# BIGALOIS OBJECTS OVER THE TAFT ALGEBRAS

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ABSTRACT. Let $k$ be a field and $H$ a Hopf algebra over $k$. We have seen in [9] that any right $H$-Galois extension of $k$ gives rise to a unique Hopf algebra $L = L(A, H)$ over which $A$ is a left $L$-Galois extension of $k$ and an $L$-$H$-bicomodule (we call this an $L$-$H$-biGalois object). The isomorphism classes of biGalois objects form a groupoid under cotensor product (whose objects are the Hopf algebras), and in particular, $H$-$H$-biGalois objects form a group $\mathrm{BiGal}(H)$ (see also [14]).

As an example of these facts we first study the two-generator Taft algebras $H_N$, using Masuoka's [5] classification of their cleft extensions. It turns out that always $L(A, H_N) \cong H_N$ in this case (in particular, any double twist of $H_N$ by a two-cocycle in the sense of Doi [1] is isomorphic to $H_N$). We find that $\mathrm{BiGal}(H_N)$ is isomorphic to a semidirect product $\dot{k} \ltimes k$ (where $\dot{k}$ denotes the unit group of the field).

For the general Taft algebras $H_{N,m}$ with $m$ grouplike and one skew primitive generator we also find that $L(A, H_{N,m}) \cong H_{N,m}$ for any Galois object $A$. If $N$ is odd, the group of biGalois objects is isomorphic to

$$(\mathrm{GL}_{m-1}(\mathbb{Z}/(N)) \ltimes (\mathbb{Z}/(N))^{m-1}) \ltimes_\omega (\mathrm{Skew}_{m-1}(\mathbb{Z}/(N)) \times (\dot{k} \ltimes k) \times (\dot{k}/\dot{k}^N)^{m-1})$$

for suitable actions and a suitable group cocycle $\omega$. Here $\mathrm{Skew}_{m-1}(\mathbb{Z}/(N))$ denotes the additive group of skew symmetric $(m-1)$-by-$(m-1)$ matrices with zero diagonal.

The result on the general Taft algebras is obtained using a description of the set $\mathrm{Gal}(H_1 \otimes H_2)$ of Galois objects over a tensor product Hopf algebra. We develop this result in the more general case that $H_i$ are flat Hopf algebras over a base ring $k$.

## 1. Preliminaries

Throughout the paper $k$ denotes a fixed base field, and $\dot{k}$ denotes its unit group; algebras, tensor products, coalgebras etc. are over $k$. Let $H$ be a Hopf algebra. A right $H$-comodule algebra $A$ is called a right $H$-Galois object if $A^{\mathrm{co}\,H} = k$ and the Galois map $\kappa_A = (\nabla \otimes H)(A \otimes \rho) : A \otimes A \to A \otimes H$ ia a bijection (here $\nabla$ and $\rho$ denote the multiplication map and right comodule structure of $A$). A right $H$-comodule algebra $A$ is *cleft* if there is a convolution invertible right colinear map $\Phi : H \to A$. A cleft right $H$-comodule algebra with $A^{\mathrm{co}\,H} = k$ is a right $H$-Galois object; the converse is true if $H$ is finite dimensional [4]. Left Galois objects are defined analogously. An $L$-$H$-biGalois object is a left $L$-Galois object and right $H$-Galois object satisfying the compatibility condition of an $L$-$H$-bicomodule. If $A$ and

$B$ are an $L$-$H$-biGalois object and an $H$-$R$-biGalois object, respectively, then the cotensor product $A \underset{H}{\square} B$ is an $L$-$R$-biGalois object, with the algebra structure of a subalgebra of $A \otimes B$, and the left and right comodule structures induced by the respective tensorands. In this way, isomorphism classes of biGalois objects form a groupoid whose objects are the $k$-Hopf algebras, see [9]. As a special case, $H$-$H$-biGalois objects form a group $\mathrm{BiGal}(H)$, see also [14]. For a Hopf algebra $H$, we denote by $\mathrm{Aut}_{\mathrm{Hopf}}(H)$ the group of its Hopf algebra automorphisms. For an algebra map $u \in \mathrm{Alg}(H,k)$ we denote by $\mathrm{coinn}(u) : H \to H$ the coinner automorphism of $H$ induced by $u$, that is, $\mathrm{coinn}(u)(h) = u^{-1}(h_{(1)})h_{(2)}u(h_{(3)})$, where $u^{-1} = uS$ denotes the convolution inverse of $u$. The set $\mathrm{CoInn}(H) := \{\mathrm{coinn}(u)|u \in \mathrm{Alg}(H,k)\}$ of coinner automorphisms of $H$ is a normal subgroup of $\mathrm{Aut}_{\mathrm{Hopf}}(H)$; we call the quotient $\mathrm{CoOut}(H) := \mathrm{Aut}_{\mathrm{Hopf}}(H)/\mathrm{CoInn}(H)$ the co-outer automorphism group of $H$. For a left $H$-comodule $V$ and a coalgebra map $f : H \to H$ we denote by ${}^fV$ the vector space $V$ with the left $H$-comodule structure $v \mapsto f(v_{(-1)}) \otimes v_{(0)}$, where $v \mapsto v_{(-1)} \otimes v_{(0)}$ denotes the original $H$-comodule structure of $V$. A group homomorphism $\mathrm{Aut}_{\mathrm{Hopf}}(H) \to \mathrm{BiGal}(H)$ is given by $f \mapsto {}^fH$. By [9, Lem.3.11] this induces an injective homomorphism $I : \mathrm{CoOut}(H) \to \mathrm{BiGal}(H)$.

Let $H$ be a Hopf algebra. A two-cocycle on $H$ is a map $\sigma : H \otimes H \to k$ satisfying

$$\sigma(f_{(1)}, g_{(1)})\sigma(f_{(2)}g_{(2)}, h) = \sigma(g_{(1)}, h_{(1)})\sigma(f, g_{(2)}h_{(2)})$$

and $\sigma(h,1) = \sigma(1,h) = 1$ for all $f,g,h \in H$. If $\sigma$ is a (convolution) invertible two-cocycle, then Doi [1] constructs a double twisted Hopf algebra $H^\sigma$ which is $H$ as a coalgebra with multiplication $g \cdot h = \sigma(g_{(1)}, h_{(1)})g_{(2)}h_{(2)}\sigma^{-1}(g_{(3)}, h_{(3)})$.

The right $H$-comodule algebra $k_\sigma[H] := k\#_\sigma H$ is defined to be $H$ with multiplication defined by $g \cdot h = \sigma(g_{(1)}, h_{(1)})g_{(2)}h_{(2)}$. It is an $H^\sigma$-$H$-bicomodule algebra via the comultiplication of $H$.

## 2. BiGalois objects over the two-generator Taft algebras

Let $N > 1$ be an integer, and assume that $k$ contains a primitive $N$-th root of unity $q$. We denote by $\langle q\rangle$ the multiplicative group of $N$-th roots of unity in $k$, which is generated by $q$. The two-generator Taft algebra [11] is

$$H_N = H_{N,q} := k\langle X, Y\rangle/(X^N - 1, Y^N, YX - qXY).$$

It is a Hopf algebra with grouplike $X$ and $(1,X)$-primitive $Y$; that is, comultiplication is given by $\Delta(X) = X \otimes X$ and $\Delta(Y) = 1 \otimes Y + Y \otimes X$, the counit is given by $\varepsilon(X) = 1$ and $\varepsilon(Y) = 0$ and the antipode is given by $S(X) = X^{N-1}$ and $S(Y) = -q^{-1}X^{N-1}Y$. A $k$-basis of $H_N$ is $\{X^iY^j | 0 \leq i < N, 0 \leq j < N\}$. The set of grouplike elements of $H_N$ is $\mathcal{G}(H_N) = \{1, X, X^2, \ldots, X^{N-1}\}$. For any $i,j$ with $j \not\equiv i+1 \mod N$, all $(X^i, X^j)$-primitives are trivial (multiples of $X^i - X^j$), while the set of $(X^i, X^j)$-primitives is $P_{X^i,X^j} = k(X^i - X^j) \oplus kX^iY$ for $j \equiv i+1 \mod N$. (See [7, Expl.1.4] and [6, Lem.16.1].)

**Lemma 1.** *We have a group isomorphism*

$$\varphi_0 : \dot{k} \ni r \mapsto f_r \in \mathrm{Aut}_{\mathrm{Hopf}}(H_N)$$

*where* $f_r(X) = X$ *and* $f_r(Y) = rY$. *Moreover,* $\mathrm{Alg}(H_N, k) = \{u_r | r \in \langle q\rangle\}$, *where* $u_r(X) = r$ *and* $u_r(Y) = 0$. *For* $r \in \langle q\rangle$ *we have* $\varphi_0(\mathrm{coinn}(u_r)) = f_r$, *so that* $\varphi_0$ *induces an isomorphism*

$$\varphi : \dot{k}/\langle q\rangle \to \mathrm{CoOut}(H_N).$$

*Proof.* Since $X$ is the unique grouplike admitting a nontrivial $(1, X)$-primitive, it is fixed by any Hopf algebra automorphism $f$. Moreover, $f(Y)$ is a nontrivial $(1, X)$-primitive, hence $f(Y) = rY + t(X - 1)$ for some $r \in \dot{k}$ and $t \in k$, and $0 = f(YX - qXY) = f(Y)X - qXf(Y) = t(1 - q)(X - 1)X$, whence $t = 0$. The remaining assertions are straightforward to check. □

The following is a special case of [5, Prop.2.17, Lem.2.19]:

**Theorem 2.** (1) *For* $\alpha \in \dot{k}$ *and* $\beta \in k$ *the algebra*

$$A_{\alpha,\beta} := A_{N,\alpha,\beta} := k\langle x, y\rangle/(x^N - \alpha, y^N - \beta, yx - qxy)$$

*is a cleft right* $H_N$*-Galois object with right comodule structure* $\rho = \rho_{\alpha,\beta}$ *defined by* $\rho(x) = x \otimes X$ *and* $\rho(y) = 1 \otimes Y + y \otimes X$. *A right colinear convolution invertible map* $\Phi : H_N \to A_{N,\alpha,\beta}$ *is given by* $\Phi(X^iY^j) = x^iy^j$ *for* $0 \le i < N$ *and* $0 \le j < N$.

(2) *Any right* $H_N$*-Galois object is of the form described in (1).*

(3) *The set* $\mathrm{Alg}^{H_N}(A_{N,\alpha',\beta'}, A_{N,\alpha,\beta})$ *of comodule algebra homomorphisms (all of which are necessarily isomorphisms) is empty if* $\beta \neq \beta'$. *If* $\beta = \beta'$, *it consists precisely of the* $g_s$ *for* $s \in \dot{k}$ *with* $\alpha' = s^N\alpha$, *where* $g_s(x) = sx$ *and* $g_s(y) = y$.

**Definition and Lemma 3.** *Let* $\alpha \in \dot{k}$ *and* $\beta \in k$. *Then* $A_{N,\alpha,\beta}$ *is an* $H_N$*-biGalois object with the right comodule structure defined above and the left comodule structure* $\lambda = \lambda_{\alpha,\beta}$ *defined by* $\lambda(x) = X \otimes x$ *and* $\lambda(y) = 1 \otimes y + Y \otimes x$.

*Proof.* That $\lambda$ is well defined is proved in the same way as that $\rho$ is well defined (or the comultiplication of $H_N$ is): For any elements $a, b$ of a $k$-algebra $R$ we have

$$ba = qab \quad \Longrightarrow \quad (a + b)^N = a^N + b^N$$

which shows that $(1 \otimes y + Y \otimes x)^N = \beta$, while $(X \otimes x)^N = \alpha$ is obvious as well as $(1\otimes y+Y\otimes x)(X\otimes x) = q(X\otimes x)(1\otimes y+Y\otimes x)$. That $A := A_{N,\alpha,\beta}$ is an $H_N$-bicomodule is easy to check. To see that it is left Galois, it is enough to show that the left Galois map $\kappa' = (H_N \otimes \nabla)(\lambda \otimes A) : A \otimes A \to H_N \otimes A$ is surjective. For this in turn it is enough to check that for each $h$ in a generating set of $H_N$ there is $\sum x_i \otimes y_i \in A \otimes A$ with $\kappa(\sum x_i \otimes y_i) = h \otimes 1$. Indeed, if we have $\sum x_i \otimes y_i, \sum x'_i \otimes y'_i \in A \otimes A$ with $\kappa'(\sum x_i \otimes y_i) = h \otimes 1$ and $\kappa'(\sum x'_i \otimes y'_i) = g \otimes 1$, then $\kappa'(\sum x_ix'_j \otimes y'_jy_i) = hg \otimes 1$, and for $a \in A$ we have $h \otimes a = \kappa'(\sum x_i \otimes y_i a)$. Now $X \otimes 1 = \kappa'(x \otimes x^{-1})$ and $Y \otimes 1 = \kappa'(y \otimes x^{-1} - 1 \otimes yx^{-1})$ completes the proof. $\square$

By [9, Thm.3.5] there is, for any right $H$-Galois object $A$, a unique up to isomorphism Hopf algebra $L(A, H)$ such that $A$ is an $L(A, H)$-$H$-biGalois object.

**Corollary 4.** *For any right $H_N$-Galois object $A$ we have $L(A_{N,\alpha,\beta}, H_N) \cong H_N$. For any invertible two-cocycle $\sigma : H_N \otimes H_N \to k$ we have $H_N^\sigma \cong H_N$. The Hopf algebra $H_N$ is determined up to isomorphism by the $k$-linear monoidal category of its comodules.*

The first assertion is contained in the preceding lemma. For an invertible cocycle $\sigma$ we have $H_N^\sigma \cong L(k\#_\sigma H_N, H_N)$ by [9, Thm.3.9]. By Theorem 2 we have $k\#_\sigma H_N \cong A_{N,\alpha,\beta}$ for suitable $\alpha, \beta$, and by Lemma 3 we have $L(A_{N,\alpha,\beta}, H_N) \cong H_N$. If $H'$ is a Hopf algebra whose category of comodules is equivalent as a monoidal category to the category of $H_N$-comodules, then there is an $H'$-$H_N$-biGalois object by [9, Cor. 5.7.], whence $H' \cong H_N$.

**Theorem 5.** *The map*

$$\begin{aligned} \psi : \dot{k} \ltimes k &\longrightarrow \mathrm{BiGal}(H_N) \\ (\alpha, \beta) &\longmapsto [A_{N,\alpha,\beta}] \end{aligned}$$

*is a group isomorphism. The diagram*

$$\begin{tikzcd} \mathrm{CoOut}(H_N) \arrow[r, "I"] & \mathrm{BiGal}(H_N) \\ \dot{k}/\langle q \rangle \arrow[u, "\varphi"] \arrow[r, "I'"'] & \dot{k} \ltimes k \arrow[u, "\psi"'] \end{tikzcd}$$

*commutes for $I'([r]) = (r^n, 0)$.*

*Proof.* Let $A$ be any $H_N$-biGalois object. By Theorem 2 we can assume that $A = A_{N,\alpha,\beta}$ as a right $H_N$-comodule algebra, with a suitable left comodule structure $\lambda'$. By [9, Thm.3.5] there is a unique automorphism $f : H_N \to H_N$ with $(f \otimes 1)\lambda = \lambda'$, that is, $A = {}^f A_{N,\alpha,\beta}$ as bicomodule algebras. Let $r \in \dot{k}$ with $f = f_r$. We claim that the right colinear isomorphism $g_r : A_{N,r^n\alpha,\beta} \to {}^{f_r}A_{N,\alpha,\beta}$ is also left colinear. In fact

$$\begin{aligned} \lambda' g_r(y) = (f_r \otimes 1)\lambda(y) = (f_r \otimes 1)(1 \otimes y + Y \otimes x) = 1 \otimes y + rY \otimes x \\ = 1 \otimes y + Y \otimes rx = (1 \otimes g_r)\lambda(y) \end{aligned}$$

and $\lambda' g_r(x) = rX \otimes x = (1 \otimes g_r)\lambda(x)$. We have shown that $\varphi$ is onto, and that the diagram in the theorem commutes.

Assume that $g : A_{N,\alpha',\beta'} \to A_{N,\alpha,\beta}$ is an isomorphism of biGalois objects. By Theorem 2 we have $\beta' = \beta$, $g = g_r$ for some $r \in \dot{k}$ and $\alpha' = r^n\alpha$. Left colinearity of $g$ implies

$$1 \otimes y + Y \otimes x = \lambda g_r(y) = (1 \otimes g_r)\lambda(y) = 1 \otimes y + Y \otimes rx$$

whence $r = 1$ and $\alpha' = \alpha$, showing injectivity of $\varphi$.

To show that $\varphi$ is a group homomorphism, we need a homomorphism of $H_N$-bicomodule algebras

$$\delta : A_{N,\alpha\alpha',\beta\alpha'+\beta'} \to A_{N,\alpha,\beta} \underset{H_N}{\square} A_{N,\alpha',\beta'}$$

which will automatically be an isomorphism because both sides are $H_N$-cleft extensions of $k$. Now an algebra homomorphism

$$\delta_0 : A_{N,\alpha\alpha',\beta\alpha'+\beta'} \to A_{N,\alpha,\beta} \otimes A_{N,\alpha',\beta'}$$

can be defined by $\delta_0(x) = x \otimes x$ and $\delta_0(y) = 1 \otimes y + y \otimes x$, since

$$(\delta_0(x))^N = (x \otimes x)^N = x^N \otimes x^N = \alpha\alpha' 1 \otimes 1,$$

$$(\delta_0(y))^N = (1 \otimes y + y \otimes x)^N = 1 \otimes y^N + y^N \otimes x^N = (\beta' + \beta\alpha')1 \otimes 1$$

and

$$\delta_0(y)\delta_0(x) = (1 \otimes y + y \otimes x)(x \otimes x) = q(x \otimes x)(1 \otimes y + y \otimes x) = q\delta_0(x)\delta_0(y).$$

It is easy to check that $\delta_0$ has its image in the cotensor product over $H_N$ and gives rise to a left and right colinear map $\delta$ as desired. $\square$

## 3. Hopf-Galois algebras over tensor products

In the next section we will compute the set of Galois objects and the group of biGalois objects over the general Taft algebras. These are tensor products of $H_N$ with the group algebra of a direct power of a cyclic group of order $N$.

The present section is concerned more generally with describing all Galois objects over a tensor product of two Hopf algebras, a problem which might be interesting in itself. We will prove a generalization of part of a theorem of Kreimer, [3, Thm. 3.7], which says that the Harrison group of a tensor product of two cocommutative Hopf algebras is

$$\mathrm{Gal}(H_1 \otimes H_2) \cong \mathrm{Gal}(H_1) \oplus \mathrm{Gal}(H_2) \oplus \mathrm{Hopf}(H_2, H_1^*).$$

Note that the group of Hopf algebra homomorphisms $\mathrm{Hopf}(H_2, H_1^*)$ can be identified with the group $\mathrm{Pair}(H_2, H_1)$ of Hopf pairings $H_2 \otimes H_1 \to k$. In this notation, Kreimer's isomorphism maps $(A_1, A_2, \tau)$ to $A_1 \otimes A_2$ endowed with the obvious right $H_1 \otimes H_2$-comodule structure and the multiplication defined by $(x \otimes y)(x' \otimes y') := x\tau(y_{(1)}, x'_{(1)})x'_{(0)} \otimes y_{(0)}y'$.

We will generalize this description of Galois objects over tensor product Hopf algebras to the non-cocommutative case (in which, however, there is no group structure on $\mathrm{Gal}(H_1 \otimes H_2)$).

To keep our results more general, we will, for this section only, assume that $k$ is any commutative ring; all Hopf algebras considered will be flat $k$-modules, and Hopf-Galois objects will be faithfully $k$-flat by definition.

Our first observation generalizes [3, Thm. 2.5 and Prop. 2.6.], where finite projective Hopf algebras are considered. Let $H = H_1 \otimes H_2$ be a tensor product of two Hopf algebras that are faithfully flat over $k$. Note that $H_i$ can be considered as a Hopf subalgebra as well as quotient Hopf algebra of $H$. It is straightforward to check that for any right $H$-comodule $V$ one has

$$V^{\mathrm{co}\, H_1} = V(H_2) := \{v \in V | v_{(0)} \otimes v_{(1)} \in V \otimes H_2\} \cong V \underset{H}{\square} H_2.$$

and vice versa.

**Lemma 6.** *Let $H_1, H_2$ be two $k$-flat Hopf algebras and $H := H_1 \otimes H_2$. Let $A$ be a faithfully flat $H_1 \otimes H_2$-Galois extension of $B := A^{\mathrm{co}\, H}$ (that is, we assume that the Galois map $\kappa : A \underset{B}{\otimes} A \ni x \otimes y \mapsto x y_{(0)} \otimes y_{(1)} \in A \otimes H$ is a bijection and $A$ is a faithfully flat left $B$-module.) Then $A_i := A(H_i)$ is a faithfully flat $H_i$-Galois extension of $B$ for $i = 1, 2$ and multiplication induces an isomorphism*

$$A(H_1) \underset{B}{\otimes} A(H_2) \cong A$$

*of right $H$-comodules.*

*Proof.* The Galois map

$$\kappa_{A \underset{H}{\square} H_1} : (A \underset{H}{\square} H_1) \otimes (A \underset{H}{\square} H_1) \to (A \underset{H}{\square} H_1) \otimes H_1$$

is a bijection because

$$\begin{tikzcd}
(A \underset{H}{\square} H_1) \otimes (A \underset{H}{\square} H_1) \arrow[r, "\kappa_{A \underset{H}{\square} H_1}"] \arrow[d, "\varphi"'] & (A \underset{H}{\square} H_1) \otimes H_1 \arrow[d, "\cong"] \\
A \underset{H}{\square} (H_1 \otimes H_1) \arrow[r, "A \,\square\, \kappa_{H_1}"'] & A \underset{H}{\square} (H_1 \otimes H_1)
\end{tikzcd}$$

commutes. Here the right hand vertical arrow is the canonical bijection, and the isomorphism $\varphi$, defined by $\varphi(\sum x_i \otimes g_i \otimes \sum y_j \otimes h_j) = \sum x_i y_j \otimes g_i \otimes h_j$, was studied by Ulbrich [12, 13]. Consequently, $A_i$ is an $H_i$-Galois algebra for $i = 1, 2$. Since $A$ can be considered a right $(A_2, H_2)$-Hopf module, it follows from [10, Thm. I] that multiplication induces an isomorphism $A_1 \otimes A_2 \to A$, which is trivially $H$-colinear. $\square$

Our next task is to describe, in the case of an $H_1 \otimes H_2$-Galois object, the algebra structure on $A_1 \otimes A_2$ resulting from that of $A$.

We first review the definition of skew pairings between Hopf algebras and the way they give rise to Hopf algebra cocycles. This was studied in [2] with applications to the Drinfeld double and other quantum group constructions (however, we switch the order of tensorands in a skew pairing.)

**Definition 7.** Let $L, H$ be two bialgebras. A skew pairing of $L$ and $H$ is a map $\tau : L \otimes H \to k$ satisfying

$$\tau(\ell\ell', h) = \tau(\ell, h_{(2)})\tau(\ell', h_{(1)})$$
$$\tau(\ell, hh') = \tau(\ell_{(1)}, h)\tau(\ell_{(2)}, h')$$

and $\tau(\ell, 1) = \varepsilon(\ell)$, $\tau(1, h) = \varepsilon(h)$ for all $\ell, \ell' \in L$ and $h, h' \in H$. A skew pairing is said to be invertible if it is as an element of $(L \otimes H)^*$. The trivial skew pairing is by definition $\varepsilon := \varepsilon_{L\otimes H}$.

Note that if $H$ is a Hopf algebra then any skew pairing $\tau : L \otimes H \to k$ is invertible with $\tau^{-1} = \tau(\mathrm{id}_L \otimes S)$.

**Remark 8.** Let $\tau : L \otimes H \to k$ be an (invertible) skew pairing. Then $\hat{\tau} := \varepsilon \otimes \tau \otimes \varepsilon : (H \otimes L) \otimes (H \otimes L) \to k$ is an (invertible) two-cocycle.

As in [2] we denote the twisted Hopf algebra $(H \otimes L)^{\hat{\tau}}$ by $H \bowtie_\tau L$. Its multiplication is given by $(h \bowtie 1)(1 \bowtie \ell) = h \bowtie \ell$ and $(1 \bowtie \ell)(h \bowtie 1) = \tau(\ell_{(1)}, h_{(1)})(h_{(2)} \bowtie \ell_{(2)})\tau^{-1}(\ell_{(3)}, h_{(3)})$.

**Lemma 9.** *Let $A$ be an $L$-$H$-bicomodule algebra and $\sigma : L \otimes L \to k$ a two-cocycle. Then $k_\sigma[A]$, defined to be $A$ with the multiplication $x \cdot y := \sigma(x_{(-1)}, y_{(-1)})x_{(0)}y_{(0)}$ is a right $H$-comodule algebra. If $\sigma$ is invertible, then $k_\sigma[A]$ is an $L^\sigma$-$H$-bicomodule algebra.*

The proof consists of straightforward computations.

**Definition 10.** Let $L_i$ be a Hopf algebra and $A_i$ a left $L_i$-comodule algebra for $i = 1, 2$. Let $\tau : L_2 \otimes L_1 \to k$ be a skew pairing. Then we define $A_1 \#_\tau A_2 := k_{\hat{\tau}}[A_1 \otimes A_2]$. If $H_i$ are also Hopf algebras such that $A_i$ are $L_i$-$H_i$-bicomodule algebras, then $A_1 \#_\tau A_2$ is a right $H_1 \otimes H_2$-comodule algebra.

After Remark 8 and Lemma 9 there is nothing to check.

For the following lemma, we will need some facts on the unique Hopf algebra $L(A, H)$ for which a given $H$-Galois object $A$ is an $L(A, H)$-$H$-biGalois object. It can be constructed as $L(A, H) := L := (A \otimes A)^{\mathrm{co}\, H}$, which is a subalgebra of $A \otimes A^{\mathrm{op}}$. We will use the notation $\xi = \xi^{(1)} \otimes \xi^{(2)} \in A \otimes A$ for an element $\xi \in L$. Then the comultiplication of $L$ is given by $\Delta(\xi) = (\xi^{(1)}{}_{(0)} \otimes \xi^{(1)}{}_{(1)}{}^{[1]}) \otimes (\xi^{(1)}{}_{(1)}{}^{[2]} \otimes \xi^{(2)})$, where for $h \in H$ we denote the image of $1 \otimes h$ under the inverse of the Galois map

$\kappa : A \otimes A \to A \otimes H$ by $h^{[1]} \otimes h^{[2]} \in A \otimes A$. Note that $h^{[1]}h^{[2]} = \varepsilon(h)$. By [9, Lem. 3.2., Lem. 3.3] $L$ can be characterized by a universal property: Any $H$-colinear map $\delta : A \to W \otimes A$ factors as $(A \otimes f)\lambda$ for a unique $k$-module map $f : L \to W$. If $W$ is a coalgebra (algebra, bialgebra), then $\delta$ is a comodule structure (algebra map, comodule algebra structure) if and only if $f$ is a coalgebra map (algebra map, bialgebra map). In case $V$ in the following lemma is finite projective, the lemma is a consequence of these results of [9], otherwise it is a generalization.

**Lemma 11.** *Let $H$ be a $k$-Hopf algebra and $A$ a faithfully flat right $H$-Galois extension of $k$. Put $L := L(A, H)$. Then for all $k$-modules $V, W$ there is a bijection*

$$\mathrm{Hom}^H(V \otimes A, W \otimes A) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathrm{Hom}_k(V \otimes L, W)$$

*given by $\Phi(\alpha)(v \otimes \xi) \otimes 1_A = \alpha(v \otimes \xi^{(1)})\xi^{(2)}$ and $\Psi(\tau)(v \otimes a) = \tau(v \otimes a_{(-1)}) \otimes a_{(0)}$.*

*Assume that $W = k$ and let $\alpha$ and $\tau$ satisfy $\alpha = \Psi(\tau)$. If $V$ is an algebra, then $\alpha$ is a module structure if and only if $\tau$ fulfills*

$$\tau(vv' \otimes \xi) = \tau(v \otimes \xi_{(2)})\tau(v' \otimes \xi_{(1)})$$

*for all $v, v' \in V$ and $\xi \in L$. If $V$ is a coalgebra, then $\alpha$ is a measuring if and only if $\tau$ satisfies*

$$\tau(v \otimes \xi\zeta) = \tau(v_{(1)} \otimes \xi)\tau(v_{(2)} \otimes \zeta)$$

*for all $v \in V$ and $\xi, \zeta \in L$. In particular, if $V$ is a bialgebra, then $\alpha$ gives $A$ the structure of a $V$-module algebra if and only if $\tau$ is a skew pairing.*

*Proof.* From Schneider's structure theorem [10, Thm. I] for Hopf modules we have the following bijection

$$\begin{aligned}\mathrm{Hom}^H(V \otimes A^{\cdot}, W \otimes A^{\cdot}) &\cong \mathrm{Hom}^H_{-A}(V \otimes A^{\cdot} \otimes A_{\cdot}^{\cdot}, W \otimes A_{\cdot}^{\cdot}) \\ &\cong \mathrm{Hom}_k((V \otimes A^{\cdot} \otimes A_{\cdot}^{\cdot})^{\mathrm{co}\,H}, W) = \mathrm{Hom}_k(V \otimes L, W)\end{aligned}$$

which is easily checked to have the claimed form. From now on let $W = k$.

Assume that $V$ is an algebra. If $\alpha$ is a module structure, then

$$\begin{aligned}\tau(vv' \otimes \xi)1_A &= \alpha(vv' \otimes \xi^{(1)})\xi^{(2)} \\ &= \alpha(v \otimes \alpha(v' \otimes \xi^{(1)}))\xi^{(2)} \\ &= \tau(v \otimes \alpha(v' \otimes \xi^{(1)}) \otimes \xi^{(2)})1_A \\ &= \tau(v \otimes \alpha(v' \otimes {\xi^{(1)}}_{(0)}){{\xi^{(1)}}_{(1)}}^{[1]}{{\xi^{(1)}}_{(1)}}^{[2]} \otimes \xi^{(2)})1_A \\ &= \tau(v \otimes \tau(v' \otimes {\xi^{(1)}}_{(0)} \otimes {{\xi^{(1)}}_{(1)}}^{[1]}){{\xi^{(1)}}_{(1)}}^{[2]} \otimes \xi^{(2)})1_A \\ &= \tau(v \otimes \xi_{(2)})\tau(v' \otimes \xi_{(1)})1_A\end{aligned}$$

Conversely, if $\tau$ fulfills this equation, then we have

$$\begin{aligned}\alpha(v \otimes \alpha(v' \otimes a)) &= \alpha(v \otimes \tau(v' \otimes a_{(-1)})a_{(0)}) \\ &= \tau(v \otimes a_{(-1)})\tau(v' \otimes a_{(-2)})a_{(0)}\end{aligned}$$

Now assume that $V$ is a coalgebra. If $\alpha$ measures, then

$$\begin{aligned}\tau(v \otimes \xi\zeta)1_A &= \tau(v \otimes \xi^{(1)}\zeta^{(1)} \otimes \zeta^{(2)}\xi^{(2)}) \otimes 1 \\ &= \alpha(v \otimes \xi^{(1)}\zeta^{(1)})\zeta^{(2)}\xi^{(2)} \\ &= \alpha(v_{(1)} \otimes \xi^{(1)})\alpha(v_{(2)} \otimes \zeta^{(1)})\zeta^{(2)}\xi^{(2)} \\ &= \alpha(v_{(1)} \otimes \xi^{(1)})\xi^{(2)}\tau(v_{(2)} \otimes \zeta) \\ &= \tau(v_{(1)} \otimes \xi)\tau(v_{(2)} \otimes \zeta)1_A\end{aligned}$$

and conversely, if $\tau$ satisfies this equation, then

$$\begin{aligned}\alpha(v \otimes xy) &= \tau(v \otimes x_{(-1)}y_{(-1)})x_{(0)}y_{(0)} \\ &= \tau(v_{(1)} \otimes x_{(-1)})\tau(v_{(2)} \otimes y_{(-1)})x_{(0)}y_{(0)} \\ &= \alpha(v_{(1)} \otimes x)\alpha(v \otimes {}_{(2)} \otimes y) \quad \square\end{aligned}$$

**Proposition 12.** *Let $H_1$ and $H_2$ be two $k$-flat Hopf algebras.*

(1) *Let $A_i$ for $i = 1, 2$ be $H_i$-Galois algebras, put $L_i := L(A_i, H_i)$ and let $\tau : L_2 \otimes L_1 \to k$ be a skew pairing. Then $A_1 \#_\tau A_2$ is an $H_1 \otimes H_2$-Galois algebra with $L(A_1 \# A_2, H_1 \otimes H_2) = L_1 \bowtie_\tau L_2$.*

(2) *Let $A$ be a right $H_1 \otimes H_2$-Galois algebra. Then there are unique up to isomorphism $H_i$-Galois algebras $A_i$ for $i = 1, 2$ and a unique skew pairing $\tau : L_2 \otimes L_1 \to k$ such that $A \cong A_1 \#_\tau A_2$.*

*Proof.* To see that $A_1 \#_\tau A_2$ is a Galois algebra, one checks that the diagram

$$\begin{tikzcd} A_1\#_\tau A_2 \otimes A_1\#_\tau A_2 \arrow[r, "\kappa_{A_1\#A_2}"] \arrow[d, "\alpha"'] & A_1\#_\tau A_2 \otimes H_1 \otimes H_2 \arrow[d, "(2\,3)"] \\ A_1 \otimes A_1 \otimes A_2 \otimes A_2 \arrow[r, "\kappa_{A_1} \otimes \kappa_{A_2}"'] & A_1 \otimes H_1 \otimes A_2 \otimes H_2 \end{tikzcd}$$

commutes, where the right hand vertical arrow just switches tensorands, and $\alpha$, defined by $\alpha(x\#y \otimes x'\#y') = x\tau(y_{(-1)}, x'_{(-1)}) \otimes x'_{(0)} \otimes y_{(0)} \otimes y'$ is an isomorphism because $\tau$ is invertible.

That the obvious left $L_1 \otimes L_2$-comodule structure on $A_1 \otimes A_2$ is a left $L_1 \bowtie_\tau L_2$-comodule algebra structure on $A_1 \#_\tau A_2$ is checked easily by direct computation. That $A_1 \#_\tau A_2$ is a left $L_1 \bowtie_\tau L_2$-Galois object follows from the commutative diagram

$$\begin{array}{ccc} A_1\#_\tau A_2 \otimes A_1\#_\tau A_2 & \xrightarrow{\tilde\kappa_{A_1\#A_2}} & L_1 \bowtie_\tau L_2 \otimes A_1\#_\tau A_2 \\ \Big\downarrow{\scriptstyle (2\,3)} & & \Big\downarrow{\scriptstyle \gamma} \\ A_1\otimes A_1\otimes A_2\otimes A_2 & \xrightarrow[\tilde\kappa_{A_1}\otimes\tilde\kappa_{A_2}]{} & L_1\otimes A_1\otimes L_2\otimes A_2 \end{array}$$

in which $\tilde\kappa : R\otimes R \to B\otimes R$ denotes the left version of the Galois map for a left $B$-comodule algebra $R$, and $\gamma$, defined by $\gamma(\xi\otimes\zeta\otimes x\otimes y) = \xi_{(1)}\otimes x\otimes\zeta_{(1)}\tau(\zeta_{(2)},\xi_{(2)})\otimes y$, is a bijection because $\tau$ is invertible.

Now let $A$ be an $H$-Galois object for $H = H_1\otimes H_2$. From Lemma 6 we have the isomorphism, induced by multiplication, $A_1\otimes A_2 \to A$.

Consider $A$ as an $H_2$-comodule algebra. Then $A_2$ is a subcomodule algebra, which means that $A$ is an algebra in the monoidal category ${}_{A_2}\mathcal{M}^{H_2}_{A_2}$ of relative Hopf bimodules, with the tensor product over $A_2$. In [8] we have proved that $({}_{A_2}\mathcal{M}^{H_2}_{A_2}, \underset{A_2}{\otimes}) \cong ({}_{L_2}\mathcal{M},\otimes)$ as monoidal categories. The equivalence maps a left $L_2$-module $V$ to the right $A_2$-module $V\otimes A_2$ with the left $A_2$-module structure $x(v\otimes y) = x_{(-1)}\cdot v\otimes x_{(0)}$. The inverse isomorphism maps $M\in {}_{A_2}\mathcal{M}^{H_2}_{A_2}$ to $M^{\mathrm{co}\,H_2}$. It follows that there is a unique left $L_2$-module structure on $A_1$ making it an $L_2$-module algebra, for which the multiplication on $A$ satisfies $(xy)(x'y') = x(y_{(-1)}\cdot x')y_{(0)}y'$ for $x,x'\in A_1$ and $y,y'\in A_2$. This left $L_2$-action is given by $\ell\cdot x = \ell^{(1)}x\ell^{(2)}$, hence is $H_1$-colinear as a map $L_2\otimes A_1\to A_1$. Lemma 11 applies to yield a unique skew pairing $\tau : L_2\otimes L_1\to k$ with $\ell\cdot x = \tau(\ell, x_{(-1)})x_{(0)}$, hence $xyx'y' = x\tau(y_{(-1)}, x'_{(-1)})x'_{(-1)}x_{(0)}y_{(0)}y'$ for all $x,x'\in A_1$ and $y,y'\in A_2$. $\square$

In general it is hard to say what the group $\mathrm{BiGal}(L_1\otimes L_2, H_1\otimes H_2)$ of biGalois objects between two tensor product Hopf algebras is. However, we can note the easy observation:

**Remark 13.** Let $H_i$, $L_i$ for $i=1,2$ be Hopf algebras. Then we have an injective map

$$\mathrm{BiGal}(L_1,H_1)\times\mathrm{BiGal}(L_2,H_2)\ni(A_1,A_2)\mapsto A_1\otimes A_2\in\mathrm{BiGal}(L_1\otimes L_2, H_1\otimes H_2)$$

which is a group homomorphism if $L_i = H_i$ for $i=1,2$.

## 4. Galois and biGalois objects over the general Taft algebras

We return to the case that $k$ is a field containing a primitive $N$-th root of unity $q$. We will determine all Galois and biGalois objects over the Taft algebras with more than one grouplike generator. These are defined [11] to be

$$H_{N,m} := k\langle X_0,\dots,X_{m-1},Y\rangle/(X_i^N-1, Y^N, YX_i - qX_iY)$$

with grouplike elements $X_i$ and $(1,X_0)$-primitive $Y$.

We let $C_N$ denote the cyclic group of order $N$. The group algebra $kC_N^m$ has commuting grouplike generators $X_1, \dots, X_m$ with defining relations $X_i^N = 1$. It is a selfdual Hopf algebra, with the isomorphism $D : kC_N^m \cong (kC_N^m)^*$ determined by $D(X_i)(X_j) = q^{\delta_{ij}}$.

Note that we have a Hopf algebra isomorphism $f : H_{N,m+1} \cong kC_N^m \otimes H_N$ determined by $f(X_0) = 1 \otimes X$, $f(X_i) = X_i \otimes X$ for $i \geq 1$ and $f(Y) = 1 \otimes Y$. Hence, instead of the Taft algebras $H_{N,m}$ we can consider the tensor product Hopf algebra $kC_N^m \otimes H_N$. We will write $X_0 := X := 1 \otimes X$ and $X_i := X_i \otimes 1$.

Let us first fix some notations. In the following, in all sums and products the index runs through $1, \dots, m$. In a noncommutative ring, products will be taken in ascending order of the indices from left to right.

We will abbreviate $\mathrm{GL}_m := \mathrm{GL}_m(\mathbb{Z}/(N))$. This group acts naturally by right matrix multiplication on $G^m$ for any $\mathbb{Z}/(N)$-module $G$. If $G$ is a multiplicative abelian group of exponent a divisor of $N$, then this right action reads $\underline{\alpha} \leftharpoonup T = (\prod_j \alpha_j^{t_{ji}})_i$ for $\underline{\alpha} = (\alpha_i) \in G^m$ and $T = (t_{ij}) \in \mathrm{GL}_m$.

We denote by $\mathrm{Skew}_m := \mathrm{Skew}_m(\mathbb{Z}/(N))$ the set of all skew symmetric matrices $R = (r_{ij}) \in M_m(\mathbb{Z}/(N))$, i.e. of matrices that satisfy $r_{ii} = 0$ and $r_{ij} = -r_{ji}$ for all $i, j$. The abelian group $\mathrm{Skew}_m$ has a right $\mathrm{GL}_m$-action defined by $R \leftharpoonup T = (\sum_{k,\ell} r_{k\ell} t_{ki} t_{\ell j})_{i,j}$. (The group $\mathrm{Skew}_m$ is naturally isomorphic to the dual group $(\wedge^2(\mathbb{Z}/(N))^m)^*$ of the exterior square of $(\mathbb{Z}/(N))^m$. With this identification, the right action of $\mathrm{GL}_m$ is the dual of the canonical left action on $\wedge^2(\mathbb{Z}/(N))^m$.)

We will now describe the groups of Galois and biGalois objects over $kC_N^m$.

**Lemma 14.** *The map*

$$\mathrm{Skew}_m \oplus (\dot{k}/\dot{k}^N)^m \ni (R, \underline{\alpha}) \longmapsto B_{R,\underline{\alpha}} \in \mathrm{Gal}(kC_N^m)$$

*defined by*

$$B_{R,\underline{\alpha}} := k\langle x_1, \dots, x_m \rangle / (x_i^N - \alpha_i, x_j x_i - q^{r_{ij}} x_j x_i)$$

*with the right $kC_N^m$-comodule algebra structure determined by $\rho(x_i) = x_i \otimes X_i$ is an isomorphism of abelian groups.*

*If $N$ is odd, then the map*

$$\mathrm{GL}_m(\mathbb{Z}/(N)) \ltimes \left(\mathrm{Skew}_m \oplus (\dot{k}/\dot{k}^N)^m\right) \ni (T, R, \underline{\alpha}) \mapsto B_{T,R,\underline{\alpha}} \in \mathrm{BiGal}(kC_N^m)$$

*defined by $B_{T,R,\underline{\alpha}} = B_{R,\underline{\alpha}}$ as right comodule algebras with left comodule structure defined by*

$$\lambda(x_i) = \prod_j X_j^{t_{ji}} \otimes x_i$$

*is a group isomorphism.*

*Proof.* The description of $\mathrm{Gal}(kC_N^m)$ is probably well known. We will sketch how to deduce it from Kreimer's result cited at the beginning of the preceding section. First, consider a $kC_N$-Galois object $A$. This is a strongly graded ring, and has a normal basis, so it is easy to see that it has the form $A \cong k[x]/(x^N - \alpha)$ for some $\alpha \in \dot{k}$. Any isomorphism $k[x]/(x^N - \alpha) \to k[x]/(x^N - \alpha')$ maps $x$ to $\lambda x$ for some $\lambda \in \dot{k}$, and one checks that $x \mapsto \lambda x$ extends to a well defined isomorphism iff $\alpha = \lambda^N \alpha'$. This is the desired description of $\mathrm{Gal}(kC_N^1)$. By induction over [3, Thm. 3.7], using $\mathrm{Pair}(kC_N^\ell, kC_N) \cong \mathrm{Pair}(kC_N, kC_N)^\ell \cong (\mathbb{Z}/(N))^\ell$, one obtains an isomorphism

$$\mathrm{Gal}(kC_N^m) \cong \mathrm{Gal}(kC_N)^m \oplus (\mathbb{Z}/(N))^{\frac{m(m-1)}{2}}$$

Identifying $(\mathbb{Z}/(N))^{\frac{m(m-1)}{2}}$ with $\mathrm{Skew}_m$ yields the claim on $\mathrm{Gal}(kC_N^m)$.

In [9, Lem. 4.7] we have shown that there is an isomorphism

$$\mathrm{Aut}_{\mathrm{Hopf}}(H) \ltimes \mathrm{Gal}(H) \ni (f, A) \mapsto {}^f A \in \mathrm{BiGal}(H)$$

for any cocommutative Hopf algebra $H$, where the action of $\mathrm{Aut}_{\mathrm{Hopf}}(H)$ on $\mathrm{Gal}(H)$ is given by $A \leftharpoonup f = A^{f^{-1}}$. In our case, clearly $\mathrm{Aut}_{\mathrm{Hopf}}(kC_N^m) \cong \mathrm{GL}_m(\mathbb{Z}/(N))$. We choose the identification

$$\mathrm{GL}_m(\mathbb{Z}/(N)) \ni T \mapsto F_T \in \mathrm{Aut}_{\mathrm{Hopf}}(kC_N^m)$$

given by $F_T(X_i) = \prod_j X_j^{t_{ji}}$. Assume $N$ is odd. We have to show $B_{R,\underline{\alpha}}^{F_T^{-1}} \cong B_{R\leftharpoonup T, \underline{\alpha}\leftharpoonup T}$ and ${}^{F_T}B_{R,\underline{\alpha}} \cong B_{T,R,\underline{\alpha}}$. The latter is straightforward, the former is proved by considering the isomorphism

$$G_T : B_{R\leftharpoonup T, \underline{\alpha}\leftharpoonup T} \to B_{R,\underline{\alpha}}^{F_T^{-1}}$$

determined by $G_T(x_i) = \prod_j x_j^{t_{ji}}$.

To see that this is really well defined, one needs the following fact: Let $a, b$ be elements in a $k$-algebra $R$ satisfying $ba = uab$ for some $u \in k$. Then one proves by induction that $(ab)^N = u^{\frac{1}{2}N(N-1)} a^N b^N$, so that, if $N$ is odd and $u$ is any $N$-th root of unity, then $(ab)^N = a^N b^N$. □

**Lemma 15.** *Let $\underline{r} \in \mathbb{Z}/(N)$. There is a unique skew pairing $\tau_{\underline{r}} : H_N \otimes kC_N^m \to k$ with $\tau_{\underline{r}}(X, X_i) = q^{-r_i}$ and $\tau_{\underline{r}}(Y, X_j) = 0$ for all $j$. Every skew pairing $\tau : H_N \otimes kC_N^m \to k$ equals $\tau_{\underline{r}}$ for a unique $\underline{r} \in (\mathbb{Z}/(N))^m$.*

The proof is immediate, since a skew pairing $\tau$ is the same as a Hopf algebra homomorphism $H_N \to kC_N^m$ in view of the self-duality of $kC_N^m$, and there is no noncentral skew primitive element in $kC_N^m$.

**Corollary 16.** *There is a bijection*

$$\begin{aligned}(\mathbb{Z}/(N))^m \times \mathrm{Skew}_m \times (\dot{k}/\dot{k}^N) \times k \times (\dot{k}/\dot{k}^N)^m &\to \mathrm{Gal}(kC_N^m \otimes H_N) \\ (R, \underline{r}, \alpha_0, \beta, \underline{\alpha}) &\mapsto \Gamma_{R,\underline{r},\alpha_0,\beta,\underline{\alpha}}\end{aligned}$$

*defined by*

$$\begin{aligned}\Gamma_{R,\underline{r},\alpha_0,\beta,\underline{\alpha}} &:= B_{R,\underline{\alpha}}\#_{\tau_{\underline{r}}}A_{\alpha_0,\beta}\\ &\cong k\langle x_0,\dots,x_m,y\rangle/(x_i^N-\alpha_i, y^N-\beta, yx_i-q^{\delta_{i0}}x_iy, x_jx_i-q^{r_{ij}}x_ix_j)\end{aligned}$$

*where we have put* $r_0 := 1$ *and* $r_{0i} := r_i =: -r_{i0}$. *The right comodule algebra structure is determined by* $\rho(x_i) = x_i \otimes X_i$ *and* $\rho(y) = 1 \otimes Y + y \otimes X_0$.

*Proof.* One only has to check that $x_i := x_i\#1$ has the indicated commutation relations with $x_0 := 1\#x_0$ and $y := 1\#y$ in the twisted product. We have $(1\#x_0)(x_i\#1) = \tau_{\underline{r}}(X_0, X_i)x_i\#x_0 = q^{-r_i}(x_i\#1)(1\#x_0)$ and

$$(1\#y)(x_i\#1) = \tau_{\underline{r}}(1, X_i)x_i\#y + \tau_{\underline{r}}(Y, X_i)x_i\#x_0 = x_i\#y = (x_i\#1)(1\#y). \quad \square$$

We know by Proposition 12 that since $L(A, H_N) \cong H_N$ for any $H_N$-Galois object $A$ and $L(B, kC_N^m) \cong kC_N^m$ for any $kC_N^m$-Galois object $B$ by cocommutativity of $kC_N^m$, we have $L(\Lambda, kC_N^m \otimes H_N) \cong kC_N^m \bowtie_\tau H_N$ for a suitable skew pairing $\tau$, for all $kC_N^m \otimes H_N$-Galois objects $\Lambda$.

**Lemma 17.** *Let* $\underline{r} \in (\mathbb{Z}/(N))^m$. *There is a Hopf algebra isomorphism*

$$\begin{aligned} kC_N^m \bowtie_{\tau_{\underline{r}}} H_N &\cong kC_N^m \otimes H_N\\ X_i \bowtie 1 &\mapsto X_i \otimes X_0^{r_i}\\ 1 \bowtie X_0 &\mapsto 1 \otimes X_0\\ 1 \bowtie Y &\mapsto 1 \otimes Y\end{aligned}$$

*Proof.* In $kC_N^m \otimes H_N$ we have $(1 \otimes Y)(X_i \otimes X_0^{r_i}) = X_i \otimes YX_0^{r_i} = X_i \otimes q^{r_i}X_0^{r_i}Y = q^{r_i}(X_i \otimes X_0^{r_i})(1 \otimes Y)$. In $kC_N^m \bowtie_{\tau_{\underline{r}}} H_N$ we have

$$\begin{aligned}(1 \bowtie Y)(X_i \bowtie 1) &= \tau_{\underline{r}}(1, X_i)X_i \bowtie 1\tau_{\underline{r}}^{-1}(Y, X_i)\\ &\quad + \tau_{\underline{r}}(1, X_i)X_i \bowtie Y\tau_{\underline{r}}^{-1}(X_0, X_i)\\ &\quad + \tau_{\underline{r}}(Y, X_i)X_i \bowtie X_0\tau_{\underline{r}}^{-1}(X_0, X_i)\\ &= q^{r_i}(X_i \bowtie 1)(1 \bowtie Y)\end{aligned}$$

We omit the rest of the proof. $\square$

**Corollary 18.** *Every right* $kC_N^m \otimes H_N$*-Galois object has the structure of a* $kC_N^m \otimes H_N$*-*$kC_N^m \otimes H_N$*-biGalois object. For every two-cocycle* $\sigma$ *on* $kC_N^m \otimes H_N$ *we have* $(kC_N^m \otimes H_N)^\sigma \cong kC_N^m \otimes H_N$. *The isomorphism class of the Hopf algebra* $kC_N^m \otimes H_N$ *(the Taft algebra) is uniquely determined by the* $k$*-linear monoidal category of its comodules.*

In particular, we can choose the left comodule structure $\lambda$ on $\Gamma_{R,\underline{r},\alpha_0,\beta,\underline{\alpha}}$ determined by $\lambda(x_0) = X_0 \otimes x_0$, $\lambda(y) = 1 \otimes y + Y \otimes x_0$ and

$$\lambda(x_i) = X_0^{r_i}X_i \otimes x_i$$

for $i > 0$ to make it a biGalois extension.

After choosing a biGalois structure for each Galois object, all possible ones are obtained by twisting the left comodule structure with a set of representatives for the coouter automorphism group of $kC_N^m$. We will also want to determine the group structure of $\mathrm{BiGal}(kC_N^m \otimes H_N)$, at least for odd $N$. To do this, we first need some more facts and notations:

(1) The group $(\mathbb{Z}/(N))^m$ acts on the group $(\dot{k} \ltimes k) \times (\dot{k}/\dot{k}^N)^m$ by
$$(\alpha_0, \beta, \underline{\alpha}) \leftharpoonup \underline{r} = (\alpha_0, \beta, (\alpha_i \alpha_0^{r_i})_i).$$
(2) Letting $\mathrm{GL}_m$ act trivially on $\dot{k} \ltimes k$, we get an action of $\mathrm{GL}_m$ on $(\dot{k} \ltimes k) \times (\dot{k}/\dot{k}^N)$.
(3) The action of $(\mathbb{Z}/(N))^m$ on $(\dot{k} \ltimes k) \times (\dot{k}/\dot{k}^N)^m$ is $\mathrm{GL}_m$-equivariant, that is
$$((\alpha_0, \beta, \underline{\alpha}) \leftharpoonup \underline{r}) \leftharpoonup T = ((\alpha_0, \beta, \underline{\alpha}) \leftharpoonup T) \leftharpoonup (\underline{r} \leftharpoonup T)$$
holds for all $T \in \mathrm{GL}_m$, $r \in \mathrm{Skew}_m$. This ensures that we get a right action of $\mathrm{GL}_m \ltimes (\mathbb{Z}/(N))^m$ on $(\dot{k} \ltimes k) \times (\dot{k}/\dot{k}^N)^m$ defined by $\xi \leftharpoonup (T, \underline{r}) = (\xi \leftharpoonup T) \leftharpoonup r$ for $\xi \in (\dot{k} \ltimes k) \times (\dot{k}/\dot{k}^N)^m$.
(4) The map
$$\omega_0 : (\mathbb{Z}/(N))^m \times (\mathbb{Z}/(N))^m \ni (\underline{r}, \underline{r}') \mapsto (r_i r'_j - r_j r'_i)_{i,j} \in \mathrm{Skew}_m$$
is biadditive, hence a two-cocycle on the group $(\mathbb{Z}/(N))^m$ with values in the trivial module $\mathrm{Skew}_m$. The map $\omega_0$ is $\mathrm{GL}_m$-equivariant, $\omega_0(\underline{r}, \underline{r}') \leftharpoonup T = \omega_0(\underline{r} \leftharpoonup T, \underline{r}' \leftharpoonup T)$, whence $\omega((T, \underline{r}), (T', \underline{r}')) := \omega_0(\underline{r} \leftharpoonup T', \underline{r}')$ defines a two-cocycle on $\mathrm{GL}_m \ltimes (Z/(N))^m$ with values in $\mathrm{Skew}_m$.

**Theorem 19.** *Assume $N$ is odd. An isomorphism*
$$\Psi : (\mathrm{GL}_m \ltimes (\mathbb{Z}/(N))^m) \ltimes_\omega \left(\mathrm{Skew}_m \times (\dot{k} \ltimes k) \times (\dot{k}/\dot{k}^N)^m\right) \to \mathrm{BiGal}(kC_N^m \otimes H_N)$$
*is given by $\Psi((T, \underline{r}), (R, (\alpha_0, \beta), \underline{\alpha})) := \Lambda_{T, \underline{r}, R, \alpha_0, \beta, \underline{\alpha}} := \Gamma_{\underline{r}, R, \alpha_0, \beta, \underline{\alpha}}$ as a right $kC_N^m \otimes H_N$-comodule algebra, with the left comodule algebra structure determined by $\lambda(x_0) = X_0 \otimes x_0, \lambda(y) = 1 \otimes y + Y \otimes x_0$ and*
$$\lambda(x_i) = X_0^{r_i} \prod_j X_j^{t_{ji}} \otimes x_i$$
*for $i > 0$.*

*Proof.* For a general Hopf algebra $H$, let $\mathcal{G}$ be a set of $H$-$H$-biGalois objects whose underlying right $H$-Galois objects are a representative system for $\mathrm{Gal}(H)$, and let $\mathcal{F}$ be a representative system for $\mathrm{CoOut}(H)$. Then by [9, Lem.3.11.] we have a bijection
$$\mathcal{F} \times \mathcal{G} \ni (\theta, A) \mapsto {}^\theta A \in \mathrm{BiGal}(H).$$

For $\underline{r} \in (\mathbb{Z}/(N))^m$, $R \in \mathrm{Skew}_m$, $\beta \in k$, $\alpha_0 \in \dot{k}$ and $\underline{\alpha} \in (\dot{k}/\dot{k}^N)^m$ we define $\mathcal{J}(\underline{r}, R, \alpha_0, \beta, \underline{\alpha}) := \Gamma_{\underline{r},R,\alpha_0,\beta,\underline{\alpha}}$ with the left comodule algebra structure satisfying $\lambda(x_0) = X_0 \otimes x_0$, $\lambda(x_i) = X_0^{r_i} X_i \otimes x_i$ for $i > 0$ and $\lambda(y) = 1 \otimes y + Y \otimes x_0$.

Choose a representative system $\mathcal{A}$ for $\dot{k}/\dot{k}^N$. Then

$$\mathcal{J} : (\mathbb{Z}/(N))^m \times \mathrm{Skew}_m \times \mathcal{A} \times k \times (\dot{k}/\dot{k}^N)^m \to \mathrm{BiGal}(kC_N^m \otimes H_N)$$

is injective and yields a bijection when composed with the forgetful map $\mathrm{BiGal}(kC_N^m \otimes H_N) \to \mathrm{Gal}(kC_N^m \otimes H_N)$.

It is straightforward to check that the canonical map

$$\mathrm{Aut}_{\mathrm{Hopf}}(kC_N^m) \times \mathrm{CoOut}(H_N) \to \mathrm{CoOut}(kC_N^m \otimes H)$$

is a bijection. Hence, choosing a representative system $\mathcal{S}$ for $\dot{k}/\langle q\rangle$, the map

$$\begin{aligned} \mathrm{GL}_m \times \mathcal{S} \times (\mathbb{Z}/(N))^m \times \mathrm{Skew}_m \times \mathcal{A} \times k \times (\dot{k}/\dot{k}^N)^m &\to \mathrm{BiGal}(kC_N^m \otimes H_N) \\ (T, v, \underline{r}, R, \alpha_0, \beta, \underline{\alpha}) &\mapsto {}^{F_T \otimes f_v}\mathcal{J}(\underline{r}, R, \alpha_0, \beta, \underline{\alpha}) \end{aligned}$$

is a bijection. Similarly to the proof of Theorem 5 one shows that there is an isomorphism of $kC_N^m \otimes H_N$-bicomodule algebras

$$\begin{aligned} \mathcal{J}(\underline{r}, R, v^N\alpha_0, \beta, \underline{\alpha}) &\to {}^{\mathrm{id}\otimes f_v}\mathcal{J}(\underline{r}, R, \alpha_0, \beta, \underline{\alpha}) \\ x_0 &\mapsto v x_0 \\ x_i &\mapsto x_i \text{ for } i > 0 \\ y &\mapsto y \end{aligned}$$

Since $\mathcal{S} \times \mathcal{A} \ni (v, \alpha) \mapsto v^N \alpha \in \dot{k}$ is a bijection, we obtain a bijection

$$\Psi : \mathrm{GL}_m \times (\mathbb{Z}/(N))^m \times \mathrm{Skew}_m \times \dot{k} \times k \times (\dot{k}/\dot{k}^N)^m \cong \mathrm{BiGal}(kC_N^m \otimes H_N)$$

by setting

$$\Psi(T, \underline{r}, R, \alpha_0, \beta, \underline{\alpha}) = {}^{F_T \otimes \mathrm{id}}\mathcal{J}(\underline{r}, R, \alpha_0, \beta, \underline{\alpha}),$$

which clearly has the form indicated in the formulation of the Theorem.

It remains to verify that $\Psi$ is a group homomorphism. Let $T, T' \in \mathrm{GL}_m$, $\underline{r}, \underline{r}' \in (\mathbb{Z}/(N)^m$, $R, R' \in \mathrm{Skew}_m$, $\beta, \beta' \in k$, $\alpha_0, \alpha_0' \in \dot{k}$ and $\underline{\alpha}, \underline{\alpha}' \in (\dot{k}/\dot{k}^N)^m$. Set $T'' := TT'$, $\underline{r}'' := \underline{r} \leftharpoonup T' + \underline{r}'$,

$$r''_{ij} := \sum_{k,\ell} r_{k\ell} t'_{ki} t'_{\ell j} + \sum_k (r_k t'_{ki} r'_j - r_k t'_{kj} r'_i) + r'_{ij},$$

$\beta'' := \beta\alpha_0' + \beta'$, $\alpha_0'' := \alpha_0 \alpha_0'$, and $\alpha_i'' := \alpha_0^{r_i'} \left(\prod_j \alpha_j^{t'_{ji}}\right) \alpha_i'$. We have to check

$$\Lambda_{T'',\underline{r}'',R'',\alpha_0'',\beta'',\underline{\alpha}''} \cong \Lambda_{T,\underline{r},R,\alpha_0,\beta,\underline{\alpha}} \mathop{\Box}_{kC_N^m \otimes H_N} \Lambda_{T',\underline{r}',R',\alpha_0',\beta',\underline{\alpha}'}.$$

We can define an isomorphism $\delta$ by $\delta(x_0) = x_0 \otimes x_0$, $\delta(x_i) = x_0^{r'_i} \prod_j x_j^{t'_{ji}} \otimes x_i$ for $i > 0$, and $\delta(y) = 1 \otimes y + y \otimes x_0$.

In fact $\delta(x_i)$ is in the cotensor product because

$$\rho(x_0^{r'_i} \prod_j x_j^{t'_{ji}}) \otimes x_i = (x_0 \otimes X_0)^{r'_i} \prod_j (x_j \otimes X_j)^{t'_{ji}} \otimes x_i$$
$$= x_0^{r'_i} \prod_j x_j^{t'_{ji}} \otimes X_0^{r'_i} \prod_j X_j^{t'_{ji}} \otimes x_i = x_0^{r'_i} \prod_j x_j^{t'_{ji}} \otimes \lambda(x_i).$$

For $i > 0$ we have

$$\delta(x_i)^N = (x_0^{r'_i} \prod_j x_j^{t'_{ji}} \otimes x_i)^N = (x_0^N)^{r'_i} \prod_j (x_j^N)^{t'_{ji}} \otimes x_i^N = \alpha_0^{r'_i} \prod_j \alpha_j^{t'_{ji}} \alpha'_i (1 \otimes 1),$$

using again that for any elements $a, b \in R$ of a $k$-algebra $R$ satisfying $ba = uab$ for some $N$-th root of unity in $k$, we have $(ab)^N = a^N b^N$. Moreover, for $i, j > 0$

$$\begin{aligned}
\delta(x_j)\delta(x_i) &= \left(x_0^{r'_j} \prod_\ell x_\ell^{t'_{\ell j}} \otimes x_j\right) \left(x_0^{r'_i} \prod_k x_k^{t'_{ki}} \otimes x_i\right) \\
&= x_0^{r'_j} \left(\prod_\ell x_\ell^{t'_{\ell j}}\right) x_0^{r'_i} \left(\prod_k x_k^{t'_{ki}}\right) \otimes x_j x_i \\
&= q^{(r'_{ij} + \sum_\ell r_\ell t'_{\ell j} r'_i)}\, x_0^{r'_j} x_0^{r'_i} \left(\prod_\ell x_\ell^{t'_{\ell j}}\right) \left(\prod_k x_k^{t'_{ki}}\right) \otimes x_i x_j \\
&= q^{(r'_{ij} + \sum_\ell r_\ell t'_{\ell j} r'_i + \sum_{k,\ell} r_{k\ell} t'_{ki} t'_{\ell j})}\, x_0^{r'_i} x_0^{r'_j} \left(\prod_k x_k^{t'_{ki}}\right) \left(\prod_\ell x_\ell^{t'_{\ell j}}\right) \otimes x_i x_j \\
&= q^{(r'_{ij} + \sum_\ell r_\ell t'_{\ell j} r'_i + \sum_{k,\ell} r_{k\ell} t'_{ki} t'_{\ell j} - \sum r_k t_{kj} r'_j)}\, x_0^{r'_i} \left(\prod_k x_k^{t'_{ki}}\right) x_0^{r'_j} \left(\prod_\ell x_\ell^{t'_{\ell j}}\right) \otimes x_i x_j \\
&= q^{r''_{ij}}\, \delta(x_i)\delta(x_j).
\end{aligned}$$

We omit the remaining details showing that $\delta$ is a well defined bicomodule algebra homomorphism. It is then an isomorphism since its domain and codomain are Galois algebras. $\square$

**Corollary 20.** *If $N$ is odd and $k$ is algebraically closed, then*

$$\mathrm{BiGal}(kC_N^m \otimes H_N) \cong ((\mathrm{GL}_m \ltimes (\mathbb{Z}/(N))^m) \ltimes_\omega \mathrm{Skew}_m) \times (\dot{k} \ltimes k).$$

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