omega \arrow[r, "\zeta"] \arrow[dr, "\vartheta"'] & \omega'\mathcal{F} \arrow[d, "\omega'\varphi"] \\ & \omega'\mathcal{G} \end{tikzcd}$$

In this situation, I will also write $(\mathcal{F}, \zeta) \overset{\varphi}{\sim} (\mathcal{G}, \vartheta)$.

Composition of morphisms in $\mathfrak{C}$ is defined by

$$[\mathcal{F}, \zeta] \circ [\mathcal{G}, \vartheta] = [\mathcal{F}\mathcal{G}, (\zeta\mathcal{G})\vartheta]$$

The category $\mathfrak{C}_0$ has the same objects as $\mathfrak{C}$, but morphisms only those functors $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ satisfying $\omega = \omega'\mathcal{F}$.

Denote by $\underline{Coalg}$ the category of coalgebras (also called comonoids) in the category $\mathcal{A}$. Denoting by $\mathcal{M}^C$ the category of all right comodules over the coalgebra C whose underlying object of $\mathcal{A}$ is actually contained in $\mathcal{A}_0$, we can define a functor $\mathcal{M}^{-} : \underline{Coalg} \to \mathfrak{C}$.

We have obtained, [24, 2.1.12]:

2.1.2. Theorem: *The functor $\mathcal{M}^{-} : \underline{Coalg} \to \mathfrak{C}$ admits a left adjoint $\underline{\mathrm{coend}} : \mathfrak{C} \to \underline{Coalg}$. The same is true for $\mathfrak{C}_0$ instead of $\mathfrak{C}$.*

Thus, we have a bijection

$$\mathcal{M}or_{\mathfrak{C}}((\mathcal{C},\omega),(\mathcal{M}^B,\mathcal{U})) \cong \underline{Coalg}(\underline{\mathrm{coend}}(\omega),B).$$

natural in $(\mathcal{C},\omega) \in \mathfrak{C}$ and $B \in \underline{Coalg}$.

$\underline{\mathrm{coend}}(\omega) := \underline{\mathrm{coend}}(\mathcal{C}) := \underline{\mathrm{coend}}(\mathcal{C},\omega)$ was constructed as a coend

$$\underline{\mathrm{coend}}(\omega) = \int^{X\in\mathcal{C}} \omega(X)^* \otimes \omega(X)$$

of the bifunctor

$$\mathcal{C}^{\mathrm{op}} \times C \ni (X,Y) \longmapsto \omega^*(X) \otimes \omega(Y) \in \mathcal{A}$$

and called the coendomorphism coalgebra of ω, since there is a bijection

$$\psi^\omega_M : \mathcal{M}or(\underline{\mathrm{coend}}(\omega), M) \cong \mathrm{Nat}(\omega, \omega \otimes M)$$

for each object $M \in \mathcal{A}$. This is given by

$$\psi^\omega_M(f) = (id \otimes f)\delta$$

for a natural transformation $\delta = \delta^\omega : \omega \to \omega \otimes \underline{\mathrm{coend}}(\omega)$ which is universal among the natural transformations $\omega \to \omega \otimes M$. It follows that $\underline{\mathrm{coend}}(\omega)$ is a coalgebra with comultiplication Δ defined by

$$\begin{tikzcd}
\omega \arrow[r, "\delta"] \arrow[d, "\delta"'] & \omega \otimes C \arrow[d, "1 \otimes \Delta"] \\
\omega \otimes C \arrow[r, "\delta \otimes 1"'] & \omega \otimes C \otimes C
\end{tikzcd}$$

Let $f : C \to M$ and $g : C \to N$. Then we have the following important formula:

$$\psi_{M\otimes N}((f \otimes g)\Delta) = (\psi_M(f) \otimes 1)\psi_N(g)$$

The counit of the adjunction stated, that is, a functor from $\mathcal{C}$ to the category of $\underline{\text{coend}}(\omega)$-comodules, is given essentially by the comodule structures δ^ω.

If $\mathcal{A} = \underline{Vec}$ is the category of vector spaces over a field k, we can make some statements about how much information we loose when going from a category to its coendomorphism coalgebra. First of all, the coendomorphism coalgebra of the underlying functor of the category $\mathcal{M}^C$ of comodules over C is isomorphic to C itself.
Call a functor *dense* if every object in its codomain is a subquotient of an object in its image. Then we have shown that the unit $\mathcal{C} \to \mathcal{M}^{\underline{\text{coend}}(C)}$ is a dense functor.
If $\mathcal{C}$ is k-linear abelian and ω is k-linear exact and faithful, then the unit of our adjunction is actually an equivalence of $\mathcal{C}$ with the category of $\underline{\text{coend}}(\omega)$-comodules.
For $f : C \to D$ a morphism of coalgebras, the functor $\mathcal{M}^f$ is dense iff f is surjective, and it is closed (that is, full and such that every subquotient of an object in the image is isomorphic to the image of a subquotient) iff f is injective.

The category $\mathfrak{C}$ can be made a symmetric monoidal one with tensor product $\times$ defined by

$$(\mathcal{C}, \omega) \times (\mathcal{D}, \nu) := (\mathcal{C} \times \mathcal{D}, \omega \otimes \nu)$$

with $\omega \otimes \nu : \mathcal{C} \times \mathcal{D} \ni (X \otimes Y) \longmapsto \omega(X) \otimes \nu(Y) \in \mathcal{A}_0$.

$\underline{Coalg}$ with the usual tensor product of coalgebras is a monoidal category anyway. It then turns out that $\underline{\text{coend}}$ is a monoidal functor which is monoidally left adjoint to the weakly monoidal functor $\mathcal{M}^{\text{—}}$. More explicitly, if $C = \underline{\text{coend}}(\omega)$ and $D = \underline{\text{coend}}(\nu)$, then

$$\omega \otimes \nu \xrightarrow{\delta^\omega \otimes \delta^\nu} \omega \otimes C \otimes \nu \otimes D \xrightarrow{1 \otimes \tau \otimes 1} \omega \otimes \nu \otimes C \otimes D$$

is a natural transformation having the same universal property as $\delta^{\omega \otimes \nu}$, whence $\underline{\text{coend}}(\omega \otimes \nu) \cong C \otimes D$.

If $\mathcal{C}$ is a monoidal category and (ω, ξ) is a monoidal functor, then $[\otimes, \xi] : (\mathcal{C}, \omega) \times (\mathcal{C}, \omega) \to (\mathcal{C}, \omega)$ makes $(\mathcal{C}, \omega)$ a monoid in $\mathfrak{C}$.

Since $\underline{\text{coend}}$ is a monoidal functor, it maps the category $\mathfrak{C}_\otimes$ of monoids in $\mathfrak{C}$ to the category of monoids in $\underline{Coalg}$, which is the category of bialgebras.

As a consequence, if $\mathcal{C}$ is a monoidal category and $\omega = (\omega, \xi)$ is a monoidal functor, then $\underline{\text{coend}}(\omega)$ is a bialgebra, the multiplication being defined by

$$\delta(\xi(x \otimes y)) = \xi(x_{(0)} \otimes y_{(0)}) \otimes x_{(1)} y_{(1)}$$

in Sweedler's notation, for $x \in \omega(X)$, $y \in \omega(Y)$.

If, moreover, $\mathcal{C}$ is a rigid monoidal category, then the bialgebra $H = \underline{\text{coend}}(\omega)$ is a Hopf algebra, the antipode s being given by

$$\langle \varphi, x_{(0)} \rangle s(x_{(1)}) = \langle \varphi_{(0)}, x \rangle \varphi_{(1)}$$

for $X \in \mathcal{C}$, $\varphi \in \omega(X^*)$ and $x \in \omega(X)$, where we identify $\omega(X^*)$ with $\omega^*(X)$.

We have also developed a notation for doing calculations with elements of $C = \underline{\text{coend}}(\omega)$ in the case that $\mathcal{A} = \underline{Vec}$. For $X \in \mathcal{C}$, and elements $v \in \omega(X)$ and $\varphi \in \omega^*(X)$ put

$$[\varphi | v] := \langle \varphi, v_{(0)} \rangle v_{(1)} \in C$$

It is a consequence of the definition of $\underline{\mathrm{coend}}(\omega)$ that the elements of this form actually generate $\underline{\mathrm{coend}}(\omega)$ as a vector space. If the original category is a category of comodules, then actually every element of the coendomorphism coalgebra (or, up to isomorphism, of the original coalgebra) has this form.

For each $X \in \mathcal{C}$ fix a basis (v_i) of $\omega(X)$, and denote the dual basis in $\omega^*(X)$ by (v^i). Then elements of the form $[v^i|v_j]$ generate C as a vector space. Coalgebra and comodule structures are given by

$$\delta(x) = v_i \otimes [v^i|x]$$
$$\Delta([\varphi|x]) = [\varphi|v_i] \otimes [v^i|x]$$
$$\varepsilon([\varphi|x]) = \langle \varphi, x \rangle$$

2.2 Inner automorphisms and a functor of 2-categories

Up to now we know $\underline{\mathrm{coend}}$ as a functor from $\mathfrak{C}$ to the category of coalgebras. Thus we know each object of $\mathfrak{C}$ (which is a category and stuff) produces a coalgebra, and each morphism in $\mathfrak{C}$ (which is a functor and stuff) produces a morphism of coalgebras. What happens if two such functors are isomorphic? We will see that in this case the coalgebra morphisms corresponding to the functors agree up to an “inner automorphism” of the source coalgebra (this result is a variation of [18, Prop. 3]).

As a special case we will see that the coendomorphism Hopf algebra of a rigid quasisymmetric monoidal category is coquasitriangular. This type of Hopf algebra frequently occurs in the theory of quantum groups. Our definition of a coquasitriangular Hopf algebra is just the formal dual of Drinfel'd's definition, [4], of a quasitriangular Hopf algebra. The reason for dualizing is that we are considering comodules, which are easier to deal with in view of the finiteness theorems. Now modules over quasitriangular Hopf algebras form a category with nice properties, as do comodules over coquasitriangular ones. The same reasons apply to the other “dualized” notions we will briefly encounter in the sequel.

We should say a word about what we mean by an “inner automorphism” of a coalgebra. As an inner automorphism of an algebra is conjugation by an invertible element, an inner automorphism of a coalgebra will be coconjugation by an invertible coelement.

Let C be a coalgebra. The set $\mathcal{M}or(C,k) =: k^C$ becomes a monoid (an algebra if $\mathcal{A}$ is k-linear), when we define multiplication of two morphisms $r, s : C \to k$ (two “coelements of C”) to give

$$r * s := (r \otimes s)\Delta : C \to C \otimes C \to k \otimes k \cong k.$$

This product is known as convolution; of course, in the case $\mathcal{A} = \underline{\mathit{Vec}}$ we have defined nothing else but the dual algebra of a coalgebra. The neutral element is ε.

Now let M be a right (resp. left) comodule over C and $X \in \mathcal{A}$ any object. Then we can define a right (resp. left) action of k^C on the set $\mathcal{M}or(M, X)$ by

$$f * r := (f \otimes r)\delta \qquad (\text{ resp. } r * f = (r \otimes f)\delta \text{ })$$

for $r \in k^C$, $f : M \to X$. That we have indeed actions of monoids on sets follows from the comodule properties of M. Of course, in the case $\mathcal{A} = \underline{Vec}$, the actions arise from the familiar left (resp. right) C^*-module structures on M; sides are changed once more, as we are acting on the contravariant argument of $\mathcal{M}or(M, X)$.

Now we can define two morphisms $f, g : C \to D$ of coalgebras to be coconjugate (or to agree up to a (co-)inner automorphism, though we have not explicitly written down the latter) iff there is a $*$-invertible $r : C \to k$ with $g * r = r * f$ or, equivalently, $g = r * f * \overline{r}$, for $\overline{r}$ the $*$-inverse of r.

The main result of [24, sec. 2.1], reviewed in 2.1, was the construction of a pair of adjoint functors between the categories $\mathfrak{C}$ and $\underline{Coalg}$. That is, we established a bijection

$$\mathcal{M}or_{\mathfrak{C}}((\mathcal{C}, \omega), (\mathcal{M}^B, \mathcal{U})) \cong \underline{Coalg}(\underline{\mathrm{coend}}(\omega), B).$$

The following lemma, which will be the main ingredient for our calculations with coinner automorphisms, extends this bijection to one also involving the natural transformations between the functors being elements of the left hand side.

2.2.1. Lemma: *Let $(\mathcal{C}, \omega) \in \mathfrak{C}$, and let $C = \underline{\mathrm{coend}}(\omega)$. Let B be a coalgebra, and denote the underlying functor $\mathcal{M}^B \to \underline{vec}$ by ν.*

Let $[\mathcal{F}, \zeta], [\mathcal{G}, \vartheta] : (\mathcal{C}, \omega) \to (\mathcal{M}^B, \nu)$ be two morphisms in $\mathfrak{C}$, and let $f, g : C \to B$ be the morphisms of coalgebras corresponding to these under the bijection established by the adjointness result [24, thm. 2.1.12].

There is a natural transformation $\varphi : \mathcal{F} \to \mathcal{G}$ if and only if there is a morphism $r : C \to k$ satisfying $g * r = r * f$.

More precisely, there is a bijection between natural morphisms $\varphi : \mathcal{F} \to \mathcal{G}$ and morphisms $r : C \to k$ satisfying $g * r = r * f$, given by

$$\nu(\varphi)\zeta = \vartheta * r \tag{1}$$

that is, by commutativity of

$$\begin{tikzcd}
\omega \arrow[r, "\delta^\omega"] \arrow[d, "\zeta"'] & \omega \otimes C \arrow[d, "\vartheta \otimes r"] \\
\nu\mathcal{F} \arrow[r, "\nu\varphi"'] & \nu\mathcal{G}
\end{tikzcd}$$

or $\psi_k^\omega(r) = \vartheta^{-1}\nu(\varphi)\zeta$.

PROOF: We have an obvious bijection between endomorphisms γ of ω and morphisms $r : C \to k$, given by

$$\psi_k^\omega(r) = \gamma.$$

We have to check that those r satisfying $g * r = r * f$ are thus put in bijection with those γ arising as $\gamma = \vartheta^{-1}\nu(\varphi)\zeta$ for some $\varphi : \mathcal{F} \to \mathcal{G}$. Indeed, if r, γ are in the relation stated, then we have

$$\begin{aligned}\psi_B(g * r) &= \psi_B((g \otimes r)\Delta)\\ &= \psi_B(g)\psi_k(r)\\ &= (\vartheta \otimes 1_B)^{-1}\delta^\nu_{\mathcal{F}'}\vartheta\gamma\end{aligned}$$

and

$$\begin{aligned}\psi_B(r * f) &= \psi_B((r \otimes f)\Delta)\\ &= (\psi_k(r) \otimes 1_B)\psi_D(f)\\ &= (\gamma \otimes 1_B)(\zeta \otimes 1_B)^{-1}\delta^\nu_{\mathcal{F}}\zeta\end{aligned}$$

As ψ_B is a bijection, we have $g*r = r*f$ iff $(\vartheta\otimes 1_B)\psi_B(g*r)\zeta^{-1} = (\vartheta\otimes 1_B)\psi_B(r*f)\zeta^{-1}$ iff $\delta^\nu_{\mathcal{G}}\vartheta\gamma\zeta^{-1} = (\vartheta\gamma\zeta^{-1} \otimes 1_B)^{-1}\delta^\nu_{\mathcal{F}}$, that is, iff $\vartheta\gamma\zeta$ is B-colinear. □

The result just obtained admits a nice interpretation in terms of 2-categories.

We can make $\mathfrak{C}_0$ a 2-category as follows: If $\mathcal{F}, \mathcal{G} : (\mathcal{C}, \omega) \to (\mathcal{D}, \nu)$ are morphisms in $\mathfrak{C}_0$, we let

$$\mathfrak{mor}(\mathcal{F}, \mathcal{G}) = \mathrm{Nat}(\mathcal{F}, \mathcal{G})$$

be the set of natural transformations, and we let composition of 2-cells be the same as in <u>*Cat*</u>. It is fairly obvious that $\mathfrak{C}_0$ is a 2-category. We have to struggle a little more if we want to define the proper notion of 2-cell between $[\mathcal{F}, \zeta], [\mathcal{G}, \vartheta] : (\mathcal{C}, \omega) \to (\mathcal{D}, \nu)$ in $\mathfrak{C}$, for now the 1-cells are equivalence classes. We let, by blatant abuse of notation,

$$\mathfrak{mor}([\mathcal{F}, \zeta], [\mathcal{G}, \vartheta]) := \mathrm{Nat}(\mathcal{F}, \mathcal{G}),$$

and, whenever $(\mathcal{F}, \zeta) \overset{\beta}{\sim} (\mathcal{F}', \zeta')$ and $(\mathcal{G}, \vartheta) \overset{\gamma}{\sim} (\mathcal{G}', \vartheta')$, we "identify" $\varphi \in \mathrm{Nat}(\mathcal{F}, \mathcal{G})$ with $\gamma\varphi\beta^{-1} \in \mathrm{Nat}(\mathcal{F}', \mathcal{G}')$ [1]. Observe that the bijection established in the preceding lemma is compatible with this identification, for if $\nu(\varphi)\zeta = \vartheta * r$, then also

$$\begin{aligned}\nu(\gamma\varphi\beta^{-1})\zeta' &= \nu(\gamma\varphi\beta^{-1})\nu(\beta)\zeta\\ &= \nu(\gamma)\nu(\varphi)\zeta\\ &= \nu(\gamma)(\vartheta * r)\\ &= (\nu(\gamma)\vartheta) * r\\ &= \vartheta' * r\end{aligned}$$

[1] In a correct version, a morphism between equivalence classes $\mathcal{S}, \mathcal{T}$ should be a mapping Γ defined on $\mathcal{S} \times \mathcal{T}$ assigning to $(\mathcal{F}, \zeta) \in \mathcal{S}$ and $(\mathcal{G}, \vartheta) \in \mathcal{T}$ a natural transformation $\mathcal{F} \to \mathcal{G}$, such that $\Gamma((\mathcal{F}', \zeta'), (\mathcal{G}', \vartheta')) = \gamma\Gamma((\mathcal{F}, \zeta), (\mathcal{G}, \vartheta))\beta^{-1}$ in the situation above.

On the other hand, we can also make the category $\underline{Coalg}$ of coalgebras a 2-category. For this we define

$$\mathfrak{mor}(f,g) := \{(g,r,f) | r : C \to k, g * r = r * f\}$$

for morphisms $f, g : C \to D$ in $\underline{Coalg}$. We defined a morphism to be a triple in order that morphism sets be disjoint, as they should be. Of course, the main ingredient is the coelement in the middle, and we shall frequently write things like $f \xrightarrow{r} g$. Of course, weak composition is given by $*$. Strong composition for 1-cells is the usual composition of coalgebra morphisms. We define

$$h \circ (g,r,f) := (hg, r, hf) \qquad (g,r,f) \circ j := (gj, rj, fj)$$

for f, g, r as above and $h : D \to E$, $j : B \to C$. It is straightforward to check that $\underline{Coalg}$ becomes a 2-category.

Note that $\mathcal{M}^{-} : \underline{Coalg} \to \mathfrak{C}_0$ extends to a 2-functor if, for a 2-cell $f \xrightarrow{r} g$, we define the natural transformation $\mathcal{M}^r : \mathcal{M}^f \to \mathcal{M}^g$ by

$$\mathcal{M}^r_M = *r = (id \otimes r)\delta_M$$

for M a C-comodule. Naturally, $\mathcal{M}^{-} : \underline{Coalg} \to \mathfrak{C}$ is a 2-functor as well.

Now by 1.3.6, lemma 2.2.1 proves

2.2.2. Theorem: $\underline{\text{coend}}$ *extends uniquely to a 2-functor* $\mathfrak{C} \to \underline{Coalg}$ *which is a left 2-adjoint for for* $\mathcal{M}^{-}$.

We put down explicitly what this means.

2.2.3. Corollary: *Let* $[\mathcal{F},\zeta], [\mathcal{G},\vartheta] : (\mathcal{C},\omega) \to (\mathcal{D},\nu)$ *be morphisms in* $\mathfrak{C}$.

(1) *Each morphism* $[\mathcal{F},\zeta] \xrightarrow{\varphi} [\mathcal{G},\vartheta]$ *yields a morphism*

$$\underline{\text{coend}}(\mathcal{F},\zeta) \xrightarrow{\underline{\text{coend}}(\varphi)} \underline{\text{coend}}(\mathcal{G},\vartheta)$$

uniquely defined by requiring

$$\begin{tikzcd} \omega \arrow[r, "\delta^\omega"] \arrow[d, "\zeta"'] & \omega \otimes \underline{\text{coend}}(\omega) \arrow[d, "\vartheta \otimes \underline{\text{coend}}(\varphi)"] \\ \nu\mathcal{F} \arrow[r, "\nu\varphi"'] & \nu\mathcal{G} \end{tikzcd}$$

to commute.

(2) $\underline{\text{coend}}(\psi\varphi) = \underline{\text{coend}}(\psi) * \underline{\text{coend}}(\varphi)$ *whenever that makes sense.*

(3) $\underline{\text{coend}}(\varphi) = \varepsilon$ *if* φ *is the identity 2-cell in* $\mathfrak{C}$, *that is, if* $\vartheta = \nu(\varphi)\,\zeta$.

(4) As a consequence, if φ is an isomorphism, then $\underline{\mathrm{coend}}(\varphi)$ is $*$-invertible, hence $\underline{\mathrm{coend}}[\mathcal{F}, \zeta]$ and $\underline{\mathrm{coend}}[\mathcal{G}, \vartheta]$ are coconjugate.

(5) For $[\mathcal{H}, \lambda] : (\mathcal{B}, \kappa) \to (\mathcal{C}, \omega)$ we have $\underline{\mathrm{coend}}(\varphi \circ \mathcal{H}) = \underline{\mathrm{coend}}(\varphi)\underline{\mathrm{coend}}[\mathcal{H}, \lambda]$.

(6) For $[\mathcal{H}, \lambda] : (\mathcal{D}, \nu) \to (\mathcal{B}, \kappa)$ we have $\underline{\mathrm{coend}}(\mathcal{H} \circ \varphi) = \underline{\mathrm{coend}}(\varphi) : C \to k$. (of course, the 2-cells *will not coincide unless their domain and codomain do).*

We have seen that the category, denote it, say, $\hat{\mathfrak{C}}$, whose objects are the objects of $\mathfrak{C}$ and whose morphisms $(\mathcal{C}, \omega) \to (\mathcal{D}, \nu)$ are pairs $(\mathcal{F}, \zeta)$ with $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\zeta : \omega \cong \nu\mathcal{F}$, is not "suitable" for our needs. We had to take equivalence classes (the morphisms of $\mathfrak{C}$) to get the adjointness results (say [24, thm. 2.1.12]) we were up to. Now we can do with $\hat{\mathfrak{C}}$ instead of $\mathfrak{C}$. Of course, we will have to take some other pains, weakening the adjointness in the sense sketched at the end of section 1.4. We state without proof:

2.2.4. Theorem: $\underline{\mathrm{coend}} : \hat{\mathfrak{C}} \to \underline{Coalg}$ *defines a 2-functor which is a weak left 2-adjoint for $\mathcal{M}^{\overline{\quad}} : \underline{Coalg} \to \hat{\mathfrak{C}}$.*

We already know from [24, 2.3] that $\underline{\mathrm{coend}}$ is a monoidal functor. To make it a nice mechanism for "translating structures" in the sequel, we have to check it to be a monoidal 2-functor. The relevant fact, which we shall not check in detail, is:

2.2.5. Corollary: *Let $[\mathcal{F}_i, \zeta_i], [\mathcal{G}_i, \vartheta_i] : (\mathcal{C}_i, \omega_i) \to (\mathcal{D}_i, \nu_i)$ be morphisms in $\mathfrak{C}$, denote $\underline{\mathrm{coend}}(\mathcal{F}_i, \zeta_i) = f_i$ and $\underline{\mathrm{coend}}(\mathcal{G}_i, \vartheta_i) = g_i$. We know that $\underline{\mathrm{coend}}(\mathcal{F}_1 \times \mathcal{F}_2, \zeta_1 \otimes \zeta_2) = f_1 \otimes f_2$ etc. Let $[\mathcal{F}_i, \zeta_i] \xrightarrow{\varphi_i} [\mathcal{G}_i, \vartheta_i]$ be 2-cells. Then*

$$\underline{\mathrm{coend}}(\varphi_1 \otimes \varphi_2) = \underline{\mathrm{coend}}(\varphi_1) \otimes \underline{\mathrm{coend}}(\varphi_2).$$

We note one little application: Monoidal transformations between morphisms in $\mathfrak{C}_\otimes$ are mapped by $\underline{\mathrm{coend}}$ to algebra morphisms $C \to k$.

Now we are provided with a nice formalism for translating some properties of categories in $\mathfrak{C}$ to properties of the coendomorphism coalgebras, since a 2-functor maps commutative diagrams of 2-cells to commutative diagrams of 2-cells.

We introduce just two more notations before proceeding to the applications we are up to:

We shall assume for the rest of the paper that $\mathcal{A}$ is symmetric. Then if we have a morphism $r : C^{\otimes n} \to k$ for C a coalgebra, then for any $m \geq n$ and $i_1, \ldots, i_n \in \{1, \ldots, m\}$ we denote by

$$r_{i_1 \cdots i_n} : C^{\otimes m} \to k$$

the morphisms obtained applying r to the i_1-th, ..., i_n-th tensorand (in this order) and ε to the remaining ones. (Note that except for special cases this makes sense only if we can coherently reorder tensorands using a symmetry.) For example, for $r : C \otimes C \to k$ we have $r_{13} : C \otimes C \otimes C \to k$ defined by $r_{13}(x \otimes y \otimes z) = r(x \otimes z)\varepsilon(y)$ for elements $x, y, z \in C$.

If $\sigma \in S_n$ is a permutation, we denote by

$$\tau_{[\sigma^{-1}(1),\ldots,\sigma^{-1}(n)]}$$

the isomorphism on an n-fold tensor product of objects of $\mathcal{A}$ obtained by composing instances of the quasisymmetry τ according to the given permutation. For example

$$\tau_{[321]}(x \otimes y \otimes z) = z \otimes y \otimes x$$

Denoting $\mathcal{T}$ the symmetry in the category of categories (which is actually just permuting factors in a cartesian product of categories), we use the same notation.

2.3 Coquasibialgebras

We have seen in [24, sec. 2.3] that a monoidal category in $\mathfrak{C}$ whose underlying functor is a monoidal one has a bialgebra as its coendomorphism coalgebra. Now of course, to induce a "multiplication" μ on $B := \underline{\mathrm{coend}}(\omega)$, it is sufficient to have a monoidal category $\mathcal{C}$ and an underlying functor ω equipped with a natural isomorphism $\xi : \omega X \otimes \omega Y \to \omega(X \otimes Y)$ but not necessarily satisfying the coherence conditions in the definition of a monoidal functor. Dito, to have a "unit" $\eta : C \to k$, it is sufficient to just have an isomorphism $\xi_0 : k \to \omega(I)$ without a coherence condition. We give the following rather dull

2.3.1. Definition: *A tensor functor is a functor $\omega : \mathcal{C} \to \mathcal{A}$ together with a natural isomorphism $\xi : \omega(X) \otimes \omega(Y) \to \omega(X \otimes Y)$ and an isomorphism $\xi_0 : k \to \omega I$. Call a tensor functor (ω, ξ, ξ_0) neutral if ξ_0 satisfies the two coherence conditions involving ξ_0 in the definition of a monoidal functor.*

Now assume we are given a neutral tensor functor $\omega : \mathcal{C} \to \mathcal{A}$, and define $\mu = \underline{\mathrm{coend}}(\otimes, \xi) : C \otimes C \to C$ and $\eta = \underline{\mathrm{coend}}(\mathrm{const}_I, \xi_0) : C \to k$. Of course, since ω is neutral, we still get the familiar equations

$$\mu(\eta \otimes id) = id = \mu(id \otimes \eta) \tag{1}$$

but now we cannot expect to have associativity of μ any more.

As $\mathcal{C}$ is a monoidal category, we have a natural isomorphism

$$\alpha : \otimes(\mathcal{I}d \times \otimes) \to \otimes(\otimes \times \mathcal{I}d)$$

And we can define

$$\mu(id \otimes \mu) \xrightarrow{a} \mu(\mu \otimes id)$$

to be the 2-cell $\underline{\mathrm{coend}}(\alpha)$. Thus we have a $*$-invertible $a : C \otimes C \otimes C \to k$ with

$$a * \ \mu(id \otimes \mu) = \mu(\mu \otimes id) \ * a \tag{2}$$

which we can also write

$$\forall x,y,z\in C:\quad a(x_{(1)}\otimes y_{(1)}\otimes z_{(1)})x_{(2)}(y_{(2)}z_{(2)})=(x_{(1)}y_{(1)})z_{(1)}a(x_{(2)}\otimes y_{(2)}\otimes z_{(2)})$$

or, for $\overline{a}$ the $*$-inverse of a,

$$\mu(\mu\otimes id)=a*\mu(id\otimes\mu)*\overline{a}$$

We get important information on a from the coherence conditions on α. The pentagon equation is a commutative diagram

$$\begin{tikzcd}
\otimes(\mathcal{I}d\times\otimes)(\mathcal{I}d\times\mathcal{I}d\times\otimes) \arrow[rr, "\alpha(\mathcal{I}d\times\mathcal{I}d\times\otimes)"] \arrow[d, "1\otimes\alpha"'] & & \otimes(\otimes\times\otimes) \arrow[dd, "\alpha(\otimes\times\mathcal{I}d\times\mathcal{I}d)"] \\
\otimes(\mathcal{I}d\times\otimes)(\mathcal{I}d\times\otimes\times\mathcal{I}d) \arrow[d, "\alpha(\mathcal{I}d\times\otimes\times\mathcal{I}d)"'] & & \\
\otimes(\otimes\times\mathcal{I}d)(\mathcal{I}d\times\otimes\times\mathcal{I}d) \arrow[rr, "\alpha\otimes 1"'] & & \otimes(\otimes\times\mathcal{I}d)(\otimes\times\mathcal{I}d\times\mathcal{I}d)
\end{tikzcd}$$

This translates immediately via the 2-functor <u>coend</u> to

$$\begin{tikzcd}
\mu(id\otimes\mu)(id\otimes id\otimes\mu) \arrow[rr, "a(id\otimes id\otimes\mu)"] \arrow[d, "\varepsilon\otimes a"'] & & \mu(\mu\otimes\mu) \arrow[dd, "a(\mu\otimes id\otimes id)"] \\
\mu(id\otimes\mu)(id\otimes\mu\otimes id) \arrow[d, "a(id\otimes\mu\otimes id)"'] & & \\
\mu(\mu\otimes id)(id\otimes\mu\otimes id) \arrow[rr, "a\otimes\varepsilon"'] & & \mu(\mu\otimes id)(\mu\otimes id\otimes id)
\end{tikzcd}$$

Thus, we have

$$a_{123}*a(id\otimes\mu\otimes id)*a_{234}=a(\mu\otimes id\otimes id)*a(id\otimes\mu\otimes id) \tag{3}$$

as an equation for morphisms $C^{4\otimes}\to k$.

The dual notion of the following definition goes back to Drinfel'd, see for example [5].

2.3.2. Definition: *A coquasibialgebra (C,μ,η,a) consists of a coalgebra C, coalgebra morphisms $\mu: C\otimes C\to C$ and $\eta: k\to C$ and a $*$-invertible $a: C\otimes C\otimes C\to k$ such that equations (1), (2) and (3) above are fulfilled.*

We have proved:

2.3.3. Theorem: *If $\mathcal{C}$ is a monoidal category and $\omega : \mathcal{C} \to \mathcal{A}$ is a neutral tensor functor, then* $\underline{\mathrm{coend}}(\omega)$ *is a coquasibialgebra.*

It is easy to see that if (C, μ, η, a) is a coquasibialgebra, then $\mathcal{M}^C$ is a monoidal category with the underlying functor a neutral tensor one. The associativity morphism is described by

$$\alpha(x \otimes (y \otimes z)) = (x_{(0)} \otimes y_{(0)}) \otimes z_{(0)}\ a(x_{(1)} \otimes y_{(1)} \otimes z_{(1)})$$

2.4 Coquasitriangular (coquasi-)bialgebras

We come to the type of bialgebras which is the main topic of this paper. It is clear that if we assume the category $\mathcal{C}$ to be a symmetric one, and the functor $\omega : \mathcal{C} \to \mathcal{A}$ a symmetric monoidal one, then the bialgebra $\underline{\mathrm{coend}}(\omega)$ is commutative. Now we can ask what happens if the category $\mathcal{C}$ is (quasi-)symmetric, but the functor ω is not.

We turn right away to the more general situation which came up in the preceding section. Thus, let $\mathcal{C}$ be a monoidal category, and let ω as above be a neutral tensor functor. Let σ be a quasisymmetry for $\mathcal{C}$. The 2-cell $\otimes \xrightarrow{\sigma} \otimes^{\mathrm{sym}}$ then translates to a 2-cell in $\underline{Coalg}$ via $\underline{\mathrm{coend}}$, which we denote $\mu \xrightarrow{r} \mu^{\mathrm{op}}$. Thus $r : H \otimes H \to k$ is a $*$-invertible morphism such that

$$r * \mu = \mu^{\mathrm{op}} * r, \tag{1}$$

or, equivalently, $\mu^{\mathrm{op}} = r * \mu * \overline{r}$. We seek conditions on r and a corresponding to the fact that σ is a quasisymmetry. We have a commutative diagram

$$\begin{tikzcd}
\otimes(\mathcal{I}d \times \otimes) \arrow[r, "\sigma(\mathcal{I}d \times \otimes)"] \arrow[d, "\alpha"'] & \otimes(\otimes \times \mathcal{I}d)\mathcal{T}_{[231]} \\
\otimes(\otimes \times \mathcal{I}d) \arrow[d, "\sigma \otimes 1"'] & \otimes(\mathcal{I}d \times \otimes)\mathcal{T}_{[231]} \arrow[u, "\alpha\mathcal{T}_{[231]}"'] \\
\otimes(\otimes \times \mathcal{I}d)\mathcal{T}_{[213]} & \otimes(\mathcal{I}d \times \otimes)\mathcal{T}_{[213]} \arrow[l, "\alpha\mathcal{T}_{[213]}"] \arrow[u, "(1 \otimes \sigma)\mathcal{T}_{[213]}"']
\end{tikzcd}$$

which translates to a commutative diagram of 2-cells

$$\begin{tikzcd}
\mu(id \otimes \mu) \arrow[r, "r(id \otimes \mu)"] \arrow[d, "a"'] & \mu(\mu \otimes id)\tau_{[231]} \\
\mu(\mu \otimes id) \arrow[d, "r_{12}"'] & \mu(id \otimes \mu)\tau_{[231]} \arrow[u, "a_{231}"'] \\
\mu(\mu \otimes id)\tau_{[213]} & \mu(id \otimes \mu)\tau_{[213]} \arrow[l, "a_{213}"] \arrow[u, "r_{13}"']
\end{tikzcd}$$

reading

$$r(id \otimes \mu) = a_{231} * r_{13} * \overline{a}_{213} * r_{12} * a \tag{2}$$

From the other diagram for a quasisymmetry we obtain

$$r(\mu \otimes id) = \overline{a}_{321} * r_{13} * a_{132} * r_{23} * \overline{a} \tag{3}$$

Thus we define

2.4.1. Definition: *A coquasitriangular coquasibialgebra (H, μ, η, a, r) consists of a coquasibialgebra (H, μ, η, a) and a $*$-invertible $r : H \otimes H \to k$ such that the equations (1), (2) and (3) hold.*

and we have proved

2.4.2. Theorem: *If $\mathcal{C}$ is a quasisymmetric category and $\omega : \mathcal{C} \to \mathcal{A}$ is a neutral tensor functor, then $\underline{\text{coend}}(\omega)$ is a coquasitriangular coquasibialgebra.*

Before we restrict *our* version to the associative setting, we recall the original definition of [4].

2.4.3. Definition: *Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra, let $R \in H \otimes H$.*
(H, R) is a **quasitriangular** *bialgebra, if*

(1) R *is invertible,*

(2) $\forall x \in H : \quad \Delta^{\text{cop}}(x) := \tau\Delta(x) = R\Delta(x)R^{-1}$

(3) $(\Delta \otimes 1)(R) = R_{13}R_{23}$

(4) $(1 \otimes \Delta)(R) = R_{13}R_{12}$

(H, R) is called **triangular**, *if in addition*

(5) $R^{-1} = R_{21} = \tau R.$

Here $\Delta^{\text{cop}} = \tau\Delta$ denotes the opposite comultiplication in H. The elements $R^{ij} \in H^{\otimes m}$ are supposed to be R's first and second tensorand at the i-th and j-th position of the m-fold tensor product and 1 at the remaining ones, say $R_{13} = \sum_{k=1}^{n} \alpha_k \otimes 1 \otimes \beta_k$ if $R = \sum_{k=1}^{n} \alpha_k \otimes \beta_k \in H \otimes H$.

The properties of a quasitriangular bialgebras ensure the category of its modules to carry the structure of a quasisymmetric monoidal category, the underlying functor being a monoidal one. (This statement can be found in [4, p.814] for the case of triangular Hopf algebras, in [12, p.71] for the general case). The quasisymmetry is given by defining, for M, N two modules over a quasitriangular Hopf algebra (H, R) and $m \in M$, $n \in N$ elements,

$$\sigma_{MN}(m \otimes n) = \tau_{MN}(R(m \otimes n)) = \sum_k \beta_k n \otimes \alpha_k m$$

where $R = \sum_k \alpha_k \otimes \beta_k$.

Since we are working with finite dimensional comodules we turn back to that setting and define the dual notion of quasitriangular bialgebra.

2.4.4. Definition: *Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra, let $r : H \otimes H \to k$.*
(H, r) is a **coquasitriangular** *bialgebra, if*

(1) r is $*$-invertible (with the $*$-inverse denoted $\overline{r}$)

(2) $\mu^{\mathrm{op}} := \mu\tau = r * \mu * \overline{r}$

(3) $r(1 \otimes \mu) = r^{13} * r^{12}$

(4) $r(\mu \otimes 1) = r^{13} * r^{23}$

(H, r) is **cotriangular**, *if in addition*

(5) $\overline{r} = r \circ \tau$.

Call a morphism of bialgebras $H \to H'$ between coquasitriangular bialgebras (H, r) and (H', r') **coquasitriangular** *if $r'(f \otimes f) = r$.*

If the category $\mathcal{C}$ is quasisymmetric as above, and the underlying functor ω is *monoidal* (but still not symmetric), we obtain, by specializing our last theorem to $a = \varepsilon^{\otimes 3}$:

2.4.5. Theorem: *Let $[\mathcal{C}, \omega] \in \mathfrak{C}_{\otimes}$, that is, $(\mathcal{C}, \otimes)$ a monoidal category, (ω, ξ) a monoidal functor.*
If $(\mathcal{C}, \sigma)$ is a quasisymmetric monoidal category, then $H := \underline{\mathrm{coend}}(\omega)$ is a coquasitriangular bialgebra.
The morphism $r : H \otimes H \to k$ from 2.4.4 can be chosen to satisfy

$$(\omega\sigma)\xi(x \otimes y) = \xi(y_{(0)} \otimes x_{(0)})r(x_{(1)} \otimes y_{(1)}).$$

Then also

$$(\omega\sigma^{-1})\xi(x \otimes y) = \xi(y_{(0)} \otimes x_{(0)})\overline{r}(y_{(1)} \otimes x_{(1)}).$$

As coquasitriangular bialgebras are our main concern, and since calculations with 2-cells and 2-functors are nice and easy, but fairly abstract, we do a few things more explicitly

2.4.6. Remark: *When evaluated on elements $x, y, z \in H$, the axioms for r in a coquasitriangular bialgebra read*

(2) $yx = r(x_{(1)} \otimes y_{(1)})x_{(2)}y_{(2)}\overline{r}(x_{(3)} \otimes y_{(3)})$

(3) $r(x \otimes yz) = r(x_{(1)} \otimes z_{(1)})\varepsilon(y_{(1)})r(x_{(2)} \otimes y_{(2)})\varepsilon(z_{(2)}) = r(x_{(1)} \otimes z)r(x_{(2)} \otimes y)$

(4) $r(xy \otimes z) = r(x \otimes z_{(1)})r(y \otimes z_{(2)})$

and in the cotriangular case

(5) $\overline{r}(x \otimes y) = r(y \otimes x)$

After $*$-multiplying with r, (2) and (5) can be written as

$$r(x_{(1)} \otimes y_{(1)})x_{(2)}y_{(2)} = y_{(1)}x_{(1)}r(x_{(2)} \otimes y_{(2)})$$

and

$$r(x_{(1)} \otimes y_{(1)})r(y_{(2)} \otimes x_{(1)}) = \varepsilon(x)\varepsilon(y),$$

respectively

We show explicitly that the category of comodules over H actually becomes a quasisymmetric monoidal category if (H, r) is coquasitriangular
For X, Y, Z three H-comodules and $x \in X, y \in Y, z \in Z$ elements, we have

$$\begin{aligned}
\sigma_{X,Y\otimes Z}(x \otimes y \otimes z) &= (y \otimes z)_{(0)} \otimes x_{(0)}r(x_{(1)} \otimes (y \otimes z)_{(1)}) \\
&= y_{(0)} \otimes z_{(0)} \otimes x_{(0)}r(x_{(1)} \otimes y_{(1)}z_{(1)}) \\
(1 \otimes \sigma_{XZ})(\sigma_{XY} \otimes 1)(x \otimes y \otimes z) &= (1 \otimes \sigma_{XZ})(y_{(0)} \otimes x_{(0)} \otimes z)r(x_{(1)}y_{(1)}) \\
&= y_{(0)} \otimes z_{(0)} \otimes x_{(0)}r(x_{(1)} \otimes z_{(1)})r(x_{(2)} \otimes y_{(1)})
\end{aligned}$$

By a similar reasoning the second diagram defining a quasisymmetry follows from axiom (4) of a coquasitriangular bialgebra.

Of course, the following result is now easy to check:

2.4.7. Remark: *Defining $\mathfrak{C}_{\otimes q}$ to be the category consisting of those objects in $\mathfrak{C}$ that are quasisymmetric monoidal categories, with morphisms those morphisms in $\mathfrak{C}_{\otimes}$ between them that have a quasisymmetric functor as a representative, and defining $\underline{CQT}$ to be the category of coquasitriangular bialgebras and coquasitriangular morphisms, we get a pair of adjoint functors $\mathcal{M}^{—} : \underline{CQT} \to \mathfrak{C}_{\otimes q}$ and $\underline{\text{coend}} : \mathfrak{C}_{\otimes q} \to \underline{CQT}$.*

2.4.8. Remark: *In the notation for elements of the coendomorphism coalgebra we have developed, we have*

$$\begin{aligned}
r([\varphi|v] \otimes [\psi|w]) &= r(\langle \varphi, v_{(0)} \rangle v_{(1)} \otimes \langle \psi, w_{(0)} \rangle w_{(1)}) \\
&= \langle \psi \otimes \varphi, w_{(0)} \otimes v_{(0)} \rangle r(v_{(1)} \otimes w_{(1)}) \\
&= \langle \psi \otimes \varphi, \sigma(v \otimes w) \rangle
\end{aligned}$$

and similarly $\overline{r}([\varphi|v] \otimes [\psi|w]) = \langle \psi \otimes \varphi, \sigma^{-1}(v \otimes w) \rangle$

2.5 Dual coboundary (coquasi-)bialgebras

We weaken the conditions necessary for obtaining a coquasitriangular coquasibialgebra a little. Remember (see 1.2.5) that a quasisymmetry for a monoidal category makes the identical endofunctor a monoidal functor $\mathcal{C}^{\mathrm{sym}} \to \mathcal{C}$.

Now assume the identity, equipped with some $\xi : X \otimes Y \to Y \otimes X$ and the identical morphism $\xi_0 = id_I$, is a monoidal functor $\mathcal{C} \to \mathcal{C}^{\mathrm{sym}}$. Thus we have a commutative diagram

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "\xi"] \arrow[d, "\alpha"'] & (Y \otimes Z) \otimes X \arrow[d, "\xi \otimes id"] \\
(X \otimes Y) \otimes Z \arrow[d, "\xi"'] & (Z \otimes Y) \otimes X \\
Z \otimes (X \otimes Y) \arrow[r, "id \otimes \xi"'] & Z \otimes (Y \otimes X) \arrow[u, "\alpha"']
\end{tikzcd}$$

This translates to a diagram of 2-cells:

$$\begin{tikzcd}
\otimes(\mathcal{I}d \times \otimes) \arrow[r, "\xi(\mathcal{I}d \times \otimes)"] \arrow[d, "\alpha"'] & \otimes(\otimes \times \mathcal{I}d)\mathcal{T}_{[231]} \arrow[d, "(\xi \otimes id)\mathcal{T}_{[231]}"] \\
\otimes(\otimes \times \mathcal{I}d) \arrow[d, "\xi(\otimes \times \mathcal{I}d)"'] & \otimes(\otimes \times \mathcal{I}d)\mathcal{T}_{[321]} \\
\otimes(\mathcal{I}d \times \otimes)\mathcal{T}_{[312]} \arrow[r, "(id \otimes \xi)\mathcal{T}_{[312]}"'] & \otimes(\mathcal{I}d \times \otimes)\mathcal{T}_{[321]} \arrow[u, "\alpha\mathcal{T}_{[321]}"']
\end{tikzcd}$$

which in turn translates via <u>coend</u> to the equation

$$r_{23} * r(id \otimes \mu) = a_{321} * r_{12} * r(\mu \otimes id) * a \tag{1}$$

if we define $r = \underline{\text{coend}}(\xi)$. Of course we still have that the multiplication and opposite multiplication are coconjugate by r. From the commutativity of the two triangle diagrams

$$\begin{tikzcd}
X \otimes I \arrow[rr, "\xi"] \arrow[dr, "\rho"'] & & I \otimes X \arrow[dl, "\lambda"] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes X \arrow[rr, "\xi"] \arrow[dr, "\lambda"'] & & X \otimes I \arrow[dl, "\rho"] \\
& X &
\end{tikzcd}$$

we also deduce that

$$r(\eta \otimes id) = \varepsilon = r(id \otimes \eta) \tag{2}$$

2.5.1. Definition: *A quasibialgebra (H, a) equipped with an invertible $r : H \otimes H \to k$ such that the equations (1) and (2) and $\mu^{\mathrm{op}} = r{*}\mu{*}\overline{r}$ hold is called a* **dual coboundary coquasibialgebra**[2].

Clearly, a dual coboundary bialgebra is supposed to be a bialgebra equipped with a $*$-invertible $r : H \otimes H \to k$ such that

$$r_{23} * r(id \otimes \mu) = r_{12} * r(\mu \otimes id)$$

$$\mu^{\mathrm{op}} = r * \mu * \overline{r}$$

and

$$r(id \otimes \eta) = \varepsilon = r(\eta \otimes id).$$

We have seen

2.5.2. Theorem: *Assume $(\mathcal{C}, \omega) \in \mathfrak{C}$ is a monoidal category with a neutral tensor underlying functor. If the identical endofunctor of $\mathcal{C}$, equipped with a suitable ξ and $\xi_0 = id_I$, is a monoidal functor $\mathcal{C}^{\mathrm{sym}} \to \mathcal{C}$, then $\underline{\mathrm{coend}}(\omega)$ is a dual coboundary coquasibialgebra.*

The reader will believe that if H is a dual coboundary coquasibialgebra then the identical endofunctor of $\mathcal{M}^H$ can be made a monoidal functor $\mathcal{C}^{\mathrm{sym}} \to \mathcal{C}$ in the same fashion.

By the way the results obtained imply of course that every coquasitriangular (coquasi-)bialgebra is a dual coboundary one (in view of 1.2.5).

[2]It would seem natural that the co-notion of a coboundary Hopf algebra is a boundary Hopf algebra. However, the term "coboundary Hopf algebra" is motivated in [4] by cohomological considerations applying to the classical limit of certain quantum deformed universal enveloping algebras. We will avoid to use the term boundary without any explicit homology behind it.

Chapter 3

Coquasitriangular Hopf algebras and the quantum Yang-Baxter equation

For this chapter, we again assume given a symmetric monoidal category $\mathcal{A}$ which is cocomplete and such that the tensor product commutes with arbitrary colimits. We denote the neutral object by k to conform with the important special case $\mathcal{A} = \underline{Vec}$, and denote by $\mathcal{A}_0$ the category of all objects of $\mathcal{A}$ having a left dual (which is the category of finite dimensional vector spaces if $\mathcal{A} = \underline{Vec}$.

3.1 The quantum Yang-Baxter equation

3.1.1. Definition: *Let $\mathcal{C}$ be a monoidal category and let $A, B, C \in \mathrm{Ob}\,\mathcal{C}$.*
Let $R : A \otimes B \to B \otimes A$, $S : A \otimes C \to C \otimes A$ and $T : B \otimes C \to C \otimes B$ be morphisms in $\mathcal{C}$. The **quantum Yang-Baxter equation** *for R, S and T is*

$$(T \otimes 1)(1 \otimes S)(R \otimes 1) = (1 \otimes R)(S \otimes 1)(1 \otimes T),$$

that is, commutativity of

$$\begin{tikzcd}
 & A \otimes B \otimes C \arrow[dl, "R \otimes 1"'] \arrow[dr, "1 \otimes T"] & \\
B \otimes A \otimes C \arrow[d, "1 \otimes S"'] & & A \otimes C \otimes B \arrow[d, "S \otimes 1"] \\
B \otimes C \otimes A \arrow[dr, "T \otimes 1"'] & & C \otimes A \otimes B \arrow[dl, "1 \otimes R"] \\
 & C \otimes B \otimes A &
\end{tikzcd}$$

Let Λ be some index set. Let the pair $\mathfrak{S} = ((V_a|a \in \Lambda), (S_{ab}|a, b \in \Lambda))$ consist of a family of objects $V_a \in \mathrm{Ob}\,\mathcal{C}$ and a family of isomorphisms $S_{ab} : V_a \otimes V_b \to V_b \otimes V_a$ in $\mathcal{C}$.

$\mathfrak{S}$ is said to be a solution of the quantum Yang-Baxter equation in $\mathcal{C}$, or a family of Yang-Baxter operators, if for all $a, b, c \in \Lambda$ the quantum Yang-Baxter equation for the morphisms S_{ab}, S_{ac} and S_{bc} is fulfilled.
$\mathcal{S}$ is **unitary**, *if $S_{ab}^{-1} = S_{ba}$ holds for all $a, b \in \Lambda$.*
$\mathcal{S}$ is **rigid**, *if all the V_a admit left duals in $\mathcal{C}$, and all the morphisms $S_{ab}^{\flat}$ and $\left(S_{ab}^{-1}\right)^{\flat}$ are isomorphisms.*

3.1.2. Remark: *Yang-Baxter operators defined on finite dimensional vector spaces can of course be expressed by their representing matrices with respect to suitable bases in the twofold tensor products. Two conventions are in use, [12, p.15]: For simplicity we consider only one morphism $S : V \otimes W \to W \otimes V$. Let $\{v_i\}$ and $\{w_j\}$ be bases in V and W, respectively.*

(1) For the so-called R-matrices we have

$$S(v_i \otimes w_j) = R_{ij}^{\ell k} w_k \otimes v_\ell,$$

thus R is the representing matrix with respect to the basis $\{v_i \otimes w_j\}$ in $V \otimes W$ of the mapping $\tilde{S} : V \otimes W \to V \otimes W$ defined by $\tilde{S} = \tau_{W,V} S$

(2) The so-called B-matrices fulfill

$$S(v_i \otimes w_j) = B_{ij}^{k\ell} w_k \otimes v_\ell$$

thus B represents S with respect to the bases $\{v_i \otimes w_j\}$ in $V \otimes W$ and $\{w_j \otimes v_i\}$ in $W \otimes V$.

Of course, $R_{ij}^{k\ell} = B_{ij}^{\ell k}$.

In [10], Lyubashenko defines a mapping

$$(-)^{\flat} : \mathrm{Hom}(V \otimes W, W \otimes V) \to \mathrm{Hom}(W^* \otimes V, V \otimes W^*)$$

in terms of the representing matrices with respect to some bases of the maps involved. We have to verify that our definition gives the correct coordinate-free translation of Lyubashenko's definition:

3.1.3. Lemma: *Let $S : V \otimes W \to W \otimes V$ be a k-linear morphism. Let for bases $\{v_i\}$ in V and $\{w_j\}$ in W*

$$S(v_i \otimes w_j) = B_{ij}^{k\ell} w_k \otimes v_\ell.$$

Let $\{v^i\}$ and $\{w^j\}$ be the dual bases in V^* and W^*, respectively. Then

$$S^{\flat}(w^j \otimes v_i) = B_{ik}^{j\ell} v_\ell \otimes w^k$$

$$S^{\flat\flat}(v^i \otimes w^j) = B_{\ell k}^{ji} w^k \otimes v^\ell$$

PROOF: Let $S' : W^* \otimes V \to V \otimes W^*$ and $S'' : V^* \otimes W^* \to W^* \otimes V^*$ be the morphisms defined by the two formulas written above. We have to verify they fulfill the conditions in definition 1.1.12. But indeed

$$\begin{aligned}(1 \otimes \mathrm{ev})(S' \otimes 1)(w^j \otimes v_i \otimes w_m) &= (1 \otimes \mathrm{ev})(B^{j\ell}_{ik} v_\ell \otimes w^k \otimes w_m) \\ &= B^{j\ell}_{im} v_\ell \\ (\mathrm{ev} \otimes 1)(1 \otimes S)(w^j \otimes v_i \otimes w_m) &= (\mathrm{ev} \otimes 1)(w^j \otimes B^{k\ell}_{im} w_k \otimes v_\ell) \\ &= B^{j\ell}_{im} v_\ell,\end{aligned}$$

proving $S' = S^\flat$, and

$$\begin{aligned}(1 \otimes \mathrm{ev})(S'' \otimes 1)(v^i \otimes w^j \otimes v_m) &= (1 \otimes \mathrm{ev})(B^{ji}_{\ell k} w^k \otimes v^\ell \otimes v_m) \\ &= B^{ji}_{mk} w^k \\ (\mathrm{ev} \otimes 1)(1 \otimes S^\flat)(v^i \otimes w^j \otimes v_m) &= (\mathrm{ev} \otimes 1)(v^i \otimes B^{j\ell}_{mk} v_\ell \otimes w^k) \\ &= B^{ji}_{mk} w^k,\end{aligned}$$

proving $S'' = S^{\flat\flat}$. □

The following lemma states that a quasisymmetric monoidal category with a monoidal underlying functor produces a solution of the quantum Yang-Baxter equation.

3.1.4. Lemma: *Let $(\mathcal{C}, (\omega, \xi)) \in \mathfrak{C}_{\otimes q}$ with quasisymmetry σ.*

Then $\mathfrak{S}_{\mathcal{C}} = ((\omega X | X \in \mathrm{Ob}\,\mathcal{C}), (\xi^{-1}_{YX}\, \omega(\sigma_{XY})\, \xi_{XY} | X, Y \in \mathrm{Ob}\,\mathcal{C}))$ is a solution of the quantum Yang-Baxter equation in $\mathcal{A}$.

If σ is a symmetry, then $\mathfrak{S}_{\mathcal{C}}$ is unitary.
If $\mathcal{C}$ is rigid, then so is $\mathfrak{S}_{\mathcal{C}}$.

PROOF: Obviously we only have to prove that $((X | X \in \mathrm{Ob}\,\mathcal{C}), (\sigma_{XY} | X, Y \in \mathrm{Ob}\,\mathcal{C}))$ is a solution of the quantum Yang-Baxter equation in $\mathcal{C}$. (That $\mathfrak{S}_{\mathcal{C}}$ is rigid if $\mathcal{C}$ is follows from 1.2.7, the statement on symmetry and unitarity is clear.)

Functoriality of σ provides us with a commutative diagram

$$\begin{tikzcd} X \otimes Y \otimes Z \arrow[r, "\sigma_{X \otimes Y, Z}"] \arrow[d, "\sigma_{XY} \otimes Z"'] & Z \otimes X \otimes Y \arrow[d, "Z \otimes \sigma_{XY}"] \\ Y \otimes X \otimes Z \arrow[r, "\sigma_{Y \otimes X, Z}"'] & Z \otimes Y \otimes X \end{tikzcd}$$

If we replace the horizontal arrows according to the definition of a quasisymmetry by $(\sigma_{XZ} \otimes Y)(X \otimes \sigma_{YZ})$ and $(\sigma_{YZ} \otimes X)(Y \otimes \sigma_{XZ})$, respectively, we get the quantum Yang-Baxter equation. □

3.1.5. Remark: *Let $\mathfrak{S} = ((V_a | a \in \Lambda), (S_{ab} | a, b \in \Lambda))$ be a solution of the quantum Yang-Baxter equation, and let $(\mathcal{C}, \omega)$ be a monoidal category with a quasisymmetry σ.*

Let $(X_a | a \in \Lambda)$ be a family of objects in $\mathcal{C}$ with $\omega(X_a) = V_a$. Now if for all $a, b \in \Lambda$ we have

$$S_{ab} = \xi^{-1}_{X_b,X_a}\, \omega(\sigma_{X_a,X_b})\, \xi_{X_a,X_b},$$

then we could say that the quasisymmetric monoidal category $(\mathcal{C}, \omega)$ "extends" $\mathfrak{S}$. We will look for extensions of a solution of the quantum Yang-Baxter equation in this sense in the next section.

By 2.4.5 the category of finite dimensional comodules over a coquasitriangular bialgebra is a quasisymmetric monoidal category. The solution of the quantum Yang-Baxter equation thus given (by lemma 3.1.4 just proved) is mirrored in the quasitriangular bialgebra as an equation for the coquasitriangular structure:

3.1.6. Corollary: *Let (B, r) be a coquasitriangular bialgebra. Then we have*

$$r^{23} * r^{13} * r^{12} = r^{12} * r^{13} * r^{23},$$

that is, evaluated on elements $a, b, c \in B$:

$$r(b_{(1)} \otimes c_{(1)})r(a_{(1)} \otimes c_{(2)})r(a_{(2)} \otimes b_{(2)}) = r(a_{(1)} \otimes b_{(1)})r(a_{(2)} \otimes c_{(2)})r(b_{(2)} \otimes c_{(2)})$$

Proof: Let σ be the quasisymmetry in $\mathcal{M}^B$ from 2.4.5. Let X, Y, Z be three (finite dimensional) right B-comodules, let $x \in X, y \in Y, z \in Z$. Then we have

$$\begin{aligned}
&(\sigma \otimes 1)(1 \otimes \sigma)(\sigma \otimes 1)(x \otimes y \otimes z) \\
&\quad = (\sigma \otimes 1)(1 \otimes \sigma)(y_{(0)} \otimes x_{(0)} \otimes z)r(x_{(1)} \otimes y_{(1)}) \\
&\quad = (\sigma \otimes 1)(y_{(0)} \otimes z_{(0)} \otimes x_{(0)})r(x_{(1)} \otimes z_{(1)})r(x_{(2)} \otimes y_{(1)}) \\
&\quad = z_{(0)} \otimes y_{(0)} \otimes x_{(0)}\, r(y_{(1)} \otimes z_{(1)})r(x_{(1)} \otimes z_{(2)})r(x_{(2)} \otimes y_{(2)})
\end{aligned}$$

and on the other hand

$$\begin{aligned}
&(1 \otimes \sigma)(\sigma \otimes 1)(1 \otimes \sigma)(x \otimes y \otimes z) \\
&\quad = (1 \otimes \sigma)(\sigma \otimes 1)(x \otimes z_{(0)} \otimes y_{(0)})r(y_{(1)} \otimes z_{(1)}) \\
&\quad = (1 \otimes \sigma)(z_{(0)} \otimes x_{(0)} \otimes y_{(0)})r(x_{(1)} \otimes z_{(1)})r(y_{(1)} \otimes z_{(2)}) \\
&\quad = z_{(0)} \otimes y_{(0)} \otimes x_{(0)}\, r(x_{(1)} \otimes y_{(1)})r(x_{(2)} \otimes z_{(1)})r(y_{(2)} \otimes z_{(2)})
\end{aligned}$$

The claim follows from the universal property of $B^{\otimes 3} \cong \underline{\mathrm{coend}}(\mathcal{U}^{\otimes 3})$, $\mathcal{U}$ denoting the forgetful functor. □

To be able to extend, in the next section, rigid solutions of the quantum Yang-Baxter equation to *rigid* quasisymmetric monoidal categories, we will, in particular, have to extend a rigid solution to a solution involving also the linear duals of the vector spaces on which the original family is defined.

3.1.7. Lemma: *Let A, B, R, S, T be as in definition 3.1.1. Let C^* be a left dual of C in $\mathcal{C}$. Then the quantum Yang-Baxter equation for R, S and T is equivalent to the quantum Yang-Baxter equation for $S^\flat, T^\flat$ and R.*

PROOF: The assertion follows directly from the formulas 1.1.15 for $(-)^\flat$: We have

$$\begin{aligned}\left(1\otimes S^\flat\right)\left(T^\flat\otimes 1\right)(1\otimes R) &= ((T\otimes 1)(1\otimes S))^\flat(1\otimes R) && \text{by 1.1.15}\\ &= ((T\otimes 1)(1\otimes S)(R\otimes 1))^\flat && \text{by 1.1.14}\end{aligned}$$

and

$$\begin{aligned}(R\otimes 1)\left(1\otimes T^\flat\right)\left(S^\flat\otimes 1\right) &= (R\otimes 1)((S\otimes 1)(1\otimes T))^\flat && \text{by 1.1.15}\\ &= ((1\otimes R)(S\otimes 1)(1\otimes T))^\flat && \text{by 1.1.14,}\end{aligned}$$

Since $(-)^\flat$ is bijective, we are done. □

3.1.8. Corollary: *Let A, B, C, R, S and T be as above, such that A, B and C admit left duals in $\mathcal{C}$. Let $R^\flat, S^\flat$ and $T^\flat$ be isomorphisms. Then the quantum Yang-Baxter equations for*

(1) R, S *and* T

(2) $\left(R^\flat\right)^{-1}, S$ *and* $\left(T^{-1}\right)^\flat$

(3) $R, \left(S^\flat\right)^{-1}$ *and* $\left(T^\flat\right)^{-1}$

(4) $\left(R^{-1}\right)^\flat, \left(S^{-1}\right)^\flat$ *and* T

are equivalent.

PROOF: If we turn around the hexagon whose commutativity is required in the quantum Yang-Baxter equation to put different vertices on top, and invert those isomorphisms involved which then go up the page instead of down, we get analogous diagrams in which the three isomorphisms occur in a different order and partly inverted. Using the lemma just proved, this procedure gives us all the equivalences stated (and quite a few more). □

3.1.9. Corollary: *Let $\mathfrak{S} = ((V_a|a\in\Lambda), (S_{ab}|a,b\in\Lambda))$ be a rigid solution of the quantum Yang-Baxter equation in $\mathcal{A}$. Let $\Lambda^* := \Lambda\dot\cup\{a^*|a\in\Lambda\}$ and $V_{a^*} := V_a^*$ for all $a\in\Lambda$. Setting, for all $a, b\in\Lambda$,*

$$S_{a^*b} = (S_{ab}^{-1})^\flat, \quad S_{ab^*} = (S_{ab}^\flat)^{-1}, \quad S_{a^*b^*} = (S_{ab}^\flat)^\flat,$$

we obtain another solution

$$\mathfrak{S}^* := ((V_x|x\in\Lambda^*), (S_{xy}|x,y\in\Lambda^*))$$

of the quantum Yang-Baxter equation.

Proof: With the definion just given the equations $S_{a^*x} = (S_{ax}^{-1})^\flat$ and $S_{xb^*} = (S_{xb}^\flat)^{-1}$ even hold for all $x \in \Lambda^*$ and $a, b \in \Lambda$. For example, we have for $x = c^*, c \in \Lambda$:

$$S_{a^*x} = S_{a^*c^*} = (S_{ac}^\flat)^\flat = (((S_{ac}^\flat)^{-1})^{-1})^\flat = (S_{ac^*}^{-1})^\flat.$$

Now we can prove the quantum Yang-Baxter equation for any three of the morphisms S_{xy}, $x, y \in \Lambda^*$, by applying the last corollary. □

3.1.10. Remark: *Let $(\mathcal{C}, \omega)$ be an extension in the sense of 3.1.5 of the solution $\mathfrak{S}$ of the quantum Yang-Baxter equation, and let $\mathcal{C}$ be rigid. Then it follows from 1.2.7 that $(\mathcal{C}, \omega)$ is also an extension of $\mathfrak{S}^*$ (Up to the functorial isomorphism $\omega(X^*) \cong \omega(X)^*$ from 1.1.8).*

3.2 Universal coquasitriangular Hopf algebras

In this section we will, given (rigid if necessary) solutions of the quantum Yang-Baxter equation, construct coquasitriangular bialgebras (and Hopf algebras) having the objects on which the solutions are given as comodules, such that the respective Yang-Baxter operators can be written as "actions"of the coquasitriangular structure of the bialgebra on tensor products (as in 2.4.5). More precisely the bialgebras (Hopf algebras) will be universal among the bialgebras (Hopf algebras) with this property.

3.2.1 Universal quasisymmetric categories

First we construct, given a solution of the quantum Yang-Baxter equation, a quasisymmetric monoidal category extending this solution.

3.2.1. Lemma: *Let $(V_a | a \in \Lambda)$ be a family of objects of $\mathcal{A}_0$. Then there is a $(\mathcal{C}, \omega) \in \mathfrak{C}_\otimes$ and a family $(X_a | a \in \Lambda)$ satisfying*

(1) $\omega(X_a) = V_a$

(2) If $(\mathcal{C}', \omega') \in \mathfrak{C}_\otimes$ and $(X'_a | a \in \Lambda)$ is a family of objects of $\mathcal{C}'$ with $\omega'(X'_a) = V_a$, then there is a unique $[\mathcal{F}, \zeta] : (\mathcal{C}, \omega) \to (\mathcal{C}', \omega')$ in $\mathfrak{C}_\otimes$ satisfying $\mathcal{F}(X_a) = X'_a$.

We write $\mathcal{C} = \langle V_a | a \in \Lambda \rangle$ and $\omega = \omega_{\langle (V_a) \rangle}$.

To be precise, we should of course, in (2), require a unique morphism in $\mathfrak{C}_\otimes$ having a representative satisfying the condition stated.

Proof: Let the set of objects of $\mathcal{C}$ be the free monoid over the set Λ. With only the identical morphisms and with the multiplication in the free monoid as the tensor product $\mathcal{C}$ then becomes a strictly monoidal category. It is a direct consequence of the coherence theorem for monoidal categories that this monoidal category has the following universal property $(*)$:

($*$) For each monoidal category $\mathcal{D}$ and each family $(Y_a|a \in \Lambda)$ of objects in $\mathcal{D}$ there is a monoidal functor $\mathcal{G} : \mathcal{C} \to \mathcal{D}$ with $\mathcal{G}(a) = Y_a$ which is unique up to a unique monoidal isomorphism.

Define $(X_a|a \in \Lambda)$ by just setting $X_a = a$. According to ($*$), choose a monoidal functor $\omega : \mathcal{C} \to \mathcal{A}_0$ with $\omega(a) = V_a$ for all $a \in \Lambda$.

Let $(\mathcal{C}', \omega')$ also be in $\mathfrak{C}_\otimes$ with $\omega'(a) = V_a$ for all $a \in \Lambda$. Then, first, there is, up to monoidal isomorphism, a unique monoidal functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ with $\mathcal{F}(a) = X_a$. Since then $\omega'\mathcal{F}$ is a monoidal functor, for which $\omega'\mathcal{F}(a) = \omega'(X_a) = V_a$ holds, there is, second, a unique monoidal isomorphism $\zeta : \omega \cong \omega'\mathcal{F}$, so that we obtain a morphism $[\mathcal{F}, \zeta]$ in $\mathfrak{C}_\otimes$ as desired. If, third, $\mathcal{G} : \mathcal{C} \to \mathcal{C}'$ is another monoidal functor with $\mathcal{G}(a) = X_a$ and $\vartheta : \omega \cong \omega'\mathcal{G}$ is a monoidal isomorphism, then there is a unique monoidal isomorphism $\varphi : \mathcal{F} \cong \mathcal{G}$, and since $\omega'(\varphi)\zeta : \omega \cong \omega'\mathcal{G}$ is a monoidal isomorphism, we have $\omega'(\varphi) \circ \zeta = \vartheta$, and thus $[\mathcal{G}, \vartheta] = [\mathcal{F}, \zeta]$ in $\mathfrak{C}_\otimes$. $\square$

3.2.2. Lemma: *Let $(\mathcal{C}, \omega)$ be the category defined in the preceding lemma. For all $a, b \in \Lambda$ let $S_{ab} : V_a \otimes V_b \to V_b \otimes V_a$ be an isomorphism. Then there is a unique family of isomorphisms*

$$S_{XY} : \omega(X) \otimes \omega(Y) \to \omega(Y) \otimes \omega(X)$$

with

- *The morphisms S_{ab} for $a, b \in \Lambda$ are the ones given.*
- $\forall X, Y, Z \in \mathcal{C}: \quad S_{X,Y\otimes Z} = (1 \otimes S_{XZ})(S_{XY} \otimes 1).$
- $\forall X, Y, Z \in \mathcal{C}: \quad S_{X\otimes Y,Z} = (S_{XZ} \otimes 1)(1 \otimes S_{YZ}).$
- *For all $X \in \mathcal{C}$ S_{XI} and S_{IX} are the coherence isomorphisms.*

Proof: Clearly there is exactly one family of isomorphisms such that only

- $\forall a, c \in \Lambda, Y \in \mathcal{C}: \quad S_{a,Y\otimes c} = (1 \otimes S_{ac})(S_{aY} \otimes 1).$
- $\forall b \in \Lambda, X, Z \in \mathcal{C}: \quad S_{X\otimes b,Z} = (S_{XZ} \otimes 1)(1 \otimes S_{bZ}).$
- For all $a \in \Lambda$ S_{aI} and S_{Ia} are the coherence isomorphisms.

hold. But then all the other equations required hold as well, as can be proved by induction on the "length" of the objects in $\mathcal{C}$ involved. As an example, we only write one of the required steps in the induction:
Assume that, for some $X, Y \in \mathcal{C}$ and $c \in \Lambda$, the equation

$$S_{X,Y\otimes c} = (1 \otimes S_{Xc})(S_{XY} \otimes 1)$$

holds, and let $a \in \Lambda$. Then we also get

$$\begin{aligned} S_{X\otimes a,Y\otimes c} &= (S_{X,Y\otimes c}\otimes 1)(1\otimes S_{a,Y\otimes c}\\ &= (1\otimes S_{Xc}\otimes 1)(S_{XY}\otimes 1\otimes 1)(1\otimes 1\otimes S_{ac})(1\otimes S_{aY}\otimes 1)\\ &= (1\otimes S_{Xc}\otimes 1)(1\otimes 1\otimes S_{ac})(S_{XY}\otimes 1\otimes 1)(1\otimes S_{aY}\otimes 1)\\ &= (1\otimes S_{X\otimes a,c})(S_{X\otimes a,Y}\otimes 1)\end{aligned}$$

Thus we see, by induction, that

$$\forall X,Y\in\mathcal{C}\forall c\in\Lambda: \quad S_{X,Y\otimes c}=(1\otimes S_{Xc})(S_{XY}\otimes 1)$$

also holds. □

The result just proved will serve to construct, from a solution of the quantum Yang-Baxter equation, a quasisymmetry for a suitable category (whose objects will just be the objectos of $\langle V_a\rangle$). This quasisymmetry will have to be a natural transformation. It will be sufficient to prove the conditions describing naturality with respect to a morphism in the category only for the "generating objects" V_a; this is made precise by the following plain result:

3.2.3. Lemma: *Let $(\mathcal{C},\omega)$ be the category $\langle V_a\rangle$ defined in lemma 3.2.1. Let $f : \omega(X)\to\omega(Y)$ be a morphism such that*

$$\forall a\in\Lambda: \begin{cases} S_{Ya}(f\otimes 1)=(1\otimes f)S_{Xa}\\ S_{aY}(1\otimes f)=(f\otimes 1)S_{aX}\end{cases}$$

holds. Then

$$\forall Z\in\mathcal{C}: \begin{cases} S_{YZ}(f\otimes 1)=(1\otimes f)S_{XZ}\\ S_{ZY}(1\otimes f)=(f\otimes 1)S_{ZX}\end{cases}$$

also holds.

Proof: By induction on the length of Z. We only do the induction step for the first of the equations. Assume this has been proved for some Z, and let $a\in\Lambda$, then we have:

$$\begin{aligned} S_{Y,Z\otimes a}(f\otimes 1_{Z\otimes a}) &= (1\otimes S_{Ya})(S_{YZ}\otimes 1)(f\otimes 1\otimes 1)\\ &= (1\otimes S_{Ya})(1\otimes f\otimes 1)(S_{XZ}\otimes 1)\\ &= (1\otimes 1\otimes f)(1\otimes S_{Xa})(S_{XZ}\otimes 1)\\ &= (1_{Z\otimes a}\otimes f)S_{X,Z\otimes a}\end{aligned}$$

□

The monoidal category constructed in 3.2.1 has almost no morphisms (only the identities). The quasisymmetric or rigid quasisymmetric monoidal categories we want to construct here, however, will have to have certain structure morphisms.

The following lemma shall provide a machinery for adding morphisms to a given monoidal category over $\mathcal{A}_0$.

3.2.4. Lemma: *Let $\mathcal{C}$ be a monoidal category and let $\omega : \mathcal{C} \to \mathcal{A}_0$ be a faithful strictly monoidal functor.*
Let $F = (f_i : \omega(Q_i) \to \omega(Z_i) | i \in N)$ be a family of morphisms (with $Q_i, Z_i \in \mathrm{Ob}\,\mathcal{C}$). Then there is a monoidal category $\mathcal{C}_F$, a faithful strictly monoidal functor $\omega_F : \mathcal{C}_F \to \mathcal{A}_0$, a strictly monoidal functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}_F$ and a family $F_i : \mathcal{F}Q_i \to \mathcal{F}Z_i$ of morphisms, such that

(1) $\omega_F(F_i) = f_i$.

(2) If $(\mathcal{C}', \omega') \in \mathfrak{C}_\otimes$ with ω' faithful, $[\mathcal{F}', \zeta'] : (\mathcal{C}, \omega) \to (\mathcal{C}', \omega')$ and if $(F_i' : \mathcal{F}'Q_i \to \mathcal{F}'Z_i)$ is a family of morphisms with $\omega'(F_i') = f_i$ (up to coherence morphisms), then there is a unique $[\mathcal{G}, \vartheta] : (\mathcal{C}_F, \omega_F) \to (\mathcal{C}', \omega')$ with $\mathcal{G}\mathcal{F} = \mathcal{F}'$ in $\mathfrak{C}$ and $\mathcal{G}(F_i) = F_i'$ up to coherence morphisms.

PROOF: Define $\mathrm{Ob}\,\mathcal{C}_F = \mathrm{Ob}\,\mathcal{C}$, let the object map of the functor ω_F be the same as the one of ω.
We define the morphism sets of $\mathcal{C}_F$ to be, by and large, subsets of the morphism sets between the underlying vector spaces, where we only have to add the domain and codomaind as indices to force the sets of morphisms to be disjoint. Thus, say,

$$\mathcal{M}\mathrm{or}_{\mathcal{C}_F}(X, Y) \subset \{(X, Y)\} \times \mathrm{Hom}_k(\omega(X), \omega(Y))$$

We define the functor ω_F as just "forgetting domain and codomain" (which we shall, as a purely technical gadget, forget anyway in the sequel.)
With these preliminaries fixed, let the morphism sets be the smallest sets such that

(1) $\omega(\mathcal{M}\mathrm{or}_{\mathcal{C}}(X, Y)) \subset \mathcal{M}\mathrm{or}_{\mathcal{C}_F}(X, Y)$

(2) $F_i := f_i \in \mathcal{M}\mathrm{or}_{\mathcal{C}_F}(Q_i, Z_i)$

(3) $g \in \mathcal{M}\mathrm{or}_{\mathcal{C}_F}(X, Y), h \in \mathcal{M}\mathrm{or}_{\mathcal{C}_F}(Y, Z) \;\Rightarrow\; hg \in \mathcal{M}\mathrm{or}_{\mathcal{C}_F}(X, Z)$

(4) $g \in \mathcal{M}\mathrm{or}_{\mathcal{C}_F}(X, Y), h \in \mathcal{M}\mathrm{or}_{\mathcal{C}_F}(A, B) \;\Rightarrow\; g \otimes h \in \mathcal{M}\mathrm{or}_{\mathcal{C}_F}(X \otimes A, Y \otimes B)$

Then $\mathcal{C}_F$ has an obvious monoidal structure, ω_F is monoidal by definition, and there is a monoidal functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}_F$ with $\omega_F \mathcal{F} = \omega$.

If we are given the set of data in part (2) of the assertion, then we construct the functor $\mathcal{G}$ required there in the obvious manner:

(1) $\mathcal{G}|_{\mathcal{C}} = \mathcal{F}'$.

(2) $\mathcal{G}(F_i) = F_i'$

(3) $\mathcal{G}(hg) = \mathcal{G}(h)\mathcal{G}(g)$

(4) $\mathcal{G}(g \otimes h) = \mathcal{G}(h) \otimes \mathcal{G}(g)$

With the natural transformation $\vartheta = id : \omega_F \to \omega'\mathcal{G}$ we see that $[\mathcal{G}, \vartheta]$ has the property required. If $[\mathcal{G}', \vartheta']$ is another such morphism in $\mathfrak{C}_{\otimes}$, then the isomorphism $\mathcal{G}\mathcal{F} \xrightarrow{\kappa} \mathcal{G}'\mathcal{F}$ for which we have $(\mathcal{G}\mathcal{F}, id) \overset{\kappa}{\sim} (\mathcal{G}'\mathcal{F}, \vartheta')$ is also a natural isomorphism $\mathcal{G} \cong \mathcal{G}'$, since it is, by assumption, compatible with the morphisms F_i, and thus with all the other morphisms in $\mathcal{C}_F$. $\square$

Since all the morphisms in the category $\mathcal{C}_F$ thus constructed arise from the morphisms in the family F by forming tensor products and composing, we note immediately

3.2.5. Lemma: *Assume the "compatibility conditions" from 3.2.3 are fulfilled for all the f_i in the family F. Then they are also fulfilled for all the morphisms in $\mathcal{C}_F$, that is, the morphisms S from 3.2.2 constitute a natural transformation*

$$S_{XY} : \omega_F(X) \otimes \omega_F(Y) \to \omega_F(Y) \otimes \omega_F(X).$$

PROOF: Assume the compatibility condition is fulfilled for morphisms g and h. Clearly it then also holds for gh. For the tensor product of morphisms, we get, for example

$$\begin{aligned}(S \otimes 1)(1 \otimes S)(\omega f \otimes \omega g \otimes 1) &= (S \otimes 1)(\omega f \otimes 1 \otimes \omega g)(1 \otimes S)\\ &= (1 \otimes \omega f \otimes \omega g)(S \otimes 1)(1 \otimes S)\end{aligned}$$

Thus the compatibility condition also holds for $f \otimes g$. $\square$

3.2.6. Corollary: *Let $\mathfrak{S} = ((V_a|a \in \Lambda), (S_{ab}|a, b \in \Lambda))$ be a solution of the quantum Yang-Baxter equation in the category $\mathcal{A}_0$. Then there is a quasisymmetric monoidal category $(\mathcal{C}, \omega) = (\langle \mathfrak{S} \rangle, \omega_{\mathfrak{S}}) \in \mathfrak{C}_{\otimes q}$ with quasisymmetry σ and family of objects X_a in $\mathcal{C}$ such that*

(1) The diagrams

$$\begin{tikzcd} \omega(X_a \otimes X_b) \arrow[r, "\sigma_{X_a,X_b}"] \arrow[d, phantom, "\wr\|"] & \omega(X_b \otimes X_a) \arrow[d, phantom, "\|\wr"] \\ V_a \otimes V_b \arrow[r, "S_{ab}"'] & V_b \otimes V_a \end{tikzcd}$$

commute (that is, $(\mathcal{C}, \omega)$ "extends the solution given").

(2) If also $(\mathcal{C}', \omega') \in \mathfrak{C}_{\otimes q}$ with quasisymmetry σ', and if there is a family X'_a of objects in $\mathcal{C}'$ such that analogous diagrams commute, then there is a unique morphism $[\mathcal{G}, \vartheta] : (\langle V_a \rangle, \omega) \to (\mathcal{C}', \omega')$ in $\mathfrak{C}_{\otimes q}$ such that $\mathcal{G}X'_a = X_a$.

PROOF: First let $(\mathcal{C}_0, \omega_0)$ be the "free" monoidal category from 3.2.1. Applying 3.2.4 to the morphisms S_{XY} from 3.2.2 yields the desired category. To see this, we only have to show (by 1.2.8) that σ defines a quasisymmetry for the category constructed, that is, that the morphisms σ_{XY} in $\mathcal{C}$ constitute a natural transformation (for they fulfill the defining equations for a quasisymmetry, anyway). For this in turn it is sufficient by 3.2.3 and 3.2.5, to check only

$$S_{a,b\otimes c}(1\otimes S_{bc})=(S_{bc}\otimes 1)S_{a,b\otimes c}$$

$$S_{a\otimes b,c}(S_{ab}\otimes 1)=(1\otimes S_{ab})S_{a\otimes b,c}.$$

But by definition of the morphisms S each of these two equations is just the quantum Yang-Baxter equation. $\square$

3.2.7. Corollary: *Let $\mathfrak{S}=((V_a|a\in\Lambda),(S_{ab}|a,b\in\Lambda))$ be a rigid solution of the quantum Yang-Baxter equation in the category $\mathcal{A}_0$. Then there is a rigid monoidal category $(\langle\mathfrak{S}\rangle_*,\omega)\in\mathfrak{C}_{\otimes *q}$ with quasisymmetry σ and a family of objects X_a in $\mathcal{C}$ such that*

(1) The diagrams

$$\begin{tikzcd}
\omega(X_a\otimes X_b) \arrow[rr, "\sigma_{X_a,X_b}"] \arrow[d, equal, "\wr"'] & & \omega(X_b\otimes X_a) \arrow[d, equal, "\wr"] \\
V_a\otimes V_b \arrow[rr, "S_{ab}"'] & & V_b\otimes V_a
\end{tikzcd}$$

commute.

(2) If also $(\mathcal{C}',\omega')\in\mathfrak{C}_{\otimes *q}$ with quasisymmetry σ', and if X'_a is a family of objects in $\mathcal{C}'$ such that analogous diagrams commute, then there is a unique morphism $[\mathcal{G},\vartheta]:(\langle V_a\rangle,\omega)\to(\mathcal{C}',\omega')$ in $\mathfrak{C}_{\otimes q}$ such that $\mathcal{G}X'_a=X_a$.

PROOF: First apply 3.2.6 to the solution $\mathfrak{S}^*$ of the quantum Yang-Baxter equation defined in 3.1.9. Denote the category obtained by $(\mathcal{C}_0,\omega_0)$. Now apply 3.2.4 once more to the morphisms $\mathrm{ev}:V_a^*\otimes V_a\to k$ and $\mathrm{db}:k\to V_a\otimes V_a^*$. This yields the desired category.

To see this, it is by 1.2.8 and 3.2.6 sufficient to check that the (unchanged) morphisms σ_{XY} still constitute a natural transformation, and that the category constructed is indeed rigid.

First σ_{XY} are still functorial, for by definition 3.1.9 the sufficient conditions for compatibility from 3.2.3 are fulfilled, e.g. for $x\in\Lambda^*$ we have

$$\begin{aligned}
(\mathrm{ev}\otimes 1)S_{x,a^*\otimes a} &= (\mathrm{ev}\otimes 1)(1\otimes S_{xa})(S_{xa^*}\otimes 1)\\
&= (1\otimes \mathrm{ev})(S^\flat_{xa}\otimes 1)(S_{xa}\otimes 1)\\
&= (1\otimes \mathrm{ev})\overset{\mathrm{coh}}{=}(1\otimes \mathrm{ev})S_{xI}
\end{aligned}$$

and the second condition on ev and the conditions on db are as easy to check.

By definition of the category $\mathcal{C}$ it is clear now that all the $a\in\Lambda$ do, considered as objects in $\mathcal{C}$, have left duals. But $\mathcal{C}$ is quasisymmetric, whence even all the objects $x\in\Lambda^*$ have left duals in $\mathcal{C}$. Since now all the objects in $\mathcal{C}$ are tensor products of objects having left duals, $\mathcal{C}$ is rigid. $\square$

3.2.8. Remark: *Since the category $\mathcal{C}$ constructed is rigid,*

$$S_{a^*X} = (S_{aX}^{-1})^\flat, \quad S_{Xa^*} = (S_{Xa}^\flat)^{-1}$$

now holds for arbitrary objects X in $\mathcal{C}$ by 1.2.7. Thus the formulas in 1.1.15 immediately yield that for the family of isomorphisms arising by 3.2.2 from the family $\mathfrak{S}^*$, the condition

$$\forall z \in \Lambda^* : \begin{cases} S_{Yz}(f \otimes 1) = (1 \otimes f)S_{Xz} \\ S_{zY}(1 \otimes f) = (f \otimes 1)S_{zX} \end{cases}$$

from 3.2.3 on some morphism f already follows from

$$\forall a \in \Lambda : \begin{cases} S_{Ya}(f \otimes 1) = (1 \otimes f)S_{Xa} \\ S_{aY}(1 \otimes f) = (f \otimes 1)S_{aX} \end{cases}$$

3.2.2 Universal coquasitriangular bialgebras and Hopf algebras

In the preceding section we have constructed universal (rigid) quasisymmetric categories from (rigid if necessary) solutions of the quantum Yang-Baxter equation. In [24] we have constructed universal coalgebras corresponding to categories with underlying functors. In the case of quasisymmetric categories, these coalgebras were shown to be coquasitriangular bialgebras in 2.4.5. The two constructions combined now easily yield universal coquasitriangular bialgebras and Hopf algebras corresponding to solutions of the quantum Yang-Baxter equation.

3.2.9. Theorem: *Let $\mathfrak{S} = ((V_a|a \in \Lambda), (S_{ab}|a, b \in \Lambda))$ be a (rigid) solution of the quantum Yang-Baxter equation in the category $\mathcal{A}_0$. Then there are a coquasitriangular bialgebra (resp. Hopf algebra) (H, r) and comodule structures $V_a \xrightarrow{\delta_a} V_a \otimes H$, written $V_a \ni x \mapsto x_{(0)} \otimes x_{(1)} \in V_a \otimes H$ such that we have*

(1) For all $a, b \in \Lambda$ the Yang-Baxter operator S_{ab} is given by

$$S_{ab}(x \otimes y) = y_{(0)} \otimes x_{(0)} r(x_{(1)} \otimes y_{(1)}).$$

(2) If (H', r') is another coquasitriangular bialgebra (resp. Hopf algebra) and δ'_a : $V_a \to V_a \otimes H'$ are comodule structures, denoted $V_a \ni x \mapsto x^{(0)} \otimes {}^{(1)} \in V_a \otimes H$, such that

$$S_{ab}(x \otimes y) = y^{(0)} \otimes x^{(0)} r'(x^{(1)} \otimes y^{(1)})$$

also holds, then there is a unique morphism $f : H \to H'$ of coquasitriangular bialgebras such that

$$\delta'_a(x) = x_{(0)} \otimes f(x_{(1)}) = (1 \otimes f)\delta_a$$

holds for all $a \in \Lambda$ and $x \in V_a$.

We denote the universal bialgebra (resp. Hopf algebra) from the theorem by $B_{\mathfrak{S}}$ (resp. $H_{\mathfrak{S}}$).

PROOF: We only prove the bialgebra version, the Hopf algebra version being perfectly analogous. Let $(\langle\mathfrak{S}\rangle, \omega)$ be the category from 3.2.6, let $H = \underline{\text{coend}}(\omega)$, equipped with the coquasitriangular structure from 2.4.5. Then in particular each V_a is a right H-comodule (via the functor $\mathcal{I}_{\mathcal{C}}$) such that the Yang-Baxter operator is given as stated, see 2.4.5. If (H', r') is another coquasitriangular bialgebra with the properties stated, then by 3.2.6 there is a unique morphism $\Phi : \mathcal{C} \to \mathcal{M}^{H'}$ in $\mathfrak{C}_{\otimes q}$ with $\Phi(V_a) = V_a$. But then there is a unique morphism of coquasitriangular bialgebras $f : H \to H'$ with the desired property. $\square$

3.2.10. Remark: *In the case $\mathcal{A} = \underline{\text{Vec}}$, we know from [24, thm. 2.2.8] that $\mathcal{M}^{H_{\mathfrak{S}}}$ is initial (in $\mathfrak{C}$) among the rigid quasisymmetric monoidal k-*linear abelian *categories that extend $\mathfrak{S}$ in the sense of 3.1.5. Of course, $\mathcal{M}^{H_{\mathfrak{S}}}$ has more objects than the category $\langle\mathfrak{S}\rangle_*$ defined in 3.2.7 (e.g. kernels of* ev *or direct sums), and also more morphisms even between objects already contained in $\langle\mathfrak{S}\rangle_*$ (e.g. zero morphisms or sums $id_{X\otimes X} + \sigma_{XX}$).*

We have described in [24, 2.3.13] how to express defining relations of the coendomorphism coalgebra in terms of representing matrices of a generating set of morphisms for the category in question. Hence we can express generators and relations for the algebras we have constructed in this section in terms of the B-matrices in the case $\mathcal{A} = \underline{\text{Vec}}$, showing $B_{\mathfrak{S}}$ to be the well known $A(R)$-construction of Fadeev-Reshetikhin-Takhtadzan:

3.2.11. Lemma: *Let $\mathfrak{S} = (V, S)$ with $S : V \otimes V \to V \otimes V$ be a solution of the quantum Yang-Baxter equation defined on a finite dimensional vector space. (just for simplicity — we could cope with several spaces as well, but where put the indices?) Let $\{v_i | i = 1, \ldots, N\}$ be a basis in V.*

(1) The algebra $B_{\mathfrak{S}}$ is generated by elements T_i^j, $(i, j = 1, ..., N)$ subject to the relations

$$T_n^\ell T_m^j B_{ik}^{nm} = B_{mn}^{\ell j} T_i^m T_k^n$$

(2) Let $\mathfrak{S}$ be rigid. Then the algebra $H_{\mathfrak{S}}$ is generated by elements T_i^j and U_i^j, $(i, j = 1, ..., N)$ subject to the relations

$$\begin{aligned} T_n^\ell T_m^j B_{ik}^{nm} &= B_{mn}^{\ell j} T_i^m T_k^n \\ U_k^j T_i^k &= \delta_i^j \\ T_k^j U_i^k &= \delta_i^j \end{aligned}$$

PROOF: We just have to apply [24, 2.3.13] to our setting. Indeed all the morphisms in $\langle\mathfrak{S}\rangle$ arise from S by forming inverses and tensor products and composition. All the morphisms in $\langle\mathfrak{S}\rangle_*$ arise by the same operations from S, ev_V and db_V. $\square$

Observe that in (2) commutation relations like $U_k^n U_i^m B_{mn}^{\ell j} = B_{ik}^{nm} U_m^j U_n^\ell$ between the U's and between the U's and T's involving the B-matrix are not needed (but hold, of course).

3.3 Some properties of coquasitriangular Hopf algebras

From this point on we restrict our "base category" $\mathcal{A}$ to be the category $\underline{Vec}$ of all vector spaces (which would, strictly speaking, not be necessary for the next section).

3.3.1 The square of the antipode

The following "computation rules" for coquasitriangular Hopf algebras are well known in the quasitriangular case and not too hard to verify, [8], [19]. I want to derive them from properties of quasisymmetric categories, and maybe thus give a kind of motivation for them.

3.3.1. Lemma: *Let (H, r) be a coquasitriangular Hopf algebra (for the first item a bialgebra would be sufficient). Then we have*

(1) $r(\eta \otimes id) = \varepsilon = r(id \otimes \eta)$

(2) $r(s \otimes id) = \overline{r}$

(3) $\overline{r}(id \otimes s) = r$

(4) $r(s \otimes s) = r$

PROOF:

(1) Since in the quasisymmetric category $\mathcal{M}^H$ we have $\sigma_{XI} = \lambda_X^{-1}\rho_X$ for each object X, we obtain for $x \in X$:

$$\begin{aligned} 1_k \otimes x_{(0)}\varepsilon(x_{(1)}) &= 1_k \otimes x \\ &= \lambda_X^{-1}\rho_X(x \otimes 1_k) \\ &= \sigma_{XI}(x \otimes 1_k) \\ &= 1_k \otimes x_{(0)} r(x_{(1)} \otimes 1_H), \end{aligned}$$

and thus $\varepsilon = r(id \otimes \eta)$. The other equation is as easy to check.

(2) We have the equation $\sigma_{X^*Y} = (\sigma_{XY}^{-1})^\flat$. For elements $x \in X, y \in Y, \varphi \in X^*$ we have

$$\begin{aligned} (1 \otimes \mathrm{ev})(\sigma_{X^*Y}(\varphi \otimes y) \otimes x) &= y_{(0)}\langle \varphi_{(0)}, x\rangle r(\varphi_{(1)} \otimes y_{(1)}) \\ &= y_{(0)}\langle \varphi, x_{(0)}\rangle r(s(x_{(1)}) \otimes y_{(1)}) \end{aligned}$$

and

$$(\mathrm{ev} \otimes 1)(\varphi \otimes \sigma_{XY}^{-1}(y \otimes x)) = \langle \varphi, x_{(0)}\rangle y_{(0)}\overline{r}(x_{(1)} \otimes y_{(1)})$$

proving our claim.

(3) follows from $\sigma_{XY^*} = (\sigma^\flat_{XY})^{-1}$ in an analogous manner.

(4) Follows immediately from the preceding two equations (and is related to the fact that $\sigma_{X^*Y^*} = \sigma^*_{XY}$).

(Note that we have proved (1) in section 2.5 already.) $\square$

It follows from 1.2.9 that for a rigid quasisymmetric monoidal category the antipode of its coendomorphism coalgebra is an isomorphism. For quasitriangular Hopf algebras a stronger result is known, [8], [19]: The square of the antipode is an inner automorphism. An analogous result can be obtained for coquasitriangular Hopf algebras, and it can be proved via properties of quasisymmetric monoidal categories.

3.3.2. Lemma: *Let (H, r) be a coquasitriangular Hopf algebra. Then the square of the antipode s of H is a coinner automorphism.*
More precisely, if we set $u := r(id \otimes s)\tau\Delta : H \to k$, that is $u(h) = r(h_{(2)} \otimes s(h_{(1)}))$, then

- *u is $*$-invertible (with $*$-inverse denoted $\overline{u}$).*
- *$s^2 = u * id * \overline{u}$*

Proof: By 1.2.10 we know that $j : X \to X^{**}$, defined by

$$j = (X \xrightarrow{1\otimes \mathrm{db}_{X^*}} X \otimes X^* \otimes X^{**} \xrightarrow{\sigma\otimes 1} X^* \otimes X \otimes X^{**} \xrightarrow{\mathrm{ev}\otimes 1} X^{**})$$

is an isomorphism, natural in X. Apply 2.2.3 (the part about coconjugate morphisms) to the morphisms $[\mathcal{I}d, id], [(—)^{**}, \iota] : \mathcal{M}^H \to \mathcal{M}^H$, where we denote by $\iota : M \to M^{**}$ the usual natural isomorphism between the underlying vector spaces of H-comodules (which is *not* colinear).

Now since

$$\begin{aligned}
j(x) &= (\mathrm{ev} \otimes 1)(\sigma \otimes 1)(1 \otimes \mathrm{db})(x)\\
&= (\mathrm{ev} \otimes 1)(\sigma \otimes 1)(x \otimes x^i \otimes \iota(x_i) \otimes x)\\
&= (\mathrm{ev} \otimes 1)(x^i{}_{(0)} \otimes x_{(0)} \otimes \iota(x_i))r(x_{(1)} \otimes x^i{}_{(1)})\\
&= \langle x^i{}_{(0)}, x_{(0)}\rangle \iota(x_i) r(x_{(1)} \otimes x^i{}_{(1)})\\
&= \iota(x_i)\langle x^i, x_{(0)}\rangle r(x_{(2)} \otimes s(x_{(1)}))\\
&= \iota(x_{(0)}) r(x_{(2)} \otimes s(x_{(1)}))
\end{aligned}$$

and $\delta(\iota(x)) = \iota(x_{(0)})s^2(x_{(1)})$, we indeed get, from 2.2.3, that the morphism of coalgebras $s^2 : H \to H$ is an inner automorphism $s^2 = u * id * \overline{u}$, where $u = r(id \otimes s)\tau\Delta$. $\square$

For (H, r) a coquasitriangular bialgebra over a field k, the bilinear form r defined on $H \otimes H$ yields two mappings from H to the dual vector space H^*. The $*$-inverse $\overline{r}$ of r yields another two such maps. A more detailed analysis shows all of those maps to be bialgebra morphisms to the dual bialgebra (defined as in [25]) of the (potentially infinite dimensional!) bialgebra H (with multiplication or comultiplication reversed). For cotriangular bialgebras or coquasitriangular Hopf algebras, part of the four maps can be expressed in terms of each other.

The maps from H to its own dual yield an H-*module* structure on each H-comodule. It is a specific property of coquasitriangular bialgebras that their comodules are modules at the same time.

3.3.2 The dual Hopf algebra

The k-linear dual of an arbitrary coalgebra C always becomes an algebra, if one defines the product of two elements $\varphi, \psi \in C^*$ by

$$\forall c \in C: \quad \langle \varphi\psi, c\rangle = \langle \varphi, c_{(1)}\rangle\langle \psi, c_{(2)}\rangle$$

The analogous construction of a dual coalgebra to an algebra fails for infinite dimensional algebras, since then the element $\Delta\varphi \in (A \otimes A)^*$ defined by

$$\forall a, b \in A: \quad \langle \Delta\varphi, a \otimes b\rangle = \langle \varphi, ab\rangle$$

in general fails to have an inverse image under the natural inclusion $A^* \otimes A^* \subset (A \otimes A)^*$.

A construction of a dual coalgebra for an infinite dimensional algebra is described, for example, in [25]. It may be fun to reconsider this construction in the context built up in the preceding chapter. It is known (see for example [1, p. 126]) that the category of finite dimensional left A-comodules over a k-algebra A is isomorphic to the category of finite dimensional right A°-comodules. I want to make this fact a definition of the dual coalgebra and briefly develop its most important properties from this starting point.

3.3.3. Lemma: *Let A be an algebra, C a coalgebra and $\mathcal{F} : \mathcal{M}^C \to {}_A\mathcal{M}$ a functor in $\mathfrak{C}_0$.*
Then there is a unique $f : C \to A^*$ such that

$$am = m_{(0)}\langle f(m_{(1)}), a\rangle$$

holds for all $m \in M \in \mathcal{M}^C$

Proof: Denote the underlying functor by $\omega : {}_A\mathcal{M} \to \underline{\text{vec}}$. Then

$$\begin{aligned} \text{Nat}(A \otimes \omega, \omega) &\cong \text{Nat}(\omega, \text{Hom}(A, \omega)) \\ &\cong \text{Nat}(\omega, \omega \otimes A^*), \end{aligned}$$

the latter since all the $\omega(X)$ for $X \in {}_A\mathcal{M}$ are finite dimensional. The claim immediately follows from the universal property of the coendomorphism coalgebra, [24, 2.1.9]. $\square$

3.3.4. Definition: *Let A be a k-Algebra. Define the dual coalgebra A by*

$$A^\circ = \underline{\text{coend}}({}_A\mathcal{M}, \mathcal{U})$$

Now we attempt to rederive the usual properties of A° from this definition.

3.3.5. Theorem: *(1) We have $A^\circ \subset A^*$, the comodule structures on the left A-modules being given by $am = m_{(0)}\langle m_{(1)}, a\rangle$. The coalgebra structure of A° is related to the algebra structure of A by*

$$\forall a, b \in A\ c \in C : \quad \langle \Delta c, a \otimes b\rangle = \langle c, ab\rangle.$$

(2) A° is the greatest subset of A^* which is a coalgebra such that the formula from (1) holds.

(3) $(\text{—})^\circ$ is a functor $\underline{\mathrm{Alg}} \to \underline{\mathrm{Coalg}}$ if we define

$$\langle f^\circ c, a\rangle = \langle c, fa\rangle.$$

(4) If A is a bialgebra, then so is A°, with

$$\langle cd, a\rangle = \langle c, a_{(1)}\rangle\langle d, a_{(2)}\rangle.$$

(5) If A is a Hopf algebra with antipode s, then s° is an antipode making A° a Hopf algebra as well.

(6) $\underline{\mathrm{Alg}}(A, C^*) \cong \underline{\mathrm{Coalg}}(C, A^\circ)$, where f corresponds to g iff

$$\langle g(c), a\rangle = \langle f(a), c\rangle$$

PROOF:

(1) By 3.3.3 there is a mapping $\iota : A^\circ \to A^*$ such that the comodule structures on the left A-modules are given by

$$am = m_{(0)}\langle \iota m_{(1)}, a\rangle.$$

In particular

$$\begin{aligned} m_{(0)}\langle \iota m_{(1)}, ab\rangle = (ab)m &= a(bm) = a(\iota m_{(0)}\langle \iota m_{(1)}, b\rangle) \\ &= m_{(0)}\langle \iota m_{(1)}, a\rangle\langle \iota m_{(2)}, b\rangle \end{aligned}$$

To exhibit A° as a subset of A^*, as desired, it remains to show injectivity of ι. First, $\operatorname{Ker}\iota$ is a coideal, for $\iota c = 0$ implies that $0 = \langle \iota c_{(1)}, a\rangle\langle \iota c_{(2)}, b\rangle$ also holds for all $a, b \in A$, and thus $\Delta(c) \in \operatorname{Ker}\iota \otimes A^\circ + A^\circ \otimes \operatorname{Ker}\iota$. This proves $C := A^\circ/\operatorname{Ker}\iota$ to be a coalgebra, and we can consider C a subset of A^*. As, by [24, thm. 2.2.8], the functor ${}_A\mathcal{M} \to \mathcal{M}^{A^\circ}$ is an equivalence, $\mathcal{M}^{A^\circ} \to \mathcal{M}^C$ is one as well, with $\mathcal{M}^C \to {}_A\mathcal{M} \to \mathcal{M}^{A^\circ}$ a quasi-inverse. Thus the natural surjection $A^\circ \to C$ is an isomorphism, proving ι injective.

(2) By lemma 3.3.3 and [24, 2.1.9] each mapping from a coalgebra to A^* satisfying the formula in (1) factors uniquely over A°.

(3) Clear, since $A \mapsto {}_A\mathcal{M}$ is a (contravariant) functor $\underline{Alg} \to \mathfrak{C}$.

(4) If A is a bialgebra, then ${}_A\mathcal{M}$ is a monoidal category by $a(m \otimes n) = (a_{(1)}m) \otimes (a_{(2)}m)$.

(5) If A is a Hopf algebra, then ${}_A\mathcal{M}$ is a rigid monoidal category.

(6) It is clear by now that one of the two sets of morphisms is in bijective correspondence with the set of $\mathfrak{C}_0$-morphisms between the categories $\mathcal{M}^C$ and ${}_A\mathcal{M}$:

$$\mathfrak{C}_0(\mathcal{M}^C, {}_A\mathcal{M}) \cong \underline{Coalg}(C, A^\circ)$$

But on the other hand

$$\begin{aligned}
\mathrm{Nat}(A \otimes \mathcal{U}^C, \mathcal{U}^C) &\cong \int_{M \in \mathcal{M}^C} \mathrm{Hom}_k(A \otimes M, M) \\
&\cong \int_{M \in \mathcal{M}^C} \mathrm{Hom}_k(A, M \otimes M^*) \\
&\cong \mathrm{Hom}_k(A, \int_{M \in \mathcal{M}^C} M \otimes M^*) \\
&\cong \mathrm{Hom}_k\left(A, \left(\int^{M \in \mathcal{M}^C} M^* \otimes M\right)^*\right) \\
&\cong \mathrm{Hom}_k(A, C^*)
\end{aligned}$$

also holds. It is straightforward to verify that this isomorphism indeed has the form stated in the theorem and puts those natural transformations $A \otimes \mathcal{U}^C \to \mathcal{U}^C$ forming modules structures on the C-comodules in bijection with the algebra morphisms from A to C^*.

□

3.3.3 Actions on comodules

It is a peculiar property of coquasitriangular Hopf algebras that they act as algebras on their *co*modules, this property being directly connected with the quasisymmetric structure of the category of H-comodules. The actions vaguely arise from the possibility to "permute" comodules past each other in a way not coinciding with the usual twist on the underlying k-modules.

3.3.6. Lemma: *Let (H, r) be a coquasitriangular bialgebra.*
Then

$$\forall M \in {}_H\mathcal{M}\ \forall x \in M\ \forall h \in H: \quad m \leftharpoonup h := m_{(0)} r(m_{(1)} \otimes h)$$

defines a monoidal functor

$$\Gamma_1 : \mathcal{M}^H \to \mathcal{M}_H$$

in $\mathfrak{C}_0$.

In the same way,

- $h \rightharpoonup m := m_{(0)}r(h \otimes m_{(1)})$ *defines a monoidal functor* $\Gamma_2 : \mathcal{M}^H \to {}_{H^{\mathrm{cop}}}\mathcal{M}$,
- $h \rightharpoondown m := m_{(0)}\overline{r}(m_{(1)} \otimes h)$ *defines a monoidal functor* $\overline{\Gamma}_1 : \mathcal{M}^H \to {}_{H^{\mathrm{cop}}}\mathcal{M}$, *and*
- $m \leftharpoondown h := m_{(0)}\overline{r}(h \otimes m_{(1)})$ *defines a monoidal functor* $\overline{\Gamma}_2 : \mathcal{M}^H \to \mathcal{M}_H$.

PROOF: We only do the necessary calculations for Γ_2:

(1) The new structure makes each H-comodule M an H-module:

$$\begin{aligned}
a \rightharpoonup (b \rightharpoonup m) &= a \rightharpoonup (m_{(0)}r(b \otimes m_{(1)})) \\
&= m_{(0)}r(a \otimes m_{(1)})r(b \otimes m_{(2)}) \\
&= m_{(0)}r(ab \otimes m_{(1)}) \\
&= (ab) \rightharpoonup m
\end{aligned}$$

$$\begin{aligned}
1 \rightharpoonup m &= m_{(0)}r(1 \otimes m_{(1)}) \\
&= m_{(0)}\varepsilon(m_{(1)}) \\
&= m
\end{aligned}$$

(2) The new structure on the tensor product of two H-comodules M and N coincides with the tensor product of the H^{cop}-module structures induced on each of the two modules:

$$\begin{aligned}
a \rightharpoonup (m \otimes n) &= (m \otimes n)_{(0)}r(a \otimes (x \otimes y)_{(1)}) \\
&= m_{(0)} \otimes n_{(0)}r(a \otimes x_{(1)}y_{(1)}) \\
&= m_{(0)} \otimes n_{(0)}r(a_{(1)} \otimes y_{(1)})r(a_{(2)} \otimes x_{(1)}) \\
&= (a_{(2)} \rightharpoonup x) \otimes (a_{(1)} \rightharpoonup y)
\end{aligned}$$

□

3.3.7. Remark: *Part of the lemma is a specialization of the following fact:*
Let C be a coalgebra, let A and B be algebras. Then a measuring ([25], pp.138/139) $\psi : C \otimes A \to B$ from A to B induces a functor in $\mathfrak{C}_0$ from the category of B-modules being C-comodules at the same time (without any compatibility conditions) to the category of A-modules via

$$am = \psi(m_{(1)} \otimes a)m_{(0)}$$

for all B-modules and C-Comodules M and all $m \in M$ and $a \in A$.

In the following we will characterize the coquasitriangular structure of a Hopf algebra by the actions just defined.

3.3.8. Definition: *Let H be a bialgebra. Call a left H-module M* **invertible** *if there is a mapping $\rho : M \otimes H \to M$ such that*

$$\rho(h_{(2)}m \otimes h_{(1)}) = \varepsilon(h)m = h_{(1)}\rho(m \otimes h_{(2)})$$

holds for all $h \in H$ and $m \in M$.
Define invertible right modules in the same fashion and denote by $\mathcal{I}\mathrm{nv}_H$ and ${}_H\mathcal{I}\mathrm{nv}$ the categories of invertible right and left modules, respectively.

3.3.9. Lemma: *If M is an invertible left H-module, then ρ is unique and defines a right module action on M (which is, of course, invertible). The category ${}_H\mathcal{I}\mathrm{nv}$ is a monoidal one and the inverse module structures define a monoidal functor ${}_H\mathcal{I}\mathrm{nv} \to \mathcal{M}_{H^{\mathrm{cop}}}$.*

PROOF: If ρ' is another inverse, then

$$\begin{aligned}
\rho'(m \otimes h) &= \rho'(\varepsilon(h_{(2)})m \otimes h_{(1)}) \\
&= \rho'(h_{(2)}\rho(m \otimes h_{(3)}) \otimes h_{(1)}) \\
&= \rho(m \otimes h)
\end{aligned}$$

ρ defines a right module structure, since

$$\begin{aligned}
\rho(m \otimes gh) &= \varepsilon(h_{(1)})\varepsilon(g_{(1)})\rho(m \otimes g_{(2)}h_{(2)}) \\
&= \rho(\varepsilon(g_{(1)})h_{(2)}\rho(m \otimes g_{(2)}h_{(3)}) \otimes h_{(1)}) \\
&= \rho(\rho(g_{(2)}h_{(2)}\rho(m \otimes g_{(3)}h_{(3)}) \otimes g_{(1)}) \otimes h_{(1)}) \\
&= \rho(\rho(\varepsilon(g_{(2)}h_{(2)})m \otimes g_{(1)}) \otimes h_{(1)}) \\
&= \rho(\rho(m \otimes g) \otimes h)
\end{aligned}$$

The remaining statements are proved by the following calculations: If M, N are invertible modules, define $\rho_{M\otimes N}(m \otimes n \otimes h) = \rho_M(m \otimes h_{(2)}) \otimes \rho_N(n \otimes h_{(1)})$. Then

$$\begin{aligned}
\rho(h_{(2)}(m \otimes n) \otimes h_{(1)}) &= \rho(h_{(2)}m \otimes h_{(3)}n \otimes h_{(1)}) \\
&= \rho(h_{(3)}m \otimes h_{(2)}) \otimes \rho(h_{(4)}n \otimes h_{(1)}) \\
&= \varepsilon(h_{(2)})m \otimes \rho(h_{(3)}n \otimes h_{(1)}) \\
&= m \otimes \rho(h_{(2)}n \otimes h_{(1)}) \\
&= \varepsilon(h)m \otimes n
\end{aligned}$$

and similarly for the second equation, proving $\rho_{M\otimes N}$ to be an inverse for the module structure on $M \otimes N$. □

3.3.10. Remark: *Any module over a Hopf algebra is invertible. Define $\rho(m \otimes h) = s(h)m$.*

We will find that the actions on an H-comodule over a coquasitriangular Hopf algebra are compatible in a certain way with the comodule structure. The relevant compatibility condition was examined, in the case of a general Hopf algebra H, by Radford in [20] and by Lambe and Radford in [9]. They proved in particular that Yang-Baxter modules carry a solution of the quantum Yang-Baxter equation.

3.3.11. Definition: *Let H be a bialgebra.*

A right Yang-Baxter H-module is a right H-module and right H-comodule M such that (denoting the module structure by $\leftharpoondown$)

$$m_{(0)} \leftharpoondown h_{(1)} \otimes m_{(1)} h_{(2)} = (m \leftharpoondown h_{(2)})_{(0)} \otimes h_{(1)} (m \leftharpoondown h_{(2)})_{(1)}$$

holds for all $m \in M$ and $h \in H$.

A left Yang-Baxter H-module is a left H-module and right (!) H-comodule M such that

$$h_{(1)} \rightharpoondown m_{(0)} \otimes h_{(2)} m_{(1)} = (h_{(2)} \rightharpoondown m)_{(0)} \otimes (h_{(2)} \rightharpoondown m)_{(1)} h_{(1)}.$$

Call a left (right) Yang-Baxter H-module invertible if the respective module structure is invertible.

Denote by ${}_H\mathcal{YB}$ and $\mathcal{YB}_H$ the categories of finite dimensional invertible left and right Yang-Baxter modules, respectively (with the morphisms that are both H-linear and H-colinear).

If H is a Hopf algebra, then ${}_H\mathcal{YB}$ and $\mathcal{YB}_H$ are, of course, just the categories of all left and right Yang-Baxter modules.

Let M and N be two right Yang-Baxter H-modules. We give $M \otimes N$ the H-comodule structure induced by μ, and the H-module structure induced by Δ. Then we have

$$\begin{aligned}
&(m \otimes n)_{(0)} \leftharpoondown h_{(1)} \otimes (m \otimes n)_{(1)} h_{(2)} \\
&= (m_{(0)} \otimes n_{(0)}) \leftharpoondown h_{(1)} \otimes m_{(1)} n_{(1)} h_{(2)} \\
&= m_{(0)} \leftharpoondown h_{(1)} \otimes n_{(0)} \leftharpoondown h_{(2)} \otimes m_{(1)} n_{(1)} h_{(3)} \\
&= m_{(0)} \leftharpoondown h_{(1)} \otimes (n \leftharpoondown h_{(3)})_{(0)} \otimes m_{(1)} h_{(2)} (n \leftharpoondown h_{(3)})_{(1)} \\
&= (m \leftharpoondown h_{(2)})_{(0)} \otimes (n \leftharpoondown h_{(3)})_{(0)} \otimes h_{(1)} (m \leftharpoondown h_{(2)})_{(1)} (n \leftharpoondown h_{(3)})_{(1)} \\
&= (m \leftharpoondown h_{(2)} \otimes n \leftharpoondown h_{(3)})_{(0)} \otimes h_{(1)} (m \leftharpoondown h_{(2)} \otimes n \leftharpoondown h_{(3)})_{(1)} \\
&= ((m \otimes n) \leftharpoondown h_{(2)})_{(0)} \otimes h_{(1)} ((m \otimes n) \leftharpoondown h_{(2)})_{(1)}
\end{aligned}$$

whence $M \otimes N$, equipped with the structures as stated, is a right Yang-Baxter H-module again.

In the same way we see that the tensor product of two left Yang-Baxter H-modules is one, now defining the left module structure on $M \otimes N$ via Δ^{cop}. We can prove more:

3.3.12. Theorem: *The categories $\mathcal{YB}_H$ and ${}_H\mathcal{YB}$ are quasisymmetric monoidal categories with quasisymmetries defined by*

$$\sigma(m \otimes n) = n_{(0)} \otimes m \leftharpoondown n_{(1)}$$

and

$$\sigma(m \otimes n) = m_{(1)} \rightharpoonup n \otimes m_{(0)}$$

respectively.

Proof: We have just defined the structures of the monoidal categories, the quasisymmetries are defined in the lemma. Now

(1) $\sigma : M \otimes N \to N \otimes M$ is H-colinear since M is a Yang-Baxter module:

$$\begin{aligned} \delta_{N \otimes M}\sigma(m \otimes n) &= \delta(n_{(0)} \otimes m \leftharpoonup n_{(1)}) \\ &= n_{(0)} \otimes (m \leftharpoonup n_{(2)})_{(0)} \otimes n_{(1)}(m \leftharpoonup n_{(2)})_{(1)} \\ &= n_{(0)} \otimes m_{(0)} \leftharpoonup n_{(1)} \otimes m_{(1)}n_{(2)} \\ &= \sigma(m_{(0)} \otimes n_{(0)}) \otimes m_{(1)}n_{(1)} \\ &= (\sigma \otimes 1)\delta(m \otimes n) \end{aligned}$$

(2) $\sigma : M \otimes N \to N \otimes M$ is H-linear since N is a Yang-Baxter module:

$$\begin{aligned} \sigma(m \otimes n) \leftharpoonup h &= (n_{(0)} \otimes m \leftharpoonup n_{(1)}) \leftharpoonup h \\ &= n_{(0)} \leftharpoonup h_{(1)} \otimes m \leftharpoonup n_{(1)}h_{(2)} \\ &= (n \leftharpoonup h_{(2)})_{(0)} \otimes m \leftharpoonup h_{(1)}(n \leftharpoonup h_{(2)})_{(1)} \\ &= \sigma(m \leftharpoonup h_{(1)} \otimes n \leftharpoonup h_{(2)}) \\ &= \sigma((m \otimes n) \leftharpoonup h) \end{aligned}$$

(3) $\sigma_{L,M\otimes N} = (1 \otimes \sigma_{LN})(\sigma_{LM} \otimes 1)$, since

$$\begin{aligned} \sigma(\ell \otimes (m \otimes n)) &= (m \otimes n)_{(0)} \otimes \ell \leftharpoonup (m \otimes n)_{(1)} \\ &= m_{(0)} \otimes n_{(0)} \otimes \ell \leftharpoonup m_{(1)} \leftharpoonup n_{(1)} \\ &= m_{(0)} \otimes \sigma(\ell \leftharpoonup m_{(1)} \otimes n) \\ &= (1 \otimes \sigma)(\sigma \otimes 1)(\ell \otimes m \otimes n) \end{aligned}$$

(4) $\sigma_{L\otimes M,N} = (\sigma_{LN} \otimes 1)(1 \otimes \sigma_{MN})$, since

$$\begin{aligned} \sigma((\ell \otimes m) \otimes n) &= n_{(0)} \otimes (\ell \otimes m) \leftharpoonup n_{(1)} \\ &= n_{(0)} \otimes \ell \leftharpoonup n_{(1)} \otimes m \leftharpoonup n_{(2)} \\ &= \sigma(\ell \otimes n_{(0)}) \otimes m \leftharpoonup n_{(1)} \\ &= (\sigma \otimes 1)(1 \otimes \sigma)(\ell \otimes m \otimes n) \end{aligned}$$

(5) σ is an isomorphism. Denote the inverse module structure on M by $\rightharpoondown$ and define $\tau : N \otimes M \to M \otimes N$ by $\tau(n \otimes m) = n_{(1)} \rightharpoondown m \otimes n_{(1)}$. Then

$$\begin{aligned} \tau\sigma(m \otimes n) &= \tau(n_{(0)} \otimes m \leftharpoonup n_{(1)}) \\ &= n_{(1)} \rightharpoondown (m \leftharpoonup n_{(2)}) \otimes n_{(0)} \\ &= \varepsilon(n_{(1)})m \otimes n_{(0)} = m \otimes n \end{aligned}$$

and similarly $\sigma\tau = id$.

(It goes without saying that σ is a natural transformation because morphisms in the category of Yang-Baxter modules are both linear and colinear.) $\square$

3.3.13. Remark: *If we denote the underlying functor $\mathcal{YB}_H \to \underline{\mathrm{vec}}$ by ω we know now that $D^*(H) := \underline{\mathrm{coend}}(\omega)$ has to be a coquasitriangular Hopf algebra, which we might call the dual Drinfeld double.*

Now we come to the main result, showing that certain actions on comodules are indeed a characteristic feature of coquasitriangular Hopf algebras.

3.3.14. Theorem: *Let H be a bialgebra. The following sets of data are equivalent:*

(1) A coquasitriangular structure $r : H \otimes H \to k$.

(2) A section $\mathcal{M}^H \to \mathcal{YB}_H$ of the forgetful functor $\mathcal{YB}_H \to \mathcal{M}^H$

(3) A section $\mathcal{M}^H \to {}_H\mathcal{YB}$ of the forgetful functor ${}_H\mathcal{YB} \to \mathcal{M}^H$

(4) A $*$-invertible bialgebra map $\gamma_1 : H \to (H^\circ)^{\mathrm{cop}}$ with

$$\langle \gamma_1(x_{(1)}), y_{(1)} \rangle x_{(2)} y_{(2)} = y_{(1)} x_{(1)} \langle \gamma_1(x_{(2)}), y_{(2)} \rangle$$

(5) A $*$-invertible bialgebra map $\gamma_2 : H \to (H^\circ)^{\mathrm{op}}$ with

$$\langle \gamma_2(y_{(1)}), x_{(1)} \rangle x_{(2)} y_{(2)} = y_{(1)} x_{(1)} \langle \gamma_1(y_{(2)}), x_{(2)} \rangle$$

PROOF: We only show (1)$\Leftrightarrow$(2)$\Leftrightarrow$(4).

(1)$\Rightarrow$(2) Define a right module structure $\leftharpoondown$ on each right H-comodule as in 3.3.6. It is invertible, an inverse being $\rightharpoondown$ as defined in the same lemma. We have

$$\begin{aligned} m_{(0)} \leftharpoondown h_{(1)} \otimes m_{(1)} h_{(2)} &= m_{(0)} r(m_{(1)} \otimes h_{(1)}) \otimes m_{(2)} h_{(2)} \\ &= m_{(0)} \otimes r(m_{(1)} \otimes h_{(1)}) m_{(2)} h_{(2)} \\ &= m_{(0)} \otimes h_{(1)} m_{(1)} r(m_{(2)} \otimes h_{(2)}) \\ &= m_{(0)} r(m_{(2)} \otimes h_{(2)}) \otimes h_{(1)} m_{(1)} \\ &= (m \leftharpoondown h_{(2)})_{(0)} \otimes h_{(1)} (m \leftharpoondown h_{(2)})_{(1)} \end{aligned}$$

and thus a Yang-Baxter module.

(2)$\Rightarrow$(4) The monoidal functor

$$\mathcal{M}^H \to \mathcal{YB}_H \to \mathcal{M}_H \cong \mathcal{M}^{(H^\circ)^{\mathrm{cop}}}$$

induces a unique bialgebra map $\gamma_1 : H \to (H^\circ)^{\mathrm{cop}}$ with

$$m \leftharpoondown h = m_{(0)} \langle \gamma_1(m_{(1)}), h \rangle.$$

Similarly, the inverse module structure $\rightharpoondown$ induces $\overline{\gamma}_1 : H \to (H^\circ)^{\mathrm{op}}$ with $h \rightharpoondown m = m_{(0)}\langle \overline{\gamma}_1(m_{(1)}), h\rangle$. Now

$$\begin{aligned}\varepsilon(h)m &= h_{(1)} \rightharpoondown (m \leftharpoonup h_{(2)})\\ &= h_{(1)} \rightharpoondown m_{(0)}\langle \gamma_1(m_{(1)}), h_{(2)}\rangle\\ &= m_{(0)}\langle \overline{\gamma}_1(m_{(1)}), h_{(1)}\rangle\langle \gamma_1(m_{(2)}), h_{(2)}\rangle\\ &= m_{(0)}\langle \overline{\gamma}_1(m_{(1)})\gamma_1(m_{(2)}), h\rangle\end{aligned}$$

Thus $m_{(0)}\mu(\overline{\gamma}_1 \otimes \gamma_1)\Delta(m_{(1)}) = m \otimes 1$ for all $m \in M$ for all $M \in \mathcal{M}^H$, whence $\overline{\gamma}_1 * \gamma_1 = \eta\varepsilon$. In the same way we can prove that $\gamma_1 * \overline{\gamma}_1 = \eta\varepsilon$, hence $\overline{\gamma}_1$ is a convolution inverse for γ_1.

To prove the formula claimed it is enough to show that for all $m \in M \in \mathcal{M}^H$ and all $h \in H$ we have

$$m_{(0)} \otimes \langle \gamma_1(m_{(1)}), h_{(1)}\rangle m_{(2)}h_{(2)} = m_{(0)} \otimes h_{(1)}m_{(1)}\langle \gamma_1(m_{(2)}), h_{(2)}\rangle.$$

Now indeed

$$\begin{aligned}m_{(0)}\langle \gamma_1(m_{(1)}), h_{(1)}\rangle \otimes m_{(2)}h_{(2)} &= m_{(0)} \leftharpoonup h_{(1)} \otimes m_{(1)}h_{(2)}\\ &= (m \leftharpoonup h_{(2)})_{(0)} \otimes h_{(1)}(m \leftharpoonup h_{(2)})_{(1)}\\ &= m_{(0)}\langle \gamma_1(m_{(2)}), h_{(2)}\rangle \otimes h_{(1)}m_{(1)}.\end{aligned}$$

(4)⇒(1) Define $r : H \otimes H \to k$ by $r(g \otimes h) = \langle \gamma_1(g), h\rangle$. A $*$-inverse for r is given by $\overline{r}(g \otimes h) = \langle \overline{\gamma}_1(g), h\rangle$. Formula (2) of 2.4.4 follows from the formula in our lemma, (3) and (4) follow from γ_1 being a bialgebra map as indicated.

□

3.3.15. Corollary: *Let (H, r) be a coquasitriangular bialgebra.*
The functor $\mathcal{M}^H \to \mathcal{YB}_H$ is a quasisymmetric functor, and thus the induced bialgebra map $H \to D^*(H)$ is a coquasitriangular bialgebra map.

Indeed we have $n_{(0)} \otimes m \leftharpoonup n_{(1)} = n_{(0)} \otimes m_{(0)}r(m_{(1)} \otimes n_{(1)})$.

If the bialgebra in consideration is a Hopf algebra, or cotriangular, or both, then the $\gamma_i, \overline{\gamma}_i$ can in part be expressed in terms of each other:

3.3.16. Remark:

(1) If H is cotriangular, then $\gamma_1 = \overline{\gamma}_2$ and $\gamma_2 = \overline{\gamma}_1$.

(2) If H is Hopf algebra, then $\overline{\gamma}_1 = \gamma_1 s$ and $\gamma_2 = \overline{\gamma}_1$.

The second part follows from 3.3.1, but also from the fact that for a bialgebra homomorphism f between Hopf algebras a $*$-inverse is given by $\overline{f} = fS = Sf$.

We can also express the various actions of a coquasitriangular bialgebra in terms of the notation for elements we have developed:

3.3.17. Lemma: *Let (H,r) be a coquasitriangular bialgebra. Let V,W be finite dimensional right H-comodules. For elements $v \in V$, $w \in W$ and $\varphi \in W^*$*

$$\begin{aligned}
[\varphi|v] \rightharpoonup w &= (id \otimes \varphi)\sigma(v \otimes w) \\
[\varphi|v] \rightharpoondown w &= (id \otimes \varphi)\sigma^{-1}(v \otimes w) \\
w \leftharpoonup [\varphi|v] &= (\varphi \otimes id)\sigma(w \otimes v) \\
w \leftharpoondown [\varphi|v] &= (\varphi \otimes id)\sigma^{-1}(w \otimes v)
\end{aligned}$$

hold (where σ denotes the quasisymmetry of $\mathcal{M}^H$)

PROOF: Indeed we have, for example:

$$\begin{aligned}
[\varphi|v] \rightharpoonup w &= w_{(0)} r([\varphi|v] \otimes w_{(1)}) \\
&= w_{(0)} r(\langle \varphi, v_{(0)} \rangle v_{(1)} \otimes w_{(1)}) \\
&= (id \otimes \varphi)\sigma(v \otimes w)
\end{aligned}$$

the rest of the claim being proved in a similar way. □

It is clear now that wether a k-linear morphism between H-comodules is H-linear with respect to the actions $\rightharpoonup$ etc. depends on wether this morphism "commutes" in a certain way via σ with the objects in $\mathcal{M}^H$; this is made precise by the next lemma:

3.3.18. Lemma: *Let H be a coquasitriangular bialgebra, let $M, N, T \in \mathcal{M}^H$ and let $f : M \to N$ be a k-linear map.*

(1) The following statements are equivalent:

(a) For all $\varphi \in T^*$, all $t \in T$ and $m \in M$

$$f([\varphi|t] \rightharpoonup m) = [\varphi|t] \rightharpoonup f(m)$$

holds

(b) For all $\varphi \in T^*$, all $t \in T$ and $m \in M$

$$f(m \leftharpoondown [\varphi|t]) = f(m) \leftharpoondown [\varphi|t]$$

holds

(c) The diagram

$$\begin{tikzcd}
T \otimes M \arrow[r, "\sigma"] \arrow[d, "1 \otimes f"'] & M \otimes T \arrow[d, "f \otimes 1"] \\
T \otimes N \arrow[r, "\sigma"'] & N \otimes T
\end{tikzcd}$$

commutes.

(2) *The following statements are equivalent*

(a) *For all $\varphi \in T^*$, all $t \in T$ and $m \in M$*

$$f([\varphi|t] \rightharpoondown m) = [\varphi|t] \rightharpoondown f(m)$$

holds.

(b) *For all $\varphi \in T^*$, all $t \in T$ and $m \in M$*

$$f(m \leftharpoonup [\varphi|t]) = f(m) \leftharpoonup [\varphi|t]$$

holds.

(c) *The diagram*

$$\begin{tikzcd} M \otimes T \arrow[r, "\sigma"] \arrow[d, "f \otimes 1"'] & T \otimes M \arrow[d, "1 \otimes f"] \\ N \otimes T \arrow[r, "\sigma"'] & T \otimes N \end{tikzcd}$$

commutes.

Proof: I only prove that $(a) \Leftrightarrow (c)$ in (1). $(b) \Leftrightarrow (c)$ is proved similarly, if we note that σ is an isomorphism and therefore nothing changes in condition (c) if we invert the horizontal arrows. (2) is proved in quite the same way as (1).

The condition required in (c) is

$$\forall T \in \mathcal{M}^H \; \forall t \in T \; \forall m \in M : \quad \sigma(t \otimes f(m)) = (f \otimes 1)\sigma(t \otimes m)$$

which is equivalent with

$$\forall T \in \mathcal{M}^H \; \forall t \in T \; \forall \varphi \in T^* \; \forall m \in M : \quad (1 \otimes \varphi)\sigma(t \otimes f(m)) = (1 \otimes \varphi)(f \otimes 1)\sigma(t \otimes m)$$

But the left hand side of this equation is just $[\varphi|t] \rightharpoonup f(m)$, the right hand side is $f([\varphi|t] \rightharpoonup m)$. $\square$

Since *every* element of H is $[\varphi|t]$ for some $T \in \mathcal{M}^H$, $t \in T$ and $\varphi \in T^*$, we deduce immediately:

3.3.19. Corollary: *Let H be a coquasitriangular bialgebra, let M and N be two right H-comodules and let $f : M \to N$ be a k-linear map.*

(1) *The following statements are equivalent:*

(a) *f is H-linear with respect to $\rightharpoonup$.*

(b) *f is H-linear with respect to $\leftharpoondown$.*

(c) *For each right H-comodule T the diagram*

$$\begin{tikzcd} T \otimes M \arrow[r, "\sigma"] \arrow[d, "1 \otimes f"'] & M \otimes T \arrow[d, "f \otimes 1"] \\ T \otimes N \arrow[r, "\sigma"'] & N \otimes T \end{tikzcd}$$

commutes.

(2) The following statements are equivalent

(a) f is H-linear with respect to $\rightharpoondown$.

(b) f is H-linear with respect to $\leftharpoonup$.

(c) For each right H-comodule T the diagram

$$\begin{tikzcd} M \otimes T \arrow[r, "\sigma"] \arrow[d, "f \otimes 1"'] & T \otimes M \arrow[d, "1 \otimes f"] \\ N \otimes T \arrow[r, "\sigma"'] & T \otimes N \end{tikzcd}$$

commutes.

The following lemma is actually a statement on when two elements of H act in the same way on all the H-comodules.

3.3.20. Lemma: *Let (H, r) be a coquasitriangular Hopf algebra, let $h \in H$. h annulates every H-comodule with respect to the action $\rightharpoonup$ iff $\gamma_1(h) = 0$. h annulates every H-comodule with respect to the action $\leftharpoondown$ iff $\gamma_2(h) = 0$.*

PROOF: Indeed $\gamma_1(h)$ is the element of H^* uniquely determined by the condition that

$$h \rightharpoonup m = m_{(0)} \langle \gamma_2(m_{(1)}), h \rangle = m_{(0)} \langle \gamma_1(h), m_{(1)} \rangle$$

hold for each H-comodule M and each $m \in M$. The second part of our claim is proved in the same way. $\square$

3.4 "Self-dual" Hopf algebras

3.4.1 A direct product of coalgebras

It is well known that the tensor product of cocommutative coalgebras is a direct product in the category of cocommutative coalgebras. I will show in the following that in the category $\underline{Coalg}$ of *all* coalgebras categorical direct products also exist. These also yield direct products of bialgebras and Hopf algebras. The new objects can be constructed using the techniques from [24, chap. 2] recalled in section 2.1.

3.4.1. Definition and Lemma: *Let C_1, C_2 be two coalgebras.*
Let $\mathcal{C}$ be the category of all finite dimensional vector spaces over k that are comodules over C_1 and C_2 at the same time (without having to satisfy any compatibility conditions). Denote the obvious underlying functor by $\omega : \mathcal{C} \to \underline{vec}$. Define the functors $\omega_i : \mathcal{C} \to \mathcal{M}^{C_i}$ as remembering only the comodule structure indicated and forgetting the other.
Let $C_1 \odot C_2 := \underline{\mathrm{coend}}(\omega)$. Define the projections $\pi_i : C_1 \odot C_2 \to C_i$ by

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathcal{I}_\mathcal{C}"] \arrow[dr, "\omega_i"'] & \mathcal{M}^{C_1 \odot C_2} \arrow[d, "\mathcal{M}^{\pi_i}"] \\ & \mathcal{M}^{C_i} \end{tikzcd}$$

Then the triple $(C_1 \odot C_2, \pi_1, \pi_2)$ is a categorical direct product of C_1 and C_2 in the category of coalgebras.

Proof: By [24, 2.1.12] $C_1 \odot C_2$ as well as the morphisms π_i are well defined. We only have to show the universal property of a product to hold. Thus let $f_i : D \to C_i$ for $i = 1, 2$ be coalgebra homomorphisms. These morphisms first define a unique functor $\mathcal{F} : \mathcal{M}^D \to \mathcal{C}$ with $\omega_i \mathcal{F} = \mathcal{M}^{f_i}$ for $i = 1, 2$. The functor $\mathcal{I}_\mathcal{C}\mathcal{F} : \mathcal{M}^D \to \mathcal{M}^{C_1 \odot C_2}$ has the form $\mathcal{I}_\mathcal{C}\mathcal{F} = \mathcal{M}^f$ for a unique morphism of coalgebras $f : D \to C_1 \odot C_2$. Since $\mathcal{M}^{\pi_i f} = \mathcal{M}^{\pi_i}\mathcal{I}_\mathcal{C}\mathcal{F} = \omega_i\mathcal{F} = \mathcal{M}^{f_i}$ holds, we obtain $\pi_i f = f_i$.
Let $g : D \to C_1 \odot C_2$ be another morphism of coalgebras satisfying $\pi_i g = f_i$ for $i = 1, 2$. The category $\mathcal{C}$ being k-linear abelian and ω being a k-linear, exact and faithful functor, there is, by [24, 2.2.8], an equivalence quasi-inverse $\mathcal{G}$ for $\mathcal{I}_\mathcal{C}$. It now follows that $\omega_i \mathcal{G}\mathcal{M}^g = \mathcal{M}^{\pi_i}\mathcal{I}_\mathcal{C}\mathcal{G}\mathcal{M}^g = \mathcal{M}^{\pi_i g} = \mathcal{M}^{f_i} = \mathcal{M}^{\pi_i f} = \cdots = \omega_i\mathcal{G}\mathcal{M}^f$ in $\mathfrak{C}$, thus we have $\mathcal{M}^g = \mathcal{M}^f$, and finally $f = g$. □

3.4.2. Corollary: *If C_1, C_2 are two bialgebras (resp. Hopf algebras), then their direct product $\mathcal{C}_1 \odot C_2$ defined in the preceding lemma is a direct product of bialgebras (resp. Hopf algebras).*

Proof: We make use of [24, sec. 2.3]: The bialgebra structures on the C_i make the category $\mathcal{C}$ from the preceding lemma a monoidal one, such that the functors ω and ω_i are monoidal. This yields a bialgebra structure on the direct product for which the projections are morphisms of bialgebras. Similar considerations show that the coalgebra map f induced by two bialgebra maps f_i in the way described in the preceding lemma is actually a bialgebra map.

For the second statement we make similar use of the results obtained in [24, sec. 2.4]. □

3.4.3. Lemma: *Let D, C_1, C_2 be coalgebras, let $f : D \to C_1 \odot C_2$ and $f_i : D \to C_i$ be coalgebra maps with $\pi_i f = f_i$. For an element $d \in D$ we have $f(d) = 0$ iff $f_i(d) = 0$ for $i = 1, 2$.*

Proof: Of course we have $f_i(d) = 0$ if $f(d) = 0$, so now we assume $f_i(d) = 0$ for $i = 1, 2$. The kernels $I_i = \mathrm{Ker}(f_i)$ of the coalgebra maps f_i are coideals, thus so is their intersection I. Since d is contained in both the I_i, it is contained in I. The coalgebra morphisms $\tilde{f}_i : D/I \to C_i$ uniquely defined by

$$\begin{tikzcd} D \arrow[r, "f_i"] \arrow[dr] & C_i \arrow[d, "\tilde{f}_i"] \\ & D/I \end{tikzcd}$$

define a unique coalgebra morphism $\tilde{f} : D/I \to C_1 \odot C_2$ with $\pi_i \tilde{f} = \tilde{f}_i$. Thus $\tilde{f}$ makes the diagram

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[dr] & C \arrow[d, "\tilde{f}"] \\ & D/I \end{tikzcd}$$

commute (since it yields two commutative diagrams after composing with π_i), in particular we have $f(I) = 0$. Since d is contained in I_i for $i = 1, 2$, it is also contained in I, whence $f(d) = 0$. □

3.4.2 Coquasitriangular quotients

Any sub-bialgebra of a coquasitriangular Hopf algebra is a fortiori coquasitriangular itself. In order that a quotient bialgebra of a coquasitriangular bialgebra be coquasitriangular as well, however, one must not mod out too much. The mapping $r : H \otimes H \to k$ has to factor through the twofold tensor product of the quotient with itself.

We will define the quotient of a coquasitriangular Hopf algebra which is produced by factoring out as great a Hopf ideal of H as possible such that the quotient still inherits the coquasitriangular structure defined on H.

3.4.4. Definition: *Let (H, r) be a coquasitriangular Hopf algebra. The Hopf algebra maps $\gamma_1 : H \to (H^\circ)^{\mathrm{cop}}, \gamma_2 : H \to (H^\circ)^{\mathrm{op}}$ define, by 3.4.1, a Hopf algebra map $\gamma : H \to (H^\circ)^{\mathrm{cop}} \odot (H^\circ)^{\mathrm{op}}$.*
Thus by $\breve{H} := \gamma(H)$ we can define a Hopf sub algebra of $(H^\circ)^{\mathrm{cop}} \odot (H^\circ)^{\mathrm{op}}$.

3.4.5. Lemma: *$\breve{H}$ is a coquasitriangular Hopf algebra and the canonical map $\nu : H \to \breve{H}$ is coquasitriangular.*

PROOF: We have to look for a suitable coquasitriangular structure $\breve{r} : \breve{H} \otimes \breve{H} \to k$.
Let $g, h \in \breve{H}$. Choose $g', h' \in H$ with $\gamma(g') = g$ and $\gamma(h') = h$.
Then we have

$$\begin{aligned} \langle \pi_1(g), h' \rangle &= \langle \pi_1 \gamma(g'), h' \rangle \\ &= \langle \gamma_1(g'), h' \rangle \\ &= \langle \gamma_2(h'), g' \rangle \\ &= \langle \pi_2 \gamma(h'), g' \rangle \\ &= \langle \pi_2(h), g' \rangle, \end{aligned}$$

the first expression being manifestly independent of the choice of g' and being linear in g, the last expression being independent of the choice of h' and linear in h.
Thus

$$\breve{r}(g \otimes h) = \langle \pi_1 g, h' \rangle = \langle \pi_2 h, g' \rangle$$

describes a well defined mapping $\breve{r} : \breve{H} \otimes \breve{H} \to k$ satisfying $\breve{r}(\nu \otimes \nu) = r$.
A mapping $\overline{\breve{r}}$ with $\overline{\breve{r}}(\nu \otimes \nu) = \overline{r}$ is obtained by $\overline{\breve{r}}(g \otimes h) = \breve{r}(\breve{s}(g) \otimes h)$ (where $\breve{s}$ denotes the antipode of $\breve{H}$).
We have to check the axioms for $\breve{r}$. Observe that it follows from $\gamma(g') = g$ and $\gamma(h') = h$ that $\gamma(g'h') = gh$ and $\gamma(g'_{(1)}) \otimes \gamma(g'_{(2)}) = g_{(1)} \otimes g_{(2)}$ also hold, since γ is a bialgebra map.
Now for $x, y, z \in \breve{H}$ and $x', y', z' \in H$ with $\gamma(x') = x$ etc. we get

$$\begin{aligned} \breve{r}(x_{(1)} \otimes y_{(1)}) x_{(2)} y_{(2)} &= r(x'_{(1)} \otimes y'_{(1)})\, \nu(x'_{(2)})\, \nu(y'_{(2)}) \\ &= \nu(\, r(x'_{(1)} \otimes y'_{(1)}) x'_{(2)} y'_{(2)}\,) \\ &= \nu(\, y'_{(1)} x'_{(1)} r(x'_{(2)} \otimes y'_{(2)})\,) \\ &= y_{(1)} x_{(1)} \breve{r}(x_{(2)} \otimes y_{(2)}) \end{aligned}$$

$$\begin{aligned} \breve{r}(xy \otimes z) &= r(x'y' \otimes z') \\ &= r(x' \otimes z'_{(1)}) r(y' \otimes z'_{(2)}) \\ &= \breve{r}(x \otimes z_{(1)}) \breve{r}(y \otimes z_{(2)}) \end{aligned}$$

And the remaining axiom as well. □

3.4.6. Remark: *Let (H, r) be cotriangular. Then we can even consider the Hopf algebra $\breve{H}$ defined in the preceding lemma as a Hopf subalgebra in $(H^\circ)^{\mathrm{op}}$, since then $\gamma_1 s = \gamma_2$.*

We turn to the properties of $\breve{H}$

By its very construction it is clear that the Hopf algebra $\breve{H}$ arises from H by factoring out as many additional relations as possible:

3.4.7. Lemma: *If $f : H \to F$ is a surjection of coquasitriangular bialgebras, then the natural surjection $\nu_H : H \to \breve{H}$ factors uniquely through f, that is, there is a unique morphism of bialgebras $g : F \to \breve{H}$ satisfying*

$$\begin{tikzcd} H \arrow[rr, "\nu_H"] \arrow[dr, "f"'] & & \breve{H} \\ & F \arrow[ur, "g"'] & \end{tikzcd}$$

Proof: The following diagram commutes:

$$\begin{tikzcd} H \arrow[r, "\gamma_H"] \arrow[d, "f"'] & (H^\circ)^{\mathrm{op}} \odot (H^\circ)^{\mathrm{cop}} \\ F \arrow[r, "\gamma_F"'] & (F^\circ)^{\mathrm{op}} \odot (F^\circ)^{\mathrm{cop}} \arrow[u, "f^\circ \odot f^\circ"'] \end{tikzcd}$$

To see this let π_i, resp. π_i' be the i-th projections of the direct products $(H^\circ)^{\mathrm{cop}} \odot (H^\circ)^{\mathrm{op}}$, resp. $(F^\circ)^{\mathrm{cop}} \odot (F^\circ)^{\mathrm{op}}$, and write $\gamma_{i,H} =: \gamma_i$ and $\gamma_{i,F} =: \gamma_i'$ for short. Then

$$\begin{aligned} \pi_i(f^\circ \odot f^\circ)\gamma' f &= f^\circ \pi_i' \gamma' f \\ &= f^\circ \gamma_i' f \\ &= \gamma_i, \end{aligned}$$

since for $x, y \in H$, for example

$$\begin{aligned} \langle f^* \gamma_1' f(x), y \rangle &= \langle \gamma_1' f(x), f(y) \rangle \\ &= r_F(f(x) \otimes f(y)) \\ &= r_H(x \otimes y) \\ &= \langle \gamma_1(x), y \rangle \end{aligned}$$

Hence $(f^\circ \odot f^\circ)\gamma_F$ is one possible choice for the morphism g in the lemma. Since f was assumed to be surjective, this is of course the only choice. □

3.4.8. Corollary: *Each surjection of coquasitriangular bialgebras $\breve{H} \to F$ is an isomorphism. In particular $\nu_{\breve{H}} \breve{H} \to \left(\breve{H}\right)^{\breve{}}$ is an isomorphism.*

Slightly reformulating this result, we arrive at the "almost self-dual" Hopf algebras we have announced above.

3.4.9. Corollary:

(1) Let F be a coquasitriangular Hopf algebra, let $H = \check{F}$.

(a) The mapping $r : H \otimes H \to k$ is nondegenerate in the following (weak) sense:

$$\forall y \in H : \; r(x \otimes y) = 0 = r(y \otimes x) \;\Rightarrow\; x = 0$$

(b) The mapping $\gamma : H \to (H^\circ)^{\mathrm{op}} \odot (H^\circ)^{\mathrm{cop}}$ is injective. In particular, if $\gamma_1(x) = 0 = \gamma_2(x)$ holds for some $x \in H$, it follows that $x = 0$.

(c) The functor Γ from $\mathcal{M}^H$ to the category of right and left H-modules, defined by the module structures $\rightharpoonup$ and $\leftharpoonup$, is full.

(2) Assume H is cotriangular (which holds automatically if F is), then:

(a) The mapping $r : H \otimes H \to k$ is nondegenerate in the following (stronger) sense:

$$\forall y \in H : \; r(x \otimes y) = 0 \;\Rightarrow\; x = 0$$

(b) The Hopf algebra map $\gamma_1 : H \to (H^\circ)^{\mathrm{op}}$ is injective, hence H is isomorphic to a Hopf subalgebra of its own dual.
If in particular H is finite dimensional, then $H \cong (H^*)^{\mathrm{op}}$.

(c) The functor Γ_1 from 3.3.6 is full.

Proof: The second part of each statement is just 3.4.7, the second is a direct consequence of the first, the third follows from the second in view of 3.3.19. □

3.4.3 Lyubashenko's construction

As the original aim of this paper was to see through Lyubashenko's paper [10], I want to show in this section that we have indeed rederived his statements in a different way.

Lyubashenko first sketches the construction of a symmetric monoidal category $\mathcal{C}$ starting from a rigid unitary solution $\mathfrak{S} = ((V_a | a \in \Lambda), (S_{ab} | a, b \in \Lambda))$ of the quantum Yang-Baxter equation, his construction being as follows:
First extend the solution to duals of the given spaces by

$$S_{a^*,b} = S_{ba}^{\flat} : V_a^* \otimes V_b \to V_b \otimes V_a^*,$$

$$S_{a,b^*} = (S_{ab}^{\flat})^{-1} : V_a \otimes V_b^* \to V_b^* \otimes V_a$$

$$S_{a^*,b^*} = S_{ab}^{\flat\flat} : V_a^* \otimes V_b^* \to V_b^* \otimes V_a^*$$

Then add tensor products and direct sums of the spaces obtained, then all the subspaces $L \subset X$ of objects X already defined for which

$$\forall a \in \Lambda: \quad S(L \otimes V_a) = V_a \otimes L$$

holds, and finally all the quotients X/L by those subspaces.

The objects thus constructed are the objects of $\mathcal{C}$, the morphisms are all the $f : X \to Y$ such that for all $Z \in \mathcal{C}$ we have

$$S(f \otimes 1) = (1 \otimes f)S.$$

Lyubashenko claims that his construction yields a symmetric monoidal category $\mathcal{C}$ (with symmetry S).

Now we have obtained above a category of this very type. First, it is clear that the equations given by Lyubashenko for the "extension of the symmetry to dual spaces" coincide in the case of a unitary solution with the definitions from 3.1.9. Furthermore:

3.4.10. Lemma: *Let $\mathfrak{S} = ((V_a|a \in \Lambda), (S_{ab}|a, b \in \Lambda))$ be a rigid solution of the quantum Yang-Baxter equation in the category of finite dimensional vector spaces. Let $\langle\mathfrak{S}\rangle_*$ be the rigid quasisymmetric monoidal category defined in 3.2.7, let $H_0 = \underline{\mathrm{coend}}(\langle\mathfrak{S}\rangle_*)$ with the coquasitriangular structure r defined in 2.4.5, and let $H = \check{H}_0$.*

All the objects of $\mathcal{M}^H$ are subquotients of finite direct sums of finite tensor products of comodules V_a, V_a^* for $a \in \Lambda$.

If U is a k-subspace of an H-comodule X, and if for all $a \in \Lambda$

$$\begin{aligned} \sigma_{V_a,X}(V_a \otimes U) &= U \otimes V_a \\ \sigma_{X,V_a}(U \otimes V_a) &= V_a \otimes U \end{aligned}$$

holds, then U is an H-subcomodule in X.

If X, Y are two H-comodules, and if $f : X \to Y$ is a k-linear map such that for all $a \in \Lambda$

$$\begin{aligned} \sigma_{V_a,Y}(1 \otimes f) &= (f \otimes 1)\sigma_{V_a,X} \\ \sigma_{Y,V_a}(f \otimes 1) &= (1 \otimes f)\sigma_{X,V_a} \end{aligned}$$

holds, then f is H-colinear.

Proof: First all the objects in $\langle\mathfrak{S}\rangle_*$ are finite tensor products of objects V_a, V_a^* for some $a \in \Lambda$. In [24, 2.2.9] we have seen that then all the objects of $\mathcal{M}^{H_0}$ are subquotients of finite sums of tensor products of that type. Since H is a quotient Hopf algebra of H_0, the corresponding functor $\mathcal{M}^{H_0} \to \mathcal{M}^H$ is dense in the sense of [24, 2.2.10] by [24, 2.2.12], that is, every object in the codomain of the functor is a subquotient of one in the image. Thus all the objects in $\mathcal{M}^H$ have the form claimed.

The condition

$$\forall a \in \Lambda \begin{cases} \sigma_{V_a,Y}(1 \otimes f) = (f \otimes 1)\sigma_{V_a,X} \\ \sigma_{Y,V_a}(f \otimes 1) = (1 \otimes f)\sigma_{X,V_a} \end{cases}$$

on a k-linear homomorphism $f : X \to Y$ between H-comodules X and Y implies by $\sigma_{V_a^*,X} = (\sigma^{-1}_{V_a,X})^\flat$ and $\sigma_{X,V_a^*} = (\sigma^\flat_{X,V_a})^{-1}$ and by the formulas 1.1.14 that

$$\forall z \in \Lambda^* \begin{cases} \sigma_{V_z,Y}(1 \otimes f) = (f \otimes 1)\sigma_{V_z,X} \\ \sigma_{Y,V_z}(f \otimes 1) = (1 \otimes f)\sigma_{X,V_z} \end{cases}$$

holds. Hence, by 3.3.18, it follows that for all $z \in \Lambda^*$, for all $\varphi \in V_z^*$, for all $t \in V_z$ and all $m \in M$

$$f([\varphi|t] \rightharpoonup m) = [\varphi|t] \rightharpoonup f(m) \quad \text{und} \quad f(m \leftharpoonup [\varphi|t]) = f(m) \leftharpoonup [\varphi|t]$$

holds for the element $[\varphi|v] \in H$. But the algebra H is generated by elements having this form, whence f is an H-linear map with respect to the actions $\rightharpoonup$ and $\leftharpoonup$. This in turn means that f is a $(H^\circ)^{\mathrm{cop}} \odot (H^\circ)^{\mathrm{op}}$-colinear map with respect to the comodule structure induced by the coalgebra map $\gamma_H : H \to (H^\circ)^{\mathrm{cop}} \odot (H^\circ)^{\mathrm{op}}$ defined in 3.4.4. But the mapping γ_H is injective by 3.4.7, since we have defined $H = \breve{H}_0$. Thus f is H-colinear.

A similar reasoning proves the desired statement about subobjects. $\square$

3.4.11. Corollary: *Assume that, in the situation of the last theorem, the given solution $\mathfrak{S} = ((V_a|a \in \Lambda), (S_{ab}|a, b \in \Lambda))$ is unitary. Then one of the two conditions on subspaces and morphisms is redundant, hence we have constructed (up to category equivalence) the same category $\mathcal{C}$ as in [10]*

After constructing his category, Lyubashenko defines the desired Hopf algebra as an algebra acting from the left on all the objects of his symmetric monoidal category. More precisely he constructs a full monoidal functor $\mathcal{C} \to {}_A\mathcal{M}$ for an algebra (which is actually a Hopf algebra) whose elements are denoted $\mathcal{T}_i^j$, where L is a vector space contained in $\mathcal{C}$, $\{\ell_i\}$ is basis in L, and where the action of $\mathcal{T}_i^j$ on a vector space X in $\mathcal{C}$ is defined by

$$S(\ell_i \otimes x) = \sum_j \mathcal{T}_i^j(x) \otimes \ell_j.$$

3.4.12. Lemma: *Let H_0, H be as in the preceding theorem. Then*

$$\begin{aligned} \sigma(\ell_i \otimes x) &= \ell_{i(1)} \rightharpoonup x \otimes \ell_{i(0)} \\ &= [\ell^j|\ell_i] \rightharpoonup x \otimes \ell_j \\ \sigma(x \otimes \ell_i) &= \ell_j \otimes x \leftharpoonup [\ell^j|\ell_i] \end{aligned}$$

holds. Furthermore, two elements of H that act the same on all the H-comodules via both the operations $\rightharpoonup$ and $\leftharpoonup$ actually are *the same.*

If, moreover, $\mathfrak{S}$ is unitary, then two elements of H that act the same on all the H-comodules with respect to only the operation $\rightharpoonup$, are the same.

PROOF: I show the first of the two equations:

$$\begin{aligned}\sigma(n \otimes x) &= x_{(0)} \otimes n_{(0)} r(n_{(1)} \otimes x_{(1)}) \\ &= n_{(1)} \rightharpoonup x \otimes n_{(0)}\end{aligned}$$

The other equation is proved in the same manner. The remaining statements follow directly from 3.3.20. □

The statement that elements of H acting in the same way actually *are* the same, obviously corresponds to the construction of the algebra H as "consisting" of elements defined only by their action on objects in the category $\mathcal{C}$.
Thus the Hopf algebra H we have constructed has, in the cotriangular case, the same form as the one described by Lyubashenko in [10]. The elements $\mathcal{T}_i^j$ in his paper are, in our notation, $[\ell_i|\ell^j]$.

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