BIALGEBRAS OVER NONCOMMUTATIVE RINGS AND A STRUCTURE THEOREM FOR HOPF BIMODULES

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1. Introduction

The following is a well known principle in the theory of quantum groups and Hopf algebras: The category of modules over an algebra L is a monoidal (or tensor) category if and only if L has a bialgebra structure. In fact, many properties of Hopf algebras (like quasitriangularity) and constructions (like the Drinfeld double) can be understood much better through this correspondence. Moreover, reconstruction theorems allow one to construct Hopf algebras from monoidal categories.

However, we have deliberately misstated the correspondence between monoidal categories and bialgebra structures. If L is a bialgebra then in fact the category ${}_L\mathcal{M}$ of left L-modules has a tensor product, namely the tensor product over the base ring k endowed with the diagonal L-module structure. Conversely, if ${}_L\mathcal{M}$ has a tensor product making it a monoidal category, then to conclude that L is a bialgebra we have to assume that the underlying k-module of the tensor product in ${}_L\mathcal{M}$ is the tensor product over k. (Otherwise, a trivial counterexample is the category of R-bimodules for any k-algebra R, which is monoidal with the tensor product over R and can be identified with the category of left $R \otimes R^{\mathrm{op}}$-modules, although $R \otimes R^{\mathrm{op}}$ is certainly not in general a bialgebra.) In other words, we have to assume that the underlying functor ${}_L\mathcal{M} \to {}_k\mathcal{M}$ is a monoidal functor. In this form the statement appears in Pareigis [7].

In the present paper we will encounter a fairly natural example of an algebra whose modules form a monoidal category, but which is not in general a bialgebra. This raises the question if in this situation there is also a structure of L which replaces the role of a bialgebra structure in being "responsible" for the tensor product in the module category. We will find the structure of a $\times_B$-bialgebra in the sense of Takeuchi to be suitable for this purpose.

The algebra L arises in the context of a faithfully flat H-Galois extension A/B (where H is a Hopf algebra, flat over k). In this situation Schneider [11] proves the following structure theorem for relative Hopf modules: There is a category equivalence $\mathcal{M}_A^H \cong \mathcal{M}_B$ between right relative Hopf modules and right B-modules, mapping

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a right B-module N to the relative Hopf module $N \underset{B}{\otimes} A$. For the special case of absolute Hopf modules, i.e. for $A = H$, this result is much older [14], and asserts that every Hopf module $M \in \mathcal{M}_H^H$ is trivial, that is, M is of the form $V \otimes H$, where V is a k-module. Woronowicz [21] has reinvented this structure theorem and extended it to the categories ${}_H\mathcal{M}_H^H$ and ${}_H^H\mathcal{M}_H^H$ of two-sided and of two-sided two-cosided Hopf modules. In [8] we provided basis-free versions of the classifications of Hopf modules in the form of category equivalences ${}_H\mathcal{M}_H^H \cong {}_H\mathcal{M}$ and ${}_H^H\mathcal{M}_H^H \cong \mathcal{YD}_H^H$ with the categories of left H-modules and of right-right Yetter-Drinfeld modules, respectively.

The motivation for studying Hopf modules in [21] comes from noncommutative geometry. The H-bimodules in consideration are generalizations of the module of differential one-forms on a Lie group, whose function algebra is replaced by a general Hopf algebra. The coactions of H on the bimodule $\Omega^1(H)$ of differential one-forms are natural generalizations of the action of the Lie group on its differential forms, on both sides.

This motivation for considering Hopf modules carries over to general Hopf Galois extensions. From the viewpoint of noncommutative geometry, a faithfully flat H-Galois extension A/B can be considered as the noncommutative analog of the affine algebraic version of a principal fiber bundle, or an H-torsor. Now if we want to consider bimodules $\Omega^1(A)$ over A generalizing the modules of differential one-forms on the total space of a fiber bundle, the action of the structure group on the differential forms should generalize to a coaction of H on $\Omega^1(A)$ making it a relative Hopf bimodule in ${}_A\mathcal{M}_A^H$. It is thus natural to try to generalize the structure theorem for Hopf bimodules to a structure theorem for relative Hopf modules over Hopf-Galois extensions.

Such a generalization will be given in section 3. We will find, for any right H-comodule algebra R and faithfully flat H-Galois extension A/B, an algebra L such that the category of relative Hopf bimodules ${}_R\mathcal{M}_A^H$ is equivalent to that of left L-modules. The equivalence is induced by Schneider's equivalence $\mathcal{M}_A^H \cong \mathcal{M}_B$. Explicitly, $L := L(A, R, H) := (R \otimes A)^{\operatorname{co} H}$, a subalgebra of $R \otimes A^{\mathrm{op}}$. For the special case $R = A$ we set $L(A, H) := L(A, A, H)$. If we specialize further to an H-Galois extension of k, that is, to the case $B = k$, the algebra $L(A, H)$ appears in [9] and is in fact a Hopf algebra. It has been constructed by Van Oystaeyen and Zhang [20] in the case where A (hence also H) is commutative. If $A = H$, then $L(A, H) = H$, so that the category equivalence ${}_A\mathcal{M}_A^H \cong {}_{L(A,H)}\mathcal{M}$ generalizes ${}_H\mathcal{M}_H^H \cong {}_H\mathcal{M}$.

Note that the category of relative Hopf bimodules ${}_A\mathcal{M}_A^H$ is in a natural way a monoidal category, namely with the tensor product over A of two relative Hopf bimodules, endowed with the codiagonal comodule structure. In particular, ${}_{L(A,H)}\mathcal{M}$ is a monoidal category as well. However, the algebra $L(A, H)$ is a Hopf algebra only if $B = k$. The reason that L is not a bialgebra otherwise is that the underlying functor ${}_L\mathcal{M} \to {}_k\mathcal{M}$ is not monoidal.

Although L is not a bialgebra, it is a $\times_B$-bialgebra. This structure on L suffices to describe the tensor product in ${}_L\mathcal{M}$ induced from the tensor product of relative Hopf bimodules. The notion of $\times_B$-bialgebras was introduced by Sweedler [15] for commutative B, and generalized to any B by Takeuchi [16]. Ordinary bialgebras are the special case $B = k$. We review the definition and notations in section 4.

In section 5 we generalize the well known correspondence between bialgebra structures and monoidal module categories to the case of modules over $\times_B$-bialgebras: An algebra over $B \otimes B^{\mathrm{op}}$ is a $\times_B$-bialgebra if and only if its module category is a monoidal category in such a way that the underlying functor to the category of left $B \otimes B^{\mathrm{op}}$-modules is monoidal. We also show that suitably defined comodules over a $\times_B$-bialgebra form a monoidal category as well. This allows one to define comodule algebras over $\times_B$-bialgebras.

In section 6 we show how these results apply to the algebra $L(A, H)$ for an H-Galois extension A/B. We determine the $\times_B$-bialgebra structure on $L(A, H)$ for which the category equivalence ${}_A\mathcal{M}_A^H \cong {}_L\mathcal{M}$ is a monoidal equivalence. In the case $B = k$ of an H-Galois extension of the base ring, we know from [9] that $L := L(A, H)$ is a Hopf algebra, and A is an L-H-bicomodule algebra and left L-Galois extension of k. Moreover, ${}^L\mathcal{M} \cong {}^H\mathcal{M}$ as monoidal categories, and the roles of L and H in the L-H-biGalois extension are completely symmetric. The rest of the paper will be devoted to partial generalizations of these facts to the case of general B. We will find that A is a left L-comodule algebra. Although there is no established definition of Hopf-Galois extensions with a $\times_B$-bialgebra instead of a Hopf algebra, we will find some evidence that A is an L-Galois extension of k: A certain map $A \otimes A \to L \diamond A$ analogous to the Galois map is a bijection, and we have ${}^{\mathrm{co}\,L}A = k$ with a suitable definition of the left hand side. If k is a field, we also have a category equivalence ${}_A^L\mathcal{M} \cong {}_k\mathcal{M}$ (note that if $B = k$ this would prove that A is L-Galois over k by Schneider's theorem [11, Thm.I].) As for the symmetry between the roles of H and L, we find that the construction $L = (A \otimes A)^{\mathrm{co}\,H}$ has a counterpart in an isomorphism $H \cong {}^{\mathrm{co}\,L}(A \underset{B}{\otimes} A)$, and the category equivalence ${}_A\mathcal{M}_A^H \cong {}_L\mathcal{M}$ has its counterpart in a category equivalence ${}_A^L\mathcal{M}_A \cong \mathcal{M}_H$. We will also prove that the categories ${}^L\mathcal{M}$ and ${}^H\mathcal{M}$ are equivalent as monoidal categories. Finally, we prove a generalization of the structure theorem for two-sided two-cosided Hopf modules in the form of a category equivalence ${}_A^L\mathcal{M}_A^H \cong \mathcal{YD}_H^H$.

2. Preliminaries

We review some definitions, notations, and facts that will be used freely throughout the paper.

A monoidal category is a category $\mathcal{C}$ with a tensor product bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is associative in the sense that there is a natural coherent associativity isomorphism $\alpha : X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z$ and a unit object I with two natural and coherent isomorphisms $I \otimes X \cong X \cong X \otimes I$. A monoidal functor $(\mathcal{F}, \xi) : \mathcal{C} \to \mathcal{D}$ between monoidal categories consists of a functor $\mathcal{F}$ and a coherent natural isomorphism $\xi : \mathcal{F}(X \otimes Y) \cong \mathcal{F}(X) \otimes \mathcal{F}(Y)$. A strict monoidal functor is a monoidal functor in which ξ is the identity. A prebraiding on a monoidal category $\mathcal{C}$ is a natural morphism $\sigma : X \otimes Y \to Y \otimes X$ satisfying $\sigma_{X \otimes Y, Z} = (\sigma_{X,Z} \otimes Y)(X \otimes \sigma_{Y,Z})$, $\sigma_{X, Y \otimes Z} = (Y \otimes \sigma_{X,Z})(\sigma_{X,Y} \otimes Z)$ and $\sigma_{I,X} = \sigma_{X,I} = \mathrm{id}_X$. A braiding is a prebraiding that is an isomorphism.

Throughout the paper k is a fixed commutative ring. All modules, algebras, coalgebras, Hopf algebras are over k, unadorned $\otimes$ means $\otimes_k$, Hom means Hom_k if nothing else is indicated. We denote the multiplication map of an algebra by $\nabla : A \otimes A \to A$, the unit by $\eta : k \to A$, the comultiplication of a coalgebra by $\Delta : C \to C \otimes C$, the counit map by $\varepsilon : C \to k$, and the antipode of a Hopf algebra by S. The categories of left A-modules, right B-modules and A-B-bimodules for algebras A and B are denoted by ${}_A\mathcal{M}$, $\mathcal{M}_B$ and ${}_A\mathcal{M}_B$. The categories of left C-comodules, right D-comodules and C-D-bicomodules for coalgebras C and D are denoted by ${}^C\mathcal{M}$, $\mathcal{M}^D$ and ${}^C\mathcal{M}^D$. A right C-comodule structure on V will usually be denoted by $\delta : V \to V \otimes C$. We make use of the following versions of Sweedler's notation: $\Delta(h) = h_{(1)} \otimes h_{(2)}$, $\delta(v) = v_{(0)} \otimes v_{(1)}$ for a right comodule, and $v \mapsto v_{(-1)} \otimes v_{(0)}$ for a left comodule structure.

Let H be a k-flat Hopf algebra. A right H-comodule algebra A is said to be an H-Galois extension of $B := A^{\mathrm{co}\,H} := \{a \in A | \delta(a) = a \otimes 1\}$ if the Galois map $\kappa : A \underset{B}{\otimes} A \to A \otimes H$, given by $\kappa(x \otimes y) = x y_{(0)} \otimes y_{(1)}$, is a bijection. We will use the following formal notation for the inverse of κ in this situation: For $h \in H$ we write $\kappa^{-1}(1 \otimes h) =: h^{[1]} \otimes h^{[2]}$, so that $h^{[1]} {h^{[2]}}_{(0)} \otimes {h^{[2]}}_{(1)} = 1 \otimes h \in A \otimes H$ by definition. The following formulas, for $g, h \in H$, $b \in B$ and $a \in A$ are in [12, (3.4)]:

$$h^{[1]} \otimes {h^{[2]}}_{(0)} \otimes {h^{[2]}}_{(1)} = {h_{(1)}}^{[1]} \otimes {h_{(1)}}^{[2]} \otimes h_{(2)} \tag{2.1}$$

$${h^{[1]}}_{(0)} \otimes h^{[2]} \otimes {h^{[1]}}_{(1)} = {h_{(2)}}^{[1]} \otimes {h_{(2)}}^{[2]} \otimes S(h_{(1)}) \tag{2.2}$$

$$h^{[1]} h^{[2]} = \varepsilon(h) 1_A \tag{2.3}$$

$$(gh)^{[1]} \otimes (gh)^{[2]} = h^{[1]} g^{[1]} \otimes g^{[2]} h^{[2]} \tag{2.4}$$

$$b h^{[1]} \otimes h^{[2]} = h^{[1]} \otimes h^{[2]} b \tag{2.5}$$

$$a_{(0)} {a_{(1)}}^{[1]} \otimes {a_{(1)}}^{[2]} = 1 \otimes a \tag{2.6}$$

Let A be a right H-comodule algebra. This is an algebra in the monoidal category of right H-comodules. Recall that a relative Hopf module $M \in \mathcal{M}_A^H$ (resp. $M \in {}_A\mathcal{M}^H$) is a right (resp. left) A-module in the category of right H-comodules, that is a right H-comodule and right (resp. left) A-module such that $\delta(ma) = m_{(0)} a_{(0)} \otimes m_{(1)} a_{(1)}$ (resp. $\delta(am) = a_{(0)} m_{(0)} \otimes a_{(1)} m_{(1)}$) holds for all $m \in M$ and $a \in A$. Given two

H-comodule algebras R and A, a relative Hopf bimodule $M \in {}_R\mathcal{M}^H_A$ is an object of both ${}_R\mathcal{M}^H$ and $\mathcal{M}^H_A$ with the same comodule structure and such that the two module structures make it an R-A-bimodule. The following structure theorem for relative Hopf modules over faithfully flat Hopf Galois extensions was proved by Schneider [11]: If A is a right H-Galois extension of B and ${}_BA$ is faithfully flat, then

$$\begin{aligned} \mathcal{M}^H_A &\to \mathcal{M}_B \\ M &\mapsto M^{\operatorname{co}H} \end{aligned} \qquad\qquad \begin{aligned} \mathcal{M}_B &\to \mathcal{M}^H_A \\ N &\mapsto N \underset{B}{\otimes} A \end{aligned}$$

are quasi-inverse category equivalences. More precisely, for any right H-comodule algebra A and $B = A^{\operatorname{co}H}$ we have a pair of adjoint functors as above, and left faithfully flat right H-Galois extensions are characterized by these functors being quasi-inverse equivalences.

The isomorphism $N \cong (N \underset{B}{\otimes} A)^{\operatorname{co}H}$ for $N \in \mathcal{M}_B$ is given by $n \mapsto n \otimes 1$, the isomorphism $\mu : M^{\operatorname{co}H} \underset{B}{\otimes} A \cong M$ for $M \in \mathcal{M}^H_A$ by $m \otimes a \mapsto ma$, and its inverse by

$$M \cong M^{\operatorname{co}H} \underset{B}{\otimes} A \qquad m \mapsto m_{(0)}{m_{(1)}}^{[1]} \otimes {m_{(1)}}^{[2]}. \tag{2.7}$$

This formula for the inverse of μ will be of key importance for many of our calculations. The map is well defined because of faithful flatness of ${}_BA$ and (2.2), and inverse to μ because of (2.3) and (2.6). If the antipode of H is bijective and A_B is faithfully flat, then we also have a category equivalence ${}_A\mathcal{M}^H \cong {}_B\mathcal{M}$.

Note that in case k is a field and H has bijective antipode, the conditions that A be left or right faithfully flat over B are equivalent for an H-Galois extension A by [11, Thm.I].

Let H be a Hopf algebra. A (right-right) Yetter-Drinfeld module $V \in \mathcal{YD}^H_H$ is a right H-module (we denote the action by $\leftharpoonup$) and H-comodule such that $v_{(0)} \leftharpoonup h_{(1)} \otimes v_{(1)}h_{(2)} = (v \leftharpoonup h_{(2)})_{(0)} \otimes h_{(1)}(v \leftharpoonup h_{(2)})_{(1)}$ holds for all $v \in V$ and $h \in H$. The category of Yetter-Drinfeld modules is monoidal, with the tensor product of two Yetter-Drinfeld modules being their tensor product over k with the diagonal module structure and codiagonal comodule structure. The category $\mathcal{YD}^H_H$ is prebraided (braided if the antipode of H is bijective) with prebraiding $V \otimes W \ni v \otimes w \mapsto w_{(0)} \otimes v \leftharpoonup w_{(1)} \in W \otimes V$ for $V, W \in \mathcal{YD}^H_H$. The category ${}^H_H\mathcal{YD}$ of left-left Yetter-Drinfeld module is defined by the compatibility condition $h_{(1)}v_{(-1)} \otimes h_{(2)} \rightharpoonup v_{(0)} = (h_{(1)} \rightharpoonup v)_{(-1)}h_{(2)} \otimes (h_{(1)} \rightharpoonup v)_{(0)}$.

Let A be an H-Galois extension of B. We denote by M^B the centralizer of B in an A-bimodule M. The Miyashita-Ulbrich action of H on A^B is defined by $a \leftharpoonup h = h^{[1]}ah^{[2]}$, see [17, 5]. It makes A^B (which is easily seen to be a subcomodule of H) a right-right Yetter-Drinfeld module, which is also a module algebra and comodule algebra. The formula $yx = x_{(0)}(y \leftharpoonup x_{(1)})$ holds for all $x \in A$ and $y \in A^B$, so that A^B is in particular a commutative algebra in the braided category of Yetter-Drinfeld modules, as observed in [3]. The following obvious generalization was noted in [10]: For any A-bimodule M, the centralizer M^B is a right H-module by $m \leftharpoonup h = h^{[1]}mh^{[2]}$, and if $M \in {}_A\mathcal{M}_A^H$ is a relative Hopf bimodule, then $M^B \in \mathcal{YD}_H^H$.

3. Hopf bimodules over a Hopf Galois extension

Throughout this section H is a k-flat Hopf algebra. Let R and A be two right H-comodule algebras. We shall extend Schneider's structure theorem for relative Hopf modules to a description of the structure of relative Hopf bimodules. First we note a simple property of the coinvariants functor:

Lemma 3.1. *Let H be a k-flat Hopf algebra. Let A be a left faithfully flat right H-Galois extension of B, let S be a k-algebra and $M \in {}_S\mathcal{M}_A^H$, where S has the trivial comodule structure. Then for any $N \in \mathcal{M}_S$ we have $(N \underset{S}{\otimes} M)^{\operatorname{co} H} \cong N \underset{S}{\otimes} (M^{\operatorname{co} H})$.*

Proof. Both sides of the claimed isomorphism carry right B-module structures induced by that of M. After tensoring with ${}_BA$, both become isomorphic to $N \underset{S}{\otimes} M \in \mathcal{M}_A^H$ by [11]. □

Definition 3.2. Let H be a Hopf algebra and let R, A be two right H-comodule algebras. Put $L(R, A, H) := (R \otimes A)^{\operatorname{co} H}$. Then $L(R, A, H)$ is a subalgebra of $R \otimes A^{\mathrm{op}}$. We write $L(A, H) := L(A, A, H)$.

We note for later use that

$$L(R, A, H) \\ = \left\{ \sum x_i \otimes y_i \in R \otimes A \middle| \sum x_i \otimes y_{i(0)} \otimes y_{i(1)} = \sum x_{i(0)} \otimes y_i \otimes S(x_{i(1)}) \right\} \quad (3.1)$$

as in [11, Lem.3.1]. Multiplication in $L(R, A, H)$ is given by $(\sum r_i \otimes x_i)(\sum s_j \otimes y_j) = \sum r_i s_j \otimes y_j x_i$. While it is easy to check that $L(R, A, H)$ is a subalgebra of $R \otimes A^{\mathrm{op}}$, it is not true in general that it is a subalgebra of $R \otimes A$. We fix the formal notation $\xi = \xi^{(1)} \otimes \xi^{(2)} \in R \otimes A$ for $\xi \in L(R, A, H)$, so that $\xi\zeta = \xi^{(1)}\zeta^{(1)} \otimes \zeta^{(2)}\xi^{(2)}$.

Theorem 3.3. *Let H be a k-flat Hopf algebra and A a right H-Galois extension of $B := A^{\operatorname{co} H}$ with ${}_BA$ faithfully flat. Let R be a right H-comodule algebra, and put $L := L(R, A, H)$. Then we have quasi-inverse category equivalences*

$$\begin{aligned} {}_R\mathcal{M}_A^H &\to {}_L\mathcal{M} \\ M &\mapsto M^{\operatorname{co} H} \end{aligned} \qquad\qquad \begin{aligned} {}_L\mathcal{M} &\to {}_R\mathcal{M}_A^H \\ N &\mapsto N \underset{B}{\otimes} A \end{aligned}$$

where

1. *The L-module structure of $M^{\operatorname{co} H}$ is defined by $\xi \triangleright m = \xi^{(1)} m \xi^{(2)}$.*

(2) *For $N \in {}_L\mathcal{M}$ we use $\triangleright$ to denote the module structure. Then N is a right B-module with $nb := (1 \otimes b) \triangleright n$. The right A-module and H-comodule structures on $N \underset{B}{\otimes} A$ are those induced by the right tensorand. The left R-module structure is given by*

$$x(n \otimes a) = (x_{(0)} \otimes {x_{(1)}}^{[1]}) \triangleright n \otimes {x_{(1)}}^{[2]}a.$$

Proof. It is straightforward to check that the functor ${}_R\mathcal{M}_A^H \to {}_L\mathcal{M}$ is well defined as stated. To construct the inverse functor, let $N \in {}_L\mathcal{M}$. We have to check that the R-module structure in (2) is well defined. First, define $\phi : L \otimes A \otimes N \to N \underset{B}{\otimes} A$ by $\phi(\xi \otimes x \otimes n) = \xi \triangleright n \otimes x$. We calculate, for $\xi \in L, b \in B, x \in A$ and $n \in N$:

$$\begin{aligned}
\phi(\xi^{(1)} \otimes \xi^{(2)}b \otimes x \otimes n) &= (\xi^{(1)} \otimes \xi^{(2)}b) \triangleright n \otimes x \\
&= (1 \otimes b)\xi \triangleright n \otimes x \\
&= (\xi \triangleright n)b \otimes x \\
&= \xi \triangleright n \otimes bx \\
&= \phi(\xi \otimes bx \otimes n)
\end{aligned}$$

and conclude that ϕ factors through $L \underset{B}{\otimes} A \otimes N$, where the right B-module structure on L is $\xi b = \xi^{(1)} \otimes \xi^{(2)}b$. For all $x \in R$ we have $x_{(0)} \otimes {x_{(1)}}^{[1]} \otimes {x_{(1)}}^{[2]} \in L \underset{B}{\otimes} A$, since this is the image of $x \otimes 1 \in R \otimes A \in \mathcal{M}_A^H$ under the isomorphism (2.7), and thus the map $\psi_0 : R \otimes N \otimes A \to N \underset{B}{\otimes} A$ with

$$\psi_0(x \otimes n \otimes a) = \phi(x_{(0)} \otimes {x_{(1)}}^{[1]} \otimes {x_{(1)}}^{[2]} \otimes n)a = (x_{(0)} \otimes {x_{(1)}}^{[1]}) \triangleright n \otimes {x_{(1)}}^{[2]}a$$

for $x \in R$, $n \in N$ and $a \in A$ is well defined. Now ψ_0 factors through a map $\psi : R \otimes N \underset{B}{\otimes} A \to N \underset{B}{\otimes} A$, since

$$\begin{aligned}
\psi_0(x \otimes nb \otimes a) &= (x_{(0)} \otimes {x_{(1)}}^{[1]}) \triangleright (nb) \otimes {x_{(1)}}^{[2]}a \\
&= (x_{(0)} \otimes {x_{(1)}}^{[1]})(1 \otimes b) \triangleright n \otimes {x_{(1)}}^{[2]}a \\
&= (x_{(0)} \otimes b{x_{(1)}}^{[1]}) \triangleright n \otimes {x_{(1)}}^{[2]}a \\
&= (x_{(0)} \otimes {x_{(1)}}^{[1]}) \triangleright n \otimes {x_{(1)}}^{[2]}ba \\
&= \psi_0(x \otimes n \otimes ba),
\end{aligned}$$

for $x \in R$, $n \in N$, $b \in B$ and $a \in A$, using (2.5). We claim next that $x(n \otimes a) := \psi(x \otimes n \otimes a)$ for $x \in R$, $n \in N$ and $a \in A$ defines a left R-module structure on $N \underset{B}{\otimes} A$. In fact

$$\begin{aligned}
(xy)(n \otimes a) &= \left(x_{(0)}y_{(0)} \otimes (x_{(1)}y_{(1)})^{[1]}\right) \triangleright n \otimes (x_{(1)}y_{(1)})^{[2]}a \\
&= (x_{(0)}y_{(0)} \otimes {y_{(1)}}^{[1]}{x_{(1)}}^{[1]}) \triangleright n \otimes {x_{(1)}}^{[2]}{y_{(1)}}^{[2]}a
\end{aligned}$$

$$
\begin{aligned}
&= (x_{(0)} \otimes x_{(1)}{}^{[1]})(y_{(0)} \otimes y_{(1)}{}^{[1]}) \triangleright n \otimes x_{(1)}{}^{[2]} y_{(1)}{}^{[2]} a \\
&= x(y(n \otimes a))
\end{aligned}
$$

holds for $x, y \in R$, $n \in N$ and $a \in A$, using (2.4). Clearly, $N \underset{B}{\otimes} A$ is an R-A-bimodule in this way, and it is also a relative Hopf module in ${}_R\mathcal{M}^H$, because

$$
\begin{aligned}
\delta(x(n \otimes a)) &= \delta\left((x_{(0)} \otimes x_{(1)}{}^{[1]}) \triangleright n \otimes x_{(1)}{}^{[2]} a\right) \\
&= (x_{(0)} \otimes x_{(1)}{}^{[1]}) \triangleright n \otimes x_{(1)}{}^{[2]}{}_{(0)} a_{(0)} \otimes x_{(1)}{}^{[2]}{}_{(1)} a_{(1)} \\
&= (x_{(0)} \otimes x_{(1)}{}^{[1]}) \triangleright n \otimes x_{(1)}{}^{[2]} a_{(0)} \otimes x_{(2)} a_{(1)} \\
&= x_{(0)}(n \otimes a_{(0)}) \otimes x_{(1)} a_{(1)}
\end{aligned}
$$

for $x \in R$, $n \in N$ and $a \in A$, using (2.1). We have proved that we have a well defined functor ${}_L\mathcal{M} \ni N \mapsto N \underset{B}{\otimes} A \in {}_R\mathcal{M}_A^H$. By [11, Thm.I] we have $N \cong (N \underset{B}{\otimes} A)^{\mathrm{co}\,H}$ as right B-modules via $n \mapsto n \otimes 1$, and $\mu : M^{\mathrm{co}\,H} \underset{B}{\otimes} A \ni m \otimes a \mapsto ma \in M$ is an isomorphism in $\mathcal{M}_A^H$. The first isomorphism is an L-linear map because, in $(N \underset{B}{\otimes} A)^{\mathrm{co}\,H}$, we have, for $n \in N$ and $\xi \in L$,

$$
\begin{aligned}
\xi \triangleright (n \otimes 1) &= \xi^{(1)}(n \otimes \xi^{(2)}) \\
&= (\xi^{(1)}{}_{(0)} \otimes \xi^{(1)}{}_{(1)}{}^{[1]}) \triangleright n \otimes \xi^{(1)}{}_{(1)}{}^{[2]} \xi^{(2)} \\
&= \left((\xi^{(1)}{}_{(0)} \otimes \xi^{(1)}{}_{(1)}{}^{[1]}) \triangleright n\right) \xi^{(1)}{}_{(1)}{}^{[2]} \xi^{(2)} \otimes 1 \\
&= (1 \otimes \xi^{(1)}{}_{(1)}{}^{[2]} \xi^{(2)})(\xi^{(1)}{}_{(0)} \otimes \xi^{(1)}{}_{(1)}{}^{[1]}) \triangleright n \otimes 1 \\
&= (\xi^{(1)}{}_{(0)} \otimes \xi^{(1)}{}_{(1)}{}^{[1]} \xi^{(1)}{}_{(1)}{}^{[2]} \xi^{(2)}) \triangleright n \otimes 1 \\
&= \xi \triangleright n \otimes 1
\end{aligned}
$$

where, in the third equality, we have used the fact that $\xi^{(1)}{}_{(0)} \otimes \xi^{(1)}{}_{(1)}{}^{[1]} \otimes \xi^{(1)}{}_{(1)}{}^{[2]} \xi^{(2)} \in R \otimes A \underset{B}{\otimes} A^{\mathrm{co}\,H} = R \otimes A \underset{B}{\otimes} B$, which follows easily from $\xi \in (R \otimes A)^{\mathrm{co}\,H}$, (2.1) and Lemma 3.1. The fifth equality is an application of (2.3).

Finally, the isomorphism μ is also left R-linear, because, for $x \in R$, $n \in N$ and $a \in A$:

$$
\begin{aligned}
\mu(x(n \otimes a)) &= \mu\left((x_{(0)} \otimes x_{(1)}{}^{[1]}) \triangleright n \otimes x_{(1)}{}^{[2]} a\right) \\
&= \mu(x_{(0)} n x_{(1)}{}^{[1]} \otimes x_{(1)}{}^{[2]} a) \\
&= x_{(0)} n x_{(1)}{}^{[1]} x_{(1)}{}^{[2]} a \\
&= xna = x\mu(n \otimes a)
\end{aligned}
$$

by (2.3). □

Theorem 3.3 contains Schneider's structure theorem for relative Hopf modules over Hopf-Galois extensions, which was used crucially in the proof, as the special case $R = k$. In the case that $B = k$, we will now give a different proof of Theorem 3.3 by showing that it is a special case of Schneider's structure theorem, applied to a smash product of R and A.

Note that in this case $A^B = A$, so that A is a right H-module algebra with the Miyashita-Ulbrich action. Hence, we can form a generalized (right) smash product $R\#A$, defined to be $R \otimes A$ with the multiplication $(r\#x)(s\#y) = rs_{(0)}\#(x \leftharpoonup s_{(1)})y$ (see [1]). Since A is a Yetter-Drinfeld module algebra, $R\#A$ is actually a right H-comodule algebra with the codiagonal comodule structure, since with this comodule structure

$$\begin{aligned}\delta\left((r\#x)(s\#y)\right) &= \delta\left(rs_{(0)}\#(x \leftharpoonup s_{(1)})y\right)\\ &= r_{(0)}s_{(0)}\#(x \leftharpoonup s_{(2)})_{(0)}y_{(0)} \otimes r_{(1)}s_{(1)}(x \leftharpoonup s_{(2)})_{(1)}y_{(1)}\\ &= r_{(0)}s_{(0)}\#(x_{(0)} \leftharpoonup s_{(1)})y_{(0)} \otimes r_{(1)}x_{(1)}s_{(2)}y_{(1)}\\ &= (r_{(0)}\#x_{(0)})(s_{(0)}\#y_{(0)}) \otimes r_{(1)}x_{(1)}s_{(1)}y_{(1)}\end{aligned}$$

holds for $r, s \in R$ and $x, y \in A$. The subalgebra $(R\#A)^{\mathrm{co}\,H}$ is easily computed to be a subalgebra not only of $R\#A$, but in fact of $R \otimes A^{\mathrm{op}}$ via the embedding into $R \otimes A$. Indeed, for $r\#x \in R\#A$ and $\sum s_i\#y_i \in (R\#A)^{\mathrm{co}\,H}$ we have

$$\begin{aligned}(r\#x)(\textstyle\sum s_i\#y_i) &= \textstyle\sum rs_{i(0)}\#(x \leftharpoonup s_{i(1)})y_i\\ &= \textstyle\sum rs_{i(0)}\#y_{i(0)}(x \leftharpoonup s_{i(1)}y_{i(1)}) = \sum rs_i\#y_ix\end{aligned}$$

where the second equality is commutativity of A in the braided category of Yetter-Drinfeld modules, and the last is coinvariance of $\sum s_i\#y_i$. Stated another way, we have $(R\#A)^{\mathrm{co}\,H} \cong L$ in the notation of Theorem 3.3. It is easy to check that $R\#A$ is a left faithfully flat H-Galois extension of its coinvariants; actually, this uses only the fact that we have an obvious H-comodule algebra homomorphism from A to $R\#A$, so that the following proposition applies:

Proposition 3.4. *Let H be a k-flat Hopf algebra, A a left faithfully flat right H-Galois extension of B, and S any right H-comodule algebra admitting a morphism $A \to S$ of comodule algebras. Then S is a left faithfully flat H-Galois extension of $S^{\mathrm{co}\,H}$.*

Proof. Via the comodule algebra homomorphism, $S \in \mathcal{M}^H_A$. In particular $S \cong S^{\mathrm{co}\,H} \underset{B}{\otimes} A$ by [11, Thm.I], which implies faithful flatness of S over its coinvariants, and the Galois map $S \underset{S^{\mathrm{co}\,H}}{\otimes} S \to S \otimes H$ is easily checked to factor as

$$S \underset{S^{\mathrm{co}\,H}}{\otimes} S \cong S \underset{S^{\mathrm{co}\,H}}{\otimes} S^{\mathrm{co}\,H} \underset{B}{\otimes} A \cong S \underset{A}{\otimes} A \underset{B}{\otimes} A \cong S \underset{A}{\otimes} A \otimes H \cong S \otimes H \qquad \square$$

Now assume k is a field and H has bijective antipode. Then by [11, Thm.I] the H-Galois extension $R\#A$ is also right faithfully flat, and the coinvariants functor ${}_{R\#A}\mathcal{M}^H \to {}_{(R\#A)^{\operatorname{co} H}}\mathcal{M}$ is an equivalence. This is another proof of Theorem 3.3 (in the case where k is a field, H has bijective antipode and $B = k$) once we note that relative Hopf bimodules in ${}_R\mathcal{M}^H_A$ can be naturally identified with relative Hopf modules in ${}_{R\#A}\mathcal{M}^H$; An R-A-Hopf bimodule structure on an H-comodule M and the corresponding left $R\#A$-Hopf module structure are related by the equations

$$(r\#x)m = rm_{(0)}(x \leftharpoonup m_{(1)})$$
$$rmx = \left(r\#x \leftharpoonup S^{-1}(m_{(1)})\right) m.$$

for $r \in R$, $x \in A$ and $m \in M$.

We now turn to the special case $R = A$ of Theorem 3.3. Let A be a right H-comodule algebra. The category ${}_A\mathcal{M}^H_A$ of relative Hopf bimodules over A is a monoidal category with the tensor product over A; the tensor product $M \underset{A}{\otimes} N$ of two Hopf bimodules is a Hopf bimodule with the codiagonal comodule structure.

The main object of study in the present paper is $L(A,H)$ for a left faithfully flat H-Galois extension A/B. For the rest of this section we specialize further to the case that $B = k$. In this case we have seen in [9], generalizing results of Van Oystaeyen and Zhang [20], that $L := L(A,H)$ is a Hopf algebra with comultiplication $\Delta(\xi) = \xi^{(1)}{}_{(0)} \otimes \xi^{(1)}{}_{(1)}{}^{[1]} \otimes \xi^{(1)}{}_{(1)}{}^{[2]} \otimes \xi^{(2)}$. In particular, the category of left L-modules has a natural structure of monoidal category: The tensor product of two left L-modules is a module with the diagonal action. It is easy to check that the equivalence ${}_A\mathcal{M}^H_A \cong {}_L\mathcal{M}$ obtained from 3.3 by specializing $R = A$ and assuming $B = k$ is then an equivalence of monoidal categories (see Theorem 6.3 for the proof of a more general statement).

Moreover, we have seen in [9] that A is a left L-comodule algebra with comodule structure $\lambda : A \to L \otimes A$ defined by $\lambda(a) := a_{(-1)} \otimes a_{(0)} := a_{(0)} \otimes a_{(1)}{}^{[1]} \otimes a_{(1)}{}^{[2]}$. In this way, A is actually a left L-Galois extension of k, with the inverse of the Galois map $\kappa_\ell : A \otimes A \ni x \otimes y \mapsto x_{(-1)} \otimes x_{(0)}y \in L \otimes A$ given by $\kappa_\ell^{-1}(\ell \otimes 1) = \ell^{(1)} \otimes \ell^{(2)}$. The left L-comodule and right H-comodule structures of A make it an L-H-bicomodule. In particular, there are relative Hopf module categories ${}^L_A\mathcal{M}^H, {}^L\mathcal{M}^H_A, {}^L_A\mathcal{M}^H_A$ etc. defined in the obvious way. Note that these categories generalize those of two-sided Hopf modules ${}_H\mathcal{M}^H_H$, two-cosided Hopf modules ${}^H\mathcal{M}^H_H, {}^H_H\mathcal{M}^H$ and two-cosided two-cosided Hopf modules ${}^H_H\mathcal{M}^H_H$, whose structure we have investigated, following Woronowicz [21], in [8]. In fact, in the special case $A = H$, we have $H \cong L(A,H)$ via $h \mapsto h_{(1)} \otimes S(h_{(2)})$.

Let V be a k-module, and consider $V \otimes A \in \mathcal{M}^H_A$. If V is an L-H-bicomodule, then $V \otimes A$ becomes a left L-comodule with the codiagonal comodule structure. It is easy to check that $V \otimes A \in {}^L\mathcal{M}^H_A$ in this way. Conversely, if $V \otimes A \in {}^L\mathcal{M}^H_A$ with a suitable L-comodule structure, then $V = (V \otimes A)^{\operatorname{co} H}$ is an L-subcomodule of $V \otimes A$. One can check that this gives a bijection between left L-comodule structures on V making it an L-H-bicomodule, and left L-comodule structures on $V \otimes A$ making it an object of ${}^L\mathcal{M}^H_A$. Theorem 3.3 tells us that giving $V \otimes A$ the structure of a relative Hopf module in ${}_A\mathcal{M}^H_A$ amounts to giving V the structure of an L-module. Now assume V is both a left L-module, so that $V \otimes A \in {}_A\mathcal{M}^H_A$, and a left L-comodule, so that $V \otimes A \in {}^L\mathcal{M}^H_A$. What condition do the two structures on V have to fulfill so that $V \otimes A \in {}^L_A\mathcal{M}^H_A$? If $V \in {}^L_L\mathcal{YD}$, then a calculation just as in [8] shows that $V \otimes A \in {}^L_A\mathcal{M}^H_A$. On the other hand, for $M \in {}^L_A\mathcal{M}^H_A$, the left action of L on $M^{\mathrm{co}\,H}$ is a special case of the Miyashita-Ulbrich action of L on a bimodule over the left L-Galois extension A. Consequently, this action makes $M^{\mathrm{co}\,H}$ a left-left Yetter-Drinfeld module.

Summarizing, we have the following results, generalizing the main result of [8]:

Theorem 3.5. *Let H be a k-flat Hopf algebra and A a faithfully flat right H-Galois extension of k. Schneider's category equivalence $\mathcal{M}^H_A \cong {}_k\mathcal{M}$ induces equivalences ${}^L\mathcal{M}^H_A \cong {}^L\mathcal{M}$ and equivalences of monoidal categories*

$${}^L_L\mathcal{YD} \cong ({}^L_A\mathcal{M}^H_A, \underset{A}{\otimes}) \cong \mathcal{YD}^H_H.$$

In particular, the category $({}^L_A\mathcal{M}^H_A, \underset{A}{\otimes})$ is prebraided (braided if the antipode of H, hence of L, is bijective). The braiding is given explicitly by

$$M \underset{A}{\otimes} N \ni m \otimes n \mapsto m_{(-1)}{}^{(1)} n_{(0)} n_{(1)}{}^{[1]} \otimes m_{(-1)}{}^{(2)} m_{(0)} n_{(1)}{}^{[2]} \in N \underset{A}{\otimes} M$$

for $M, N \in {}^L_A\mathcal{M}^H_A$.

The equivalence ${}^L_A\mathcal{M}^H_A \cong \mathcal{YD}^H_H$ follows by switching the roles of L and H in the above. This is possible because A is an L-H-biGalois extension and this determines L uniquely. We omit checking that we have monoidal functors, and verifying the explicit form of the braiding.

Note that ${}_A\mathcal{M}^H_A$ is a monoidal category with the tensor product over A *regardless* of whether $B = k$. However, $L(A, H)$ cannot be expected in general to be a bialgebra, so that it does not seem to make sense to ask for a *monoidal* equivalence ${}_A\mathcal{M}^H_A \cong {}_L\mathcal{M}$. However, ${}_L\mathcal{M}$ is clearly a monoidal category, with the monoidal structure induced by that of ${}_A\mathcal{M}^H_A$. In the following we will find a structure of $\times_B$-bialgebra on $L(A, H)$ which is "responsible" for this monoidal structure of ${}_L\mathcal{M}$. We will also be concerned with partial generalizations of Theorem 3.5 to the case when L is no longer a Hopf algebra.

4. Bialgebras over $B \otimes \overline{B}$

In this section we shall fix the (necessarily somewhat tedious) notations surrounding $\times_B$-bialgebras, following closely [16].

Throughout the section B is a fixed k-algebra and $\overline{B} = B^{\mathrm{op}}$. We denote by $B \ni b \mapsto \overline{b} \in \overline{B}$ the obvious k-linear isomorphism.

Let M be a $\overline{B}$-$\overline{B}$-bimodule and and N a B-B-bimodule. We use the following notations, due to Mac Lane, see [15, 16] (we shall only "explain" them here by giving a couple of examples):

$$\int_x {}_{\overline{x}}M \otimes {}_xN = M \otimes N \Big/ \langle \{ \overline{x}m \otimes n - m \otimes xn | m \in M, n \in N, x \in B \} \rangle$$

$$\int^x M_{\overline{x}} \otimes N_x = \{ \textstyle\sum m_i \otimes n_i \in M \otimes N | \forall x \in B : \sum m_i \overline{x} \otimes n_i = \sum m_i \otimes n_i x \}$$

$$\int^y \int_x {}_{\overline{x}}M_{\overline{y}} \otimes {}_xN_y = \left\{ \textstyle\sum m_i \otimes n_i \in \displaystyle\int_x {}_{\overline{x}}M \otimes {}_xN \right| \left. \forall y \in B : \textstyle\sum m_i \overline{y} \otimes n_i = \sum m_i \otimes n_i y \right\}$$

$$\int_x \int^y {}_{\overline{x}}M_{\overline{y}} \otimes {}_xN_y = \left(\int^y M_{\overline{y}} \otimes N_y \right) \Big/ \Big\langle \{ \textstyle\sum \overline{x} m_i \otimes n_i - \sum m_i \otimes x n_i | x \in B \} \Big\rangle$$

There is a natural k-linear homomorphism from the fourth to the third expression, mapping the class of $\sum m_i \otimes n_i$ to the class of $\sum m_i \otimes n_i$. The modules involving only the $\int_x$ symbol are of course variations on the ordinary tensor product over B, in fact $\int_x M_x \otimes {}_xN \cong M \underset{B}{\otimes} N$ for $M \in \mathcal{M}_B$ and $N \in {}_B\mathcal{M}$. If B is finitely generated projective over k, then $\int^x M_x \otimes {}_xN \cong N \,\square_{B^*}\, M$ in this situation, where B^* is the dual coalgebra of B, over which N and M are comodules in the natural way, and $\square_{B^*}$ denotes cotensor product.

Let M be a $\overline{B}$-$\overline{B}$-bimodule and P a B-B-bimodule. Then

$$M \times_B P := \int^y \int_x {}_{\overline{x}}M_{\overline{y}} \otimes {}_xN_y.$$

If $M \in {}_{B\otimes\overline{B}}\mathcal{M}_{B\otimes\overline{B}}$, then $M \times_B P \in {}_B\mathcal{M}_B$ with B acting on the left factor, and if also $P \in {}_{B\otimes\overline{B}}\mathcal{M}_{B\otimes\overline{B}}$, then $M \times_B P \in {}_{B\otimes\overline{B}}\mathcal{M}_{B\otimes\overline{B}}$. Thus, $\times_B$ defines a product on the category ${}_{B\otimes\overline{B}}\mathcal{M}_{B\otimes\overline{B}}$, which, however, is neither associative nor unitary. Assume that $M \in {}_{\overline{B}}\mathcal{M}_{\overline{B}}$, $P \in {}_{B\otimes\overline{B}}\mathcal{M}_{B\otimes\overline{B}}$ and $N \in {}_B\mathcal{M}_B$. Then

$$M \times_B P \times_B N := \int^{y,c} \int_{x,b} {}_{\overline{x}}M_{\overline{y}} \otimes {}_{x,\overline{b}}P_{y,\overline{c}} \otimes {}_bN_c$$

(where, since $\int^y$ and $\int^c$ "commute" (being limits), we have abbreviated $\int^{y,c} := \int^y \int^c = \int^c \int^y$. There are associativity maps

$$(M \times_B P) \times_B N \xrightarrow{\alpha} M \times_B P \times_B N$$

$$\sum_j (\sum_i m_{ij} \otimes p_{ij}) \otimes n_j \mapsto \sum_{i,j} m_{ij} \otimes p_{ij} \otimes n_j$$

$$M \times_B (P \times_B N) \xrightarrow{\alpha'} M \times_B P \times_B N$$

$$\sum_i m_i \otimes \sum_j p_{ij} \otimes n_{ij} \mapsto \sum_{i,j} m_i \otimes p_{ij} \otimes n_{ij}$$

If M and N are $B \otimes \overline{B}$-bimodules, then so are $(M \times_B P) \times_B N$, $M \times_B (P \times_B N)$ and $M \times_B P \times_B N$, and the maps α, α' are bimodule maps.

A $B \otimes \overline{B}$-bimodule structure on $\mathrm{End}(B)$ is induced by the left $B \otimes \overline{B}$-module structure on B. We have, for any $M \in {}_{\overline{B}}\mathcal{M}_{\overline{B}}$ and $N \in {}_B\mathcal{M}_B$, the following maps

$$\begin{aligned}
\theta : M \times_B \mathrm{End}(B) &\to M \\
\sum m_i \otimes f_i &\mapsto \sum \overline{f_i(1)} m_i \\
\theta' : \mathrm{End}(B) \times_B N &\to N \\
\sum f_i \otimes n_i &\mapsto \sum f_i(1) n_i
\end{aligned}$$

θ is a $\overline{B}$-bimodule map and θ' is a B-bimodule map. If $M, N \in {}_{B\otimes\overline{B}}\mathcal{M}_{B\otimes\overline{B}}$, then these maps are bimodule maps. We will write $E := \mathrm{End}(B)$ below.

Definition 4.1. A $\times_B$-coalgebra is a $B \otimes \overline{B}$-bimodule L together with a comultiplication $\Delta : L \to L \times_B L$ and a counit $\varepsilon : L \to E$, both of which are $B \otimes \overline{B}$-bimodule maps, such that the diagrams

$$\begin{tikzcd}
L \arrow[rr, "\Delta"] \arrow[d, "\Delta"'] & & L \times_B L \arrow[d, "L \times_B \Delta"] \\
L \times_B L \arrow[d, "\Delta \times_B L"'] & & L \times_B (L \times_B L) \arrow[d, "\alpha'"] \\
(L \times_B L) \times_B L \arrow[rr, "\alpha"'] & & L \times_B L \times_B L
\end{tikzcd}$$

$$\begin{tikzcd}
L \arrow[r, "\Delta"] \arrow[d, "\mathrm{id}"'] & L \times_B L \arrow[d, "L \times_B \varepsilon"] \\
L & L \times_B E \arrow[l, "\theta"]
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[r, "\Delta"] \arrow[d, "\mathrm{id}"'] & L \times_B L \arrow[d, "\varepsilon \times_B L"] \\
L & E \times_B L \arrow[l, "\theta'"]
\end{tikzcd}$$

commute.

For $\times_B$-coalgebras we will use the same "Sweedler" notation as for ordinary coalgebras, writing $\Delta(\xi) =: \xi_{(1)} \otimes \xi_{(2)} \in L \times_B L$.

The category of left $B \otimes \overline{B}$-modules is monoidal with the tensor product $M \diamond N := \int_x {}_{\overline{x}}M \otimes {}_x N$ (of course, this category is naturally isomorphic to $({}_B\mathcal{M}_B, \underset{B}{\otimes})$). In this notation a map $\Delta : L \to L \times_B L$ in ${}_{B\otimes\overline{B}}\mathcal{M}_{B\otimes\overline{B}}$ induces a map $\Delta_0 : L \to L \diamond L$ in ${}_{B\otimes\overline{B}}\mathcal{M}$. By composing with $E \ni f \to f(1) \in B$, a map $\varepsilon : L \to E$ in ${}_{B\otimes\overline{B}}\mathcal{M}_{B\otimes\overline{B}}$ induces a map $\varepsilon_0 : L \to B$ in ${}_{B\otimes\overline{B}}\mathcal{M}$. It is straightforward to check that the maps Δ and ε give L the structure of a $\times_B$-coalgebra if and only if Δ_0 and ε_0 give L the structure of a coalgebra in $({}_{B\otimes\overline{B}}\mathcal{M}, \diamond)$, that is, if and only if $(\Delta_0 \diamond L)\Delta_0 = (L \diamond \Delta_0)\Delta_0$ and $(\varepsilon_0 \diamond L)\Delta_0 = \mathrm{id}_L = (L \diamond \varepsilon_0)\Delta_0$ hold.

If $M, M', N, N' \in {}_{B\otimes\overline{B}}\mathcal{M}_{B\otimes\overline{B}}$, then we have a well defined $B \otimes \overline{B}$-bimodule map

$$\tau : (M \times_B M') \underset{B\otimes\overline{B}}{\otimes} (N \times_B N') \to (M \underset{B\otimes\overline{B}}{\otimes} N) \times_B (M' \underset{B\otimes\overline{B}}{\otimes} N')$$
$$(\sum m_i \otimes m_i') \otimes (\sum n_i \otimes n_i') \mapsto \sum m_i \otimes n_i \otimes m_i' \otimes n_i'$$

Definition 4.2. Let S be any k-algebra. An algebra over S is an algebra L together with an algebra map $\iota : S \to L$.

In particular, an algebra over S is an S-bimodule in a natural way. Clearly, algebras over S are precisely the algebras in the monoidal category $({}_S\mathcal{M}_S, \underset{S}{\otimes})$.

Using the map τ above, we see that for any algebra L over $B \otimes \overline{B}$ the $B \otimes \overline{B}$-bimodule $L \times_B L$ is again an algebra over $B \otimes \overline{B}$.

Definition 4.3. Let B be a k-algebra. A $\times_B$-bialgebra is an algebra L over $B \otimes \overline{B}$ that has the structure of a $\times_B$-coalgebra such that the comultiplication $\Delta : L \to L \times_B L$ and counit $\varepsilon : L \to E$ are maps of algebras over $B \otimes \overline{B}$.

The following example is a common generalization of [16, Expl.4.6] (where the cocycle is trivial), and [4, Thm.1.6.(b)] (where $B = k$; in the present case, we do not deal with an antipode):

Example 4.4. Let B be an algebra and H a bialgebra over k. Let $\rightharpoonup: H \otimes B \to B$ be a measuring, that is, assume

$$h \rightharpoonup (bc) = (h_{(1)} \rightharpoonup b)(h_{(2)} \rightharpoonup c) \quad \text{and} \quad h \rightharpoonup 1 = 1$$

hold for all $b, c \in B$ and $h \in H$. Assume that $\sigma : H \otimes H \to B$ is a convolution invertible cocycle, that is, assume $\sigma(1, h) = \sigma(h, 1) = 1$ and

$$(f_{(1)} \rightharpoonup \sigma(g_{(1)}, h_{(1)}))\sigma(f_{(2)}, g_{(2)}h_{(2)}) = \sigma(f_{(1)}, g_{(1)})\sigma(f_{(2)}g_{(2)}, h)$$

hold for all $f, g, h \in H$. Finally, assume that $\rightharpoonup$ is a σ-twisted H-module structure, that is, assume $1 \rightharpoonup b = b$ and

$$(g_{(1)} \rightharpoonup h_{(1)} \rightharpoonup b)\sigma(g_{(2)}, h_{(2)}) = \sigma(g_{(1)}, h_{(1)})(g_{(2)}h_{(2)} \rightharpoonup b)$$

hold for all $g, h \in H$ and $b \in B$.

$R := R(H, B, \rightharpoonup, \sigma) := B \otimes H \otimes \overline{B}$ is a $\times_B$-bialgebra with multiplication

$$(b \otimes g \otimes \overline{c})(d \otimes h \otimes \overline{e}) = b(g_{(1)} \rightharpoonup d)\sigma(g_{(2)}, h_{(1)}) \otimes g_{(3)}h_{(2)} \otimes \overline{\sigma^{-1}(g_{(4)}, h_{(3)})(g_{(5)} \rightharpoonup e)c}$$

comultiplication

$$\Delta_R(b \otimes g \otimes \overline{c}) = b \otimes g_{(1)} \otimes \overline{1} \otimes 1 \otimes g_{(2)} \otimes \overline{c}$$

and counit

$$\varepsilon(b \otimes g \otimes \overline{c})(d) = b(g \rightharpoonup d)c$$

for $b, c, d, e \in B$ and $g, h \in H$. The algebra map $B \otimes \overline{B} \to R$ making R an algebra over $B \otimes \overline{B}$ is given by $b \otimes \overline{c} \mapsto b \otimes 1 \otimes \overline{c}$ for $b, c \in B$.

(The conditions on $\rightharpoonup$ and σ listed above, except for invertibility of σ, are precisely those needed to equip $B\#_\sigma H := B \otimes H$ with the associative multiplication of a crossed product. We shall see the relation of the construction above with crossed products in Example 6.4.)

5. Modules and comodules over $\times_B$-bialgebras

The following characterization of bialgebras goes back to Pareigis [7], and is now widely used in quantum group theory: An algebra B is a bialgebra if and only if the category ${}_B\mathcal{M}$ is a monoidal category such that the underlying functor ${}_B\mathcal{M} \to {}_k\mathcal{M}$ is monoidal. Dually (this is really a rigorous duality in view of the categorical techniques used in [7]) a coalgebra B is a bialgebra if and only if the category ${}^B\mathcal{M}$ is a monoidal category such that the underlying functor ${}^B\mathcal{M} \to {}_k\mathcal{M}$ is monoidal. The assumptions on the underlying functors are really essential. For example the category of modules over $L := L(A, H)$ from section 3 is monoidal, but L need not be a bialgebra since the underlying functor to k-modules is not monoidal. As a trivial example, let B be an algebra and consider the category ${}_{B\otimes\overline{B}}\mathcal{M}$, which is monoidal with tensor product $\diamond$, although $B \otimes \overline{B}$ certainly need not be a bialgebra.

It turns out, however, that the rule "if the category of modules over a algebra is monoidal, then the algebra is a bialgebra" can be carried over to a more general situation suitable for our example $L(A, H)$. The "bialgebras" to consider are $\times_B$-bialgebras, and the underlying functors to the category of $B \otimes \overline{B}$-modules have to be required to be monoidal. In this section we shall prove the equivalence between $\times_B$-bialgebra structures and suitable monoidal category structures before verifying in the next section that this applies to the setup discussed in section 3.

Theorem 5.1. *The following data are equivalent for an algebra L over $B \otimes \overline{B}$:*

(1) *A $\times_B$-bialgebra structure on L.*
(2) *The structure of a monoidal category on ${}_L\mathcal{M}$ such that the underlying functor ${}_L\mathcal{M} \to {}_{B\otimes\overline{B}}\mathcal{M}$ is strictly monoidal.*

Proof. First, assume that (L, Δ, ε) is a bialgebra over $B \otimes \overline{B}$. For $M, N \in {}_L\mathcal{M}$ we define an L-module structure on $M \diamond N$ by $\xi \triangleright (m \otimes n) := \xi_{(1)} \triangleright m \otimes \xi_{(2)} \triangleright n$ for $\xi \in L$, $m \in M$ and $n \in N$. It is not difficult to show directly that this is a well defined module structure, but we shall give a slightly more conceptual argument. For $M, M' \in {}_{B\otimes\overline{B}}\mathcal{M}$, we have $\mathrm{Hom}(M, M') \in {}_{B\otimes\overline{B}}\mathcal{M}_{B\otimes\overline{B}}$. In this way, $\mathrm{End}(M)$ is an algebra over $B \otimes \overline{B}$. A left L-module structure inducing a given left $B \otimes \overline{B}$-module structure on M can be described equivalently by a map $\rho : L \to \mathrm{End}(M)$ of algebras over $B \otimes \overline{B}$.

Lemma 5.2. *For $M, M', N, N' \in {}_{B\otimes\overline{B}}\mathcal{M}$ we have a map*

$$\tau : \mathrm{Hom}(M, M') \times_B \mathrm{Hom}(N, N') \to \mathrm{Hom}(M \diamond N, M' \diamond N')$$

$$\sum f_i \otimes g_i \mapsto \Big(m \otimes n \mapsto \sum f_i(m) \otimes g_i(n)\Big)$$

Proof. This follows from the behavior of Hom on colimits; there is a natural map $\varinjlim \operatorname{Hom}(X, Y_i) \to \operatorname{Hom}(X, \varinjlim Y_i)$ and a natural isomorphism $\varprojlim \operatorname{Hom}(Y_i, X) \cong \operatorname{Hom}(\varinjlim Y_i, X)$. Thus, the natural map $\operatorname{Hom}(M, M') \otimes \operatorname{Hom}(N, N') \to \operatorname{Hom}(M \otimes N, M' \otimes N')$ induces

$$\begin{aligned}
&\operatorname{Hom}(M, M') \times_B \operatorname{Hom}(N, N') \\
&\quad = \int^y \int_x \operatorname{Hom}({}_{\overline{y}}M, {}_{\overline{x}}M') \otimes \operatorname{Hom}({}_yN, {}_xN') \\
&\quad \to \operatorname{Hom}(\int_y {}_{\overline{y}}M \otimes {}_yN, \int_x {}_{\overline{x}}M' \otimes {}_xN') \\
&\quad = \operatorname{Hom}(M \diamond N, M' \diamond N') \qquad \square
\end{aligned}$$

Now for $M, N \in {}_L\mathcal{M}$ we can rewrite the desired L-module structure on $M \diamond N$ as corresponding to the algebra map

$$L \xrightarrow{\Delta} L \times_B L \xrightarrow{\rho\times\rho} \operatorname{End}(M) \times_B \operatorname{End}(N) \xrightarrow{\tau} \operatorname{End}(M \diamond N)$$

Clearly, coassociativity of Δ implies that the product functor thus described on ${}_L\mathcal{M}$ is associative.

Now assume that $({}_L\mathcal{M}, \diamond)$ is a monoidal category. Then in particular, $L \diamond L$ is a left L-module. Since for $y \in B$ the maps $L \ni \xi \mapsto \xi y \in L$ and $L \ni \xi \mapsto \xi\overline{y} \in L$ are left L-linear, $L \times_B L = \int^y L_{\overline{y}} \diamond L_y$ is a limit (in fact an end) of maps in ${}_L\mathcal{M}$, hence a left L-module. We can define $\Delta : L \to L \times_B L$ by $\Delta(\xi) = \xi \triangleright (1 \otimes 1)$. If $M, N \in {}_L\mathcal{M}$ and $m \in M, n \in N$, then the map $f : L \diamond L \ni \xi \otimes \zeta \mapsto \xi \triangleright m \otimes \zeta \triangleright n \in M \diamond N$ is in ${}_L\mathcal{M}$ (being the image under the bifunctor $\diamond$ of two left L-linear maps $L \to M$ resp. $L \to N$). It follows that the left L-module structure on $M \diamond N$ fulfills

$$\xi \triangleright (m \otimes n) = \xi \triangleright f(1 \otimes 1) = f(\xi \triangleright (1 \otimes 1)) = f(\xi_{(1)} \otimes \xi_{(2)}) = \xi_{(1)} \triangleright m \otimes \xi_{(2)} \triangleright n$$

for $\xi \in L$, $m \in M$ and $n \in N$, where $\xi_{(1)} \otimes \xi_{(2)}$ denotes $\Delta(\xi) \in L \times_B L \subset L \diamond L$.

Specializing $M = N = L$, this formula and associativity of the left L-module $L \diamond L$ imply that Δ is an algebra map.

To check that Δ is coassociative in the sense of the definition of a $\times_B$-coalgebra, it is enough to verify that $\Delta_0 : L \to L \diamond L$ defined by $\Delta_0(\xi) = \Delta(\xi) = \xi \triangleright (1_L \otimes 1_L) \in L \diamond L$ is coassociative. We write $\Delta_0(\xi) = \Delta(\xi) =: \xi_{(1)} \otimes \xi_{(2)}$. We know that in the monoidal category ${}_L\mathcal{M}$ the canonical isomorphism $L \diamond (L \diamond L) \cong (L \diamond L) \diamond L$ is L-linear. Hence for $\xi \in L$ we have, in $L \diamond L \diamond L$:

$$\begin{aligned}
(\Delta_0 \diamond L)\Delta_0(\xi) &= \Delta_0(\xi_{(1)}) \otimes \xi_{(2)} \\
&= \xi_{(1)} \triangleright (1_L \otimes 1_L) \otimes \xi_{(2)} \\
&= \xi \triangleright ((1_L \otimes 1_L) \otimes 1_L) \\
&= \xi \triangleright (1_L \otimes (1_L \otimes 1_L))
\end{aligned}$$

$$= \xi_{(1)} \otimes \xi_{(2)} \triangleright (1_L \otimes 1_L)$$
$$= (L \diamond \Delta_0)\Delta_0(\xi)$$

We omit checking that the representation $L \to \operatorname{End}(B)$ of L on the neutral object B with respect to $\diamond$ gives a counit for L. $\square$

Remark 5.3. In the preceding theorem, we have restricted ourselves to the case where the underlying functor ${}_L\mathcal{M} \to {}_{B\otimes\overline{B}}\mathcal{M}$ is *strictly* monoidal. This is not really important: Given a structure $\diamond'$ of a monoidal category on ${}_L\mathcal{M}$ for which the underlying functor $\mathcal{U} : {}_L\mathcal{M} \to {}_{B\otimes\overline{B}}\mathcal{M}$ together with a natural coherent isomorphism $\xi : \mathcal{U}(M) \diamond \mathcal{U}(N) \cong \mathcal{U}(M \diamond' N)$ is monoidal, we can always construct another tensor product $\diamond$ on ${}_L\mathcal{M}$ such that the identity is a monoidal equivalence $({}_L\mathcal{M}, \diamond') \cong ({}_L\mathcal{M}, \diamond)$, and the underlying functor $({}_L\mathcal{M}, \diamond) \to ({}_{B\otimes\overline{B}}\mathcal{M}, \diamond)$ is strictly monoidal. One just takes $M \diamond N$ for $M, N \in {}_L\mathcal{M}$ to be $\mathcal{U}(M) \diamond \mathcal{U}(N)$ endowed with the L-module structure that makes ξ an L-linear isomorphism.

Example 5.4. Assume the situation of example 4.4. We define a category $\mathcal{C}$ whose objects are pairs consisting of

(1) a B-bimodule N and
(2) a map $\rightharpoonup: H \otimes N \to N$ such that, for all $g, h \in H$, $b, c \in B$ and $n \in N$:
  (a) $g \rightharpoonup (bnc) = (g_{(1)} \rightharpoonup b)(g_{(2)} \rightharpoonup n)(g_{(3)} \rightharpoonup c)$
  (b) $(g_{(1)} \rightharpoonup h_{(1)} \rightharpoonup n)\sigma(g_{(2)}, h_{(2)}) = \sigma(g_{(1)}, h_{(1)})(g_{(2)}h_{(2)} \rightharpoonup n)$.

and morphisms are those B-bimodule morphisms compatible with the respective $\rightharpoonup$-maps. We claim that $\mathcal{C} \cong {}_R\mathcal{M}$, where $R = R(H, B, \rightharpoonup, \sigma)$.

In fact, a map $\mu : (B \otimes H \otimes \overline{B}) \otimes N \to N$ is clearly described equivalently by maps $\mu_\ell : B \otimes N \to N$, denoted by $b \otimes n \mapsto bn$, $\mu_r : N \otimes B \to N$, denoted by $n \otimes b \mapsto nb$, and $H \otimes N \to N$, denoted by $h \otimes n \mapsto h \rightharpoonup n$, via the formula $\mu(b \otimes h \otimes \overline{c} \otimes n) = b(h \rightharpoonup n)c$ for $b, c \in B$, $h \in H$ and $n \in N$. It is straightforward (but a little tedious) to check that μ makes N a left R-module if and only if μ_ℓ and μ_r make N a B-bimodule, and the conditions for $\rightharpoonup$ stated above hold.

The monoidal structure on $\mathcal{C}$ corresponding to the monoidal structure on ${}_R\mathcal{M}$ is described as follows: The tensor product of $(N, \rightharpoonup), (N', \rightharpoonup) \in \mathcal{C}$ is $(N \underset{B}{\otimes} N', \rightharpoonup)$ with $h \rightharpoonup (n \otimes n') = h_{(1)} \rightharpoonup n \otimes h_{(2)} \rightharpoonup n'$.

For ordinary bialgebras it is also well known that their comodule categories are monoidal. This holds true for $\times_B$-bialgebras as well, but we will have to give the suitable definition of comodules over them first.

Definition 5.5. Let L be a $\times_B$-coalgebra. A left L-comodule is a B-bimodule M together with a map $\lambda : M \to L \times_B M$ of B-bimodules making the diagrams

$$\begin{tikzcd}
M \arrow[r, "\lambda"] \arrow[d, "\lambda"'] & L \times_B M \arrow[d, "L \times_B \lambda"] \\
L \times_B M \arrow[d, "\Delta \times_B M"'] & L \times_B (L \times_B M) \arrow[d, "\alpha'"] \\
(L \times_B L) \times_B M \arrow[r, "\alpha"'] & L \times_B L \times_B M
\end{tikzcd}$$

$$\begin{tikzcd}
M \arrow[r, "\lambda"] \arrow[d, "\mathrm{id}"'] & L \times_B M \arrow[d, "\varepsilon \times_B M"] \\
L & E \times_B M \arrow[l, "\theta'"]
\end{tikzcd}$$

commute. We will denote by ${}^L\mathcal{M}$ the category of left L-comodules.

As a rule, we shall use the same notation for the action of a left L-comodule structure on elements as for usual comodules: $\lambda(m) = m_{(-1)} \otimes m_{(0)}$.

Proposition 5.6. *Let L be a $\times_B$-bialgebra. Then the category ${}^L\mathcal{M}$ is monoidal.*

Proof. Let $M, N \in {}^L\mathcal{M}$ and denote the left L-comodule structures by $M \ni m \mapsto m_{(-1)} \otimes m_{(0)} \in L \times_B M$ and $N \ni n \mapsto n_{(-1)} \otimes n_{(0)} \in L \times_B M$. Then an L-comodule structure on $M \underset{B}{\otimes} N$ will be given by

$$\begin{aligned} M \underset{B}{\otimes} N &\to L \times_B (M \underset{B}{\otimes} N) \\ m \otimes n &\mapsto m_{(-1)}n_{(-1)} \otimes m_{(0)} \otimes n_{(0)} \end{aligned}$$

To see that this is well defined, we have to show that the map

$$\begin{gathered} \phi : (L \times_B M) \underset{B}{\otimes} (L \times_B N) \to (L \underset{B \otimes \overline{B}}{\otimes} L) \times_B (M \underset{B}{\otimes} N) \\ \left(\sum \xi_i \otimes m_i\right) \otimes \left(\sum \zeta_j \otimes n_j\right) \mapsto \sum \xi_i \otimes \zeta_j \otimes m_i \otimes n_j \end{gathered}$$

is well defined. Clearly,

$$\begin{gathered} \phi_0 : (L \times_B M) \otimes L \otimes N \to \int_x {}_{\overline{x}}L \underset{B \otimes \overline{B}}{\otimes} L \otimes {}_xM \underset{B}{\otimes} N \\ \sum \xi_i \otimes m_i \otimes \zeta \otimes n \mapsto \sum \xi_i \otimes \zeta \otimes m_i \otimes n \end{gathered}$$

is well defined. For $b \in B$, $\sum \xi_i \otimes m_i \in L \times_B M$, $\zeta \in L$ and $n \in N$ we have

$$\begin{aligned} \phi_0\left(\sum \xi_i \otimes m_i \otimes \overline{b}\zeta \otimes n\right) &= \sum \xi_i \otimes \overline{b}\zeta \otimes m_i \otimes n \\ &= \sum \xi_i\overline{b} \otimes \zeta \otimes m_i \otimes n \end{aligned}$$

$$
\begin{aligned}
&= \sum \xi_i \otimes \zeta \otimes m_i b \otimes n \\
&= \sum \xi_i \otimes \zeta \otimes m_i \otimes bn \\
&= \phi_0 \left(\sum \xi_i \otimes m_i \otimes \zeta \otimes bn \right)
\end{aligned}
$$

and

$$
\begin{aligned}
\phi_0 \left((\sum \xi_i \otimes m_i) b \otimes \zeta \otimes n \right) &= \phi_0 \left(\sum \xi_i b \otimes m_i \otimes \zeta \otimes n \right) \\
&= \sum \xi_i b \otimes \zeta \otimes m_i \otimes n \\
&= \sum \xi_i \otimes b\zeta \otimes m_i \otimes n \\
&= \phi_0 \left(\sum \xi_i \otimes m_i \otimes b\zeta \otimes n \right).
\end{aligned}
$$

If also $\sum \zeta_j \otimes n_j \in L \times_B N$, then

$$
\phi_0 \left(\sum \xi_i \otimes m_i \otimes \sum \zeta_j \otimes n_j \right) \in (L \underset{B \otimes \overline{B}}{\otimes} L) \times_B (M \underset{B}{\otimes} N)
$$

since $\sum \overline{b}\xi_i \otimes \zeta_j \otimes m_i \otimes n_j = \sum \xi_i \otimes \zeta_j \otimes b m_i \otimes n_j$. Consequently, ϕ_0 induces the desired map ϕ.

We omit checking that the tensor product on ${}^L\mathcal{M}$ thus defined is associative and unitary (the unit object is B with the left comodule structure $B \ni b \mapsto b \cdot 1_L \otimes 1_B \in L \times_B B$). $\square$

Definition 5.7. Let L be a $\times_B$-bialgebra and A an algebra over B that is also a left L-comodule. The following are equivalent

1. The comodule structure map $\lambda : A \to L \times_B A$ is an algebra map.
2. A is an algebra in the monoidal category ${}^L\mathcal{M}$, that is, the multiplication map of A and the map $B \to A$ are left L-colinear.

If these conditions hold, we shall call A a left L-comodule algebra.

Note that the first condition makes sense, since by [16, Prop.3.1] the B-bimodule $L \times_B A$ is an algebra over B with $(\sum \xi_i \otimes x_i)(\sum \zeta_j \otimes y_j) = \sum \xi_i \zeta_j \otimes x_i y_j$. The equivalence of the two conditions is proved by the same calculation as for ordinary comodule algebras.

6. The $\times_B$-bialgebra associated to a Hopf Galois extension

Throughout this section H is a k-flat Hopf algebra and A is a right H-Galois extension of B with ${}_BA$ faithfully flat.

We will see that the algebra $L(A,H)$ considered in section 3 has the structure of a $\times_B$-bialgebra, which can be expected from Theorem 5.1 since the category of left $L(A,H)$-modules is monoidal.

Lemma 6.1. *Let H be a k-flat Hopf algebra and A a left faithfully flat right H-Galois extension of B. Then the functor ${}_A\mathcal{M}_A^H \ni M \mapsto M^{\mathrm{co}\,H} \in {}_B\mathcal{M}_B$ is monoidal.*

Proof. for $M, N \in {}_A\mathcal{M}_A^H$ we have

$$(M \underset{A}{\otimes} N)^{\mathrm{co}\,H} \cong \left((M^{\mathrm{co}\,H} \underset{B}{\otimes} A) \underset{A}{\otimes} N\right)^{\mathrm{co}\,H} \cong (M^{\mathrm{co}\,H} \underset{B}{\otimes} N)^{\mathrm{co}\,H} \cong M^{\mathrm{co}\,H} \underset{B}{\otimes} N^{\mathrm{co}\,H},$$

using Lemma 3.1. □

Let $L = L(A, H)$ as in 3.2. Clearly, L is an algebra over $B \otimes \overline{B}$. To match the results of section 3 to the notations suitable for $\times_B$-bialgebras, we have to rephrase:

Corollary 6.2. *Let H be a k-flat Hopf algebra, and A a left faithfully flat H-Galois extension of B. We have inverse category equivalences*

$$\begin{aligned} {}_A\mathcal{M}_A^H &\to {}_L\mathcal{M} & \qquad {}_L\mathcal{M} &\to {}_A\mathcal{M}_A^H \\ M &\mapsto M^{\mathrm{co}\,H} & N &\mapsto N \diamond A \end{aligned}$$

and the underlying functor ${}_A\mathcal{M}_A^H \to {}_L\mathcal{M} \to {}_{B\otimes\overline{B}}\mathcal{M}$ is monoidal from $({}_A\mathcal{M}_A^H, \underset{A}{\otimes})$ to $({}_{B\otimes\overline{B}}\mathcal{M}, \diamond)$.

In particular, Theorem 5.1 tells us that there is a unique structure of a $\times_B$-bialgebra on L such that the equivalence ${}_A\mathcal{M}_A^H \cong {}_L\mathcal{M}$ is monoidal.

Theorem 6.3. *Let H be a k-flat Hopf algebra and A an H-Galois extension of $B := A^{\mathrm{co}\,H}$ with ${}_BA$ faithfully flat. Then $L := L(A, H)$ is a $\times_B$-bialgebra with $\Delta(\xi) = \xi^{(1)}{}_{(0)} \otimes \xi^{(1)}{}_{(1)}{}^{[1]} \otimes \xi^{(1)}{}_{(1)}{}^{[2]} \otimes \xi^{(2)}$ and $\varepsilon(\xi)(b) = \xi^{(1)} b \xi^{(2)}$ for $\xi \in L$ and $b \in B$. The equivalence ${}_A\mathcal{M}_A^H \cong {}_L\mathcal{M}$ is monoidal.*

Proof. In the proof of 5.1, we have seen that the unique Δ for which the given equivalence is monoidal is given by $\Delta(\xi) = \xi \triangleright (1_L \otimes 1_L) \in L \times_B L \subset L \diamond L$. Now we have $L \diamond L \subset (A \otimes A) \underset{A}{\otimes} (A \otimes A)$, and by Theorem 3.3 the action of L on the coinvariants of a relative Hopf bimodule M is given by $\xi \triangleright m = \xi^{(1)} m \xi^{(2)}$ for $\xi \in L$ and $m \in M^{\mathrm{co}\,H}$. We calculate

$$\begin{aligned} \xi \triangleright (1_L \otimes 1_L) &= \xi \triangleright (1_A \otimes 1_A \otimes 1_A \otimes 1_A) \\ &= \xi^{(1)} \otimes 1_A \otimes 1_A \otimes \xi^{(2)} \\ &= \xi^{(1)}{}_{(0)} \otimes \xi^{(1)}{}_{(1)}{}^{[1]} \xi^{(1)}{}_{(1)}{}^{[2]} \otimes 1_A \otimes \xi^{(2)} \\ &= \xi^{(1)}{}_{(0)} \otimes \xi^{(1)}{}_{(1)}{}^{[1]} \otimes \xi^{(1)}{}_{(1)}{}^{[2]} \otimes \xi^{(2)} \end{aligned}$$

in $A \otimes A \underset{A}{\otimes} A \otimes A$, using (2.3) in the third equality. This proves our claim on the form of Δ, provided we can show that $\Gamma := \xi^{(1)}{}_{(0)} \otimes \xi^{(1)}{}_{(1)}{}^{[1]} \otimes \xi^{(1)}{}_{(1)}{}^{[2]} \otimes \xi^{(2)} \in L \diamond L$. Now since $\xi \in L = (A \otimes A)^{\mathrm{co}\,H}$, and we have the colinear isomorphism (2.7), we can conclude $\Gamma \in (L \diamond A \otimes A)^{\operatorname{co} H}$, which is, by Lemma 3.1, naturally isomorphic to $L \diamond (A \otimes A)^{\operatorname{co} H} = L \diamond L$.

The claim on the form of ε is clear. $\square$

Example 6.4. We continue in the situation of example 4.4. Put $A = B\#_\sigma H$. This is known [2] to be a (certainly faithfully flat) H-Galois extension of B; moreover, it is a cleft H-comodule algebra, that is, there is a convolution invertible H-colinear map $\gamma : H \to A$, namely $\gamma(h) = 1\#h$. We have an isomorphism of $\times_B$-bialgebras

$$\begin{aligned}\psi : R(H, B, \rightharpoonup, \sigma) &\to L(A, H)\\ b \otimes h \otimes \overline{c} &\mapsto b\gamma(h_{(1)}) \otimes \gamma(h_{(2)})c\\ &= b\#h_{(1)} \otimes \sigma^{-1}(S(h_{(4)}), h_{(5)})(S(h_{(3)}) \rightharpoonup c)\#S(h_{(2)})\end{aligned}$$

with inverse $\psi^{-1} = \phi$ defined by

$$\phi(\sum b_i\#g_i \otimes c_i\#h_i) = \sum b_i \otimes g_{i(1)} \otimes \overline{(g_{i(2)} \rightharpoonup c_i)\sigma(g_{i(3)}, h_i)}$$

for $\sum b_i\#g_i \otimes c_i\#h_i \in (A \otimes A)^{\operatorname{co} H}$.

For details on crossed products and cleft extensions we refer to [6] and the literature cited therein. Recall that multiplication in $B\#_\sigma H$ is given by

$$(b\#g)(c\#h) = b(g_{(1)} \rightharpoonup c)\sigma(g_{(2)}, h_{(1)})\#g_{(3)}h_{(2)}.$$

The equality in the definition of ψ is a direct consequence of the explicit form of the convolution inverse of γ, namely $\gamma^{-1}(h) = \sigma^{-1}(S(h_{(2)}), h_{(3)})\#S(h_{(1)})$ for $h \in H$. The comodule algebra A is an H-Galois extension of B (identifying B with a subalgebra of A via $b \mapsto b\#1$) with $h^{[1]} \otimes h^{[2]} = \gamma^{-1}(h_{(1)}) \otimes \gamma(h_{(2)})$ for $h \in H$. The cocycle and the measuring action of H on B are related to γ through the equations $\sigma(g_{(1)}, h_{(1)})\gamma(g_{(2)}h_{(2)}) = \gamma(g)\gamma(h)$, and $(g_{(1)} \rightharpoonup b)\gamma(g_{(2)}) = \gamma(g)b$, which entail $\gamma^{-1}(g_{(1)}h_{(1)})\sigma^{-1}(g_{(2)}, h_{(2)}) = \gamma^{-1}(h)\gamma^{-1}(g)$, and $\gamma^{-1}(g_{(1)})(g_{(2)} \rightharpoonup b) = b\gamma^{-1}(g)$ for $g, h \in H$ and $b \in B$.

ψ is an algebra map since for $b, c, d, e \in B$ and $g, h \in H$ we have

$$\begin{aligned}&\psi((b \otimes g \otimes \overline{c})(d \otimes h \otimes \overline{e}))\\ &= \psi\left(b(g_{(1)} \rightharpoonup d)\sigma(g_{(2)}, h_{(1)}) \otimes g_{(3)}h_{(2)} \otimes \overline{\sigma^{-1}(g_{(4)}, h_{(3)})(g_{(5)} \rightharpoonup e)c}\right)\\ &= b(g_{(1)} \rightharpoonup d)\sigma(g_{(2)}, h_{(1)})\gamma(g_{(3)}h_{(2)}) \otimes \gamma^{-1}(g_{(4)}h_{(3)})\sigma^{-1}(g_{(5)}, h_{(4)})(g_{(6)} \rightharpoonup e)c\\ &= b(g_{(1)} \rightharpoonup d)\gamma(g_{(2)})\gamma(h_{(1)}) \otimes \gamma^{-1}(h_{(2)})\gamma^{-1}(g_{(3)})(g_{(4)} \rightharpoonup e)c\\ &= b\gamma(g_{(1)})d\gamma(h_{(1)}) \otimes \gamma^{-1}(h_{(2)})e\gamma^{-1}(g_{(2)})c\\ &= \left(b\gamma(g_{(1)}) \otimes \gamma^{-1}(g_{(2)})c\right)\left(d\gamma(h_{(1)}) \otimes \gamma^{-1}(h_{(2)})e\right)\\ &= \psi(b \otimes g \otimes \overline{c})\psi(d \otimes h \otimes \overline{e}).\end{aligned}$$

It is obvious that ψ is compatible with the respective embeddings of $B\otimes\overline{B}$, hence a map of algebras over $B\otimes\overline{B}$. To check that it is a map of $\times_B$-coalgebras, we calculate

$$\begin{aligned}(\psi\times_B\psi)\Delta_R(b\otimes h\otimes\overline{c}) &= \psi(b\otimes h_{(1)}\otimes\overline{1})\otimes\psi(1\otimes h_{(2)}\otimes\overline{c})\\ &= b\gamma(h_{(1)})\otimes\gamma^{-1}(h_{(2)})\otimes\gamma(h_{(3)})\otimes\gamma^{-1}(h_{(4)})c\\ &= b\gamma(h_{(1)})\otimes {h_{(2)}}^{[1]}\otimes {h_{(2)}}^{[2]}\otimes\gamma^{-1}(h_{(3)})c\\ &= \Delta_L(b\gamma(h_{(1)})\otimes\gamma^{-1}(h_{(2)})c)\\ &= \Delta_L\psi(b\otimes h\otimes\overline{c})\end{aligned}$$

for $b,c\in B$ and $h\in H$. We omit checking that ψ preserves counits.

The calculations showing that ϕ as defined above is an inverse for ψ can be simplified considerably using the actions $\wedge,\vee: H\otimes B\to B$ introduced by Montgomery and Schneider [13, Def.1.1]. We have $h\wedge b=\gamma(h_{(1)})b\gamma(S(h_{(2)}))$ and $h\vee b=\gamma^{-1}(h_{(2)})b\gamma^{-1}(S(h_{(1)}))$. Note that for $\sum b_i\#g_i\otimes c_i\#h_i\in L(A,H)$ we have

$$\sum b_i\#g_i\otimes c_i\#h_i=\sum b_i\#g_i\otimes c_i\#\varepsilon(h_{i(1)})h_{i(2)}=\sum b_i\#g_{i(1)}\otimes c_i\#\varepsilon(h_i)S(g_{i(2)})$$

by (3.1) so that $L(A,H)$ is spanned by elements of the form $b\#h_{(1)}\otimes c\#S(h_{(2)})$ for $h\in H$, $b,c\in B$. Now

$$\phi(b\#h_{(1)}\otimes c\#S(h_{(2)}))=b\otimes h_{(1)}\otimes\overline{(h_{(2)}\rightharpoonup c)\sigma(h_{(3)},S(h_{(4)}))}=b\otimes h_{(1)}\otimes\overline{h_{(2)}\wedge c}$$

by [13, Lem.1.2.(1) and Lem.1.6.(1)], and

$$\psi(b\otimes h\otimes\overline{c})=b\gamma(h_{(1)})\otimes(h_{(3)}\vee c)\gamma(Sh_{(2)})=b\#h_{(1)}\otimes h_{(3)}\vee c\#Sh_{(2)}$$

by [13, Lem.1.2.(2)]. Consequently

$$\begin{aligned}\phi\psi(b\otimes h\otimes\overline{c}) &= \phi(b\#h_{(1)}\otimes h_{(3)}\vee c\#Sh_{(2)})\\ &= b\otimes h_{(1)}\otimes\overline{h_{(2)}\wedge(h_{(3)}\vee c)}=b\otimes h\otimes\overline{c}\end{aligned}$$

and

$$\begin{aligned}\psi\phi(b\#h_{(1)}\otimes c\#S(h_{(2)})) &= \psi(b\otimes h_{(1)}\otimes\overline{h_{(2)}\wedge c})\\ &= b\#h_{(1)}\otimes h_{(3)}\vee(h_{(4)}\wedge c)\#S(h_{(2)})=b\#h_{(1)}\otimes c\#S(h_{(2)})\end{aligned}$$

by [13, Lem.1.2.(6)].

For the special case $A=B\#_\sigma H$ the structure theorem for relative Hopf bimodules thus means that ${}_A\mathcal{M}_A^H\cong\mathcal{C}$ for the category $\mathcal{C}$ introduced in example 5.4. We state without proof that the equivalence assigns to an object $(N,\rightharpoonup)$ of $\mathcal{C}$ the relative Hopf bimodule $N\#_\sigma H$ defined to be $N\otimes H$ as a right H-comodule with the left and right $B\#_\sigma H$-module structures

$$\begin{aligned}(b\#g)(n\#h) &= b(g_{(1)}\rightharpoonup n)\sigma(g_{(2)},h_{(1)})\#g_{(3)}h_{(2)}\\ (n\#g)(b\#h) &= n(g_{(1)}\rightharpoonup b)\sigma(g_{(2)},h_{(1)})\#g_{(3)}h_{(2)}\end{aligned}$$

for $b \in B$, $n \in N$ and $g, h \in H$.

Note that in the case where the cocycle σ is trivial, $\mathcal{C}$ is just the category of B-B-bimodules in the monoidal category of H-modules.

Corollary 6.5. *Let H be a k-finitely generated projective Hopf algebra, and B a left H-module algebra. Then we have a category equivalence*

$$_{B\#H}\mathcal{M}^{H}_{B\#H} \cong {}_B\mathcal{M}^{H^*}_B,$$

where $B\#H$ denotes the smash product.

In [10] we have shown that the center of the monoidal category $_A\mathcal{M}^H_A$ is equivalent to the category $\mathcal{YD}^H_H$ of Yetter-Drinfeld modules. Using the preceding corollary we can extend this result to any H-comodule algebra A, provided H is finite:

Corollary 6.6. *Let H be a k-finitely generated projective Hopf algebra and A a right H-comodule algebra. Then the center of the monoidal category of relative Hopf bimodules over A is*

$$\mathcal{Z}({}_A\mathcal{M}^H_A) \cong \mathcal{YD}^H_H$$

Indeed, we have $\mathcal{Z}({}_A\mathcal{M}^H_A) \cong \mathcal{Z}({}_{A\#H^*}\mathcal{M}^{H^*}_{A\#H^*}) \cong \mathcal{YD}^{H^*}_{H^*} \cong \mathcal{YD}^H_H$, because $A\#H^*$ is an H^*-Galois extension, so that [10, Thm.4.3] applies.

We now return to the general case of a faithfully flat H-Galois extension A/B. In the proof of Theorem 6.3 we have avoided showing that Δ is coassociative and an algebra map by making use of the "explanation" given for its existence via Theorem 5.1. This can of course also be done explicitly using the formulas for the $h^{[1]} \otimes h^{[2]}$ notation listed in section 2. We shall have to do part of the necessary calculations to prove the following lemma, which says that A is a left L-comodule algebra with a right H-colinear comodule structure (a kind of bicomodule algebra).

Lemma 6.7. *The map*

$$\begin{aligned}\lambda : A &\to L \times_B A\\ a &\mapsto a_{(0)} \otimes {a_{(1)}}^{[1]} \otimes {a_{(1)}}^{[2]}\end{aligned}$$

is a morphism of H-comodule algebras (where $L \times_B A$ has the comodule structure induced by that of A), and a left L-comodule structure for A.

Proof. That $\lambda_0 : A \ni a \mapsto a_{(0)} \otimes {a_{(1)}}^{[1]} \otimes {a_{(1)}}^{[2]} \in L \diamond A$ is well defined follows from (2.7). That the image of λ_0 is contained in $L \times_B A$ is a direct consequence of (2.5). That the right comodule structure of $L \times_B A$ is well defined follows from the assumption that H is k-flat. The fact that λ is a colinear algebra map follows from (2.1) and (2.4). To see that λ is a comodule structure, it is enough to check that

$$(\Delta_0 \diamond A)\lambda_0 = (L \diamond \lambda_0)\lambda_0 : A \to L \diamond L \diamond A.$$

For $a \in A$ we have

$$\begin{aligned}(L \diamond \lambda_0)\lambda_0(a) &= a_{(0)} \otimes a_{(1)}{}^{[1]} \otimes \lambda_0(a_{(1)}{}^{[2]})\\ &= a_{(0)} \otimes a_{(1)}{}^{[1]} \otimes a_{(1)}{}^{[2]}{}_{(0)} \otimes a_{(1)}{}^{[2]}{}_{(1)}{}^{[1]} \otimes a_{(1)}{}^{[2]}{}_{(1)}{}^{[2]}\\ &= a_{(0)} \otimes a_{(1)}{}^{[1]} \otimes a_{(1)}{}^{[2]} \otimes a_{(2)}{}^{[1]} \otimes a_{(2)}{}^{[2]}\\ &= \Delta_0(a_{(0)} \otimes a_{(1)}{}^{[1]}) \otimes a_{(1)}{}^{[2]}\\ &= (\Delta_0 \diamond A)\lambda_0(a) \quad \square\end{aligned}$$

In particular, A is a left L-comodule algebra; we can thus define ${}^L_A\mathcal{M}$, ${}^L\mathcal{M}_A$ and ${}^L_A\mathcal{M}_A$ as the categories of left, right, and bimodules over the algebra A in the monoidal category of left L-comodules. The last of the three categories is monoidal with the tensor product over A. We can define an L-H-bicomodule M to be a left L-comodule and right H-comodule such that the left comodule structure $M \to L \times_B M$ is right H-colinear. Thus, we can also define mixed relative Hopf module categories ${}^L_A\mathcal{M}^H$ etc. We will now be concerned with the structure of all these relative Hopf module categories.

Proposition 6.8. *Let H be a k-flat Hopf algebra, A a left faithfully flat right H-Galois extension of B, and put $L := L(A, H)$. Schneider's category equivalence $\mathcal{M}^H_A \cong \mathcal{M}_B$ induces an equivalence ${}^L\mathcal{M}^H_A \cong {}^L\mathcal{M}$. If A is also right faithfully flat over B and the antipode of H is bijective, then the equivalence ${}_A\mathcal{M}^H \cong {}_B\mathcal{M}$ induces an equivalence ${}^L_A\mathcal{M}^H \cong {}^L\mathcal{M}$.*

Proof. If $M \in {}^L\mathcal{M}^H_A$, then clearly $M^{\mathrm{co}\,H}$ is an L-subcomodule of M. Conversely, if $N \in {}^L\mathcal{M}$ then $A \underset{B}{\otimes} N \in {}^L\mathcal{M}$ by Proposition 5.6. For $M \in {}^L\mathcal{M}^H_A$ the multiplication map $M \underset{B}{\otimes} A \to M$ is L-colinear by definition of a relative Hopf module in ${}^L\mathcal{M}_A$, hence so is the isomorphism $M^{\mathrm{co}\,H} \underset{B}{\otimes} A \cong M$. Clearly, for $N \in {}^L\mathcal{M}$ the inclusion $N \to N \underset{B}{\otimes} A$ is L-colinear, hence so is the isomorphism $N \cong (N \underset{B}{\otimes} A)^{\mathrm{co}\,H}$. $\square$

Lemma 6.9. *Let H be a k-flat Hopf algebra with bijective antipode and A a right H-Galois extension of B which is faithfully flat as both a right and left B-module. Then $M \in {}_A\mathcal{M}^H_A$ is faithfully flat as a right A-module if and only if $M^{\mathrm{co}\,H}$ is faithfully flat as a right B-module.*

Proof. Since $M \cong M^{\mathrm{co}\,H} \underset{B}{\otimes} A$ as a right A-module by [11, Thm.I], faithful flatness of $M^{\mathrm{co}\,H}$ over B implies faithful flatness of M over A. Conversely, let f be an injective left B-module map. With the obvious meaning of $\cong$, we have $A \underset{B}{\otimes} (M^{\mathrm{co}\,H} \underset{B}{\otimes} f) \cong M \underset{B}{\otimes} f \cong M \underset{A}{\otimes} A \underset{B}{\otimes} f$, which is injective since A_B and M_A are flat; thus, since A_B is faithfully flat, $M^{\mathrm{co}\,H} \underset{B}{\otimes} f$ is injective. If N is a left B-module with $M^{\mathrm{co}\,H} \underset{B}{\otimes} N = 0$, then $0 = A \underset{B}{\otimes} (M^{\operatorname{co}H} \underset{B}{\otimes} N) \cong M \underset{B}{\otimes} N \cong M \underset{A}{\otimes} A \underset{B}{\otimes} N$, whence $N = 0$ since M_A and A_B are faithfully flat. □

Definition 6.10. Let L be a $\times_B$-bialgebra and let N be a left L-comodule. Then

$$^{\operatorname{co}L}N := \{n \in N | \lambda(n) = 1_L \otimes n\}.$$

By our definition, $^{\operatorname{co}L}N$ does not have to be a sub B-bimodule of N.

Remark 6.11. For a left L-comodule N we have $^{\operatorname{co}L}N \subset N^B$.

In fact, $\lambda : N \to L \times_B N$ is injective by commutativity of the second diagram in Definition 5.5 of a comodule, and $1_L \otimes n \in (L \times_B N)^B$ for $n \in N$, since $b(1_L \otimes n) = b \cdot 1_L \otimes n = 1_L \cdot b \otimes n = (1_L \otimes n)b$ for $b \in B$.

Proposition 6.12. *Let H be a k-flat Hopf algebra and A a left faithfully flat H-Galois extension of B which is also faithfully k-flat. Put $L := L(A, H)$. Then $^{\operatorname{co}L}A = k$, and we have an isomorphism*

$$A \otimes A \ni x \otimes y \mapsto x_{(-1)} \otimes x_{(0)}y \in L \diamond A$$

which induces an isomorphism $A \otimes A^B \cong L \times_B A$.

Proof. Let $x \in {}^{\operatorname{co}L}A$. Then, in $L \times_B A \subset L \diamond A \subset A \otimes A \underset{B}{\otimes} A$ we have $1_A \otimes 1_A \otimes x = 1_L \otimes x = x_{(-1)} \otimes x_{(0)} = x_{(0)} \otimes {x_{(1)}}^{[1]} \otimes {x_{(1)}}^{[2]}$ so that, in $A \otimes A$, $1 \otimes x = x_{(0)} \otimes {x_{(1)}}^{[1]}{x_{(1)}}^{[2]} = x \otimes 1$ by (2.3), whence $x \in k$ by faithful flatness of A over k. The claimed isomorphism $\tilde{\kappa} : A \otimes A \cong L \diamond A$ is indeed nothing but the isomorphism (2.7). By faithful flatness of A over k, we have $A \otimes A^B = \{\sum x_i \otimes y_i \in A \otimes A | \forall b \in B : \sum x_i \otimes by_i = \sum x_i \otimes y_i b$. Now $\tilde{\kappa}(x \otimes by) = x_{(-1)} \otimes x_{(-1)}by = x_{(-1)}\overline{b} \otimes x_{(0)}y$ and $\tilde{\kappa}(x \otimes yb) = x_{(-1)} \otimes x_{(-1)}yb$ imply $\tilde{\kappa}(A \otimes A^B) = L \times_B A$. □

Obviously the preceding proposition states roughly that A is a left L-Galois extension of k. However, we are not really sure if this assumes the right definition of Galois extensions over $\times_B$-bialgebras. But we can continue to check if A as an L-comodule algebra has properties analogous to those of an ordinary Hopf-Galois extension. One of these properties, actually characterizing faithfully flat Hopf-Galois extensions by [11, Thm.I], is a structure theorem for relative Hopf modules.

Theorem 6.13. *Assume that k is a field, H is a Hopf algebra and A is a left faithfully flat H-Galois extension of B. Put $L := L(A, H)$. Then the following are quasi-inverse category equivalences:*

$$\begin{array}{ccc} {}^L_A\mathcal{M} \to {}_k\mathcal{M} & \qquad & {}_k\mathcal{M} \to {}^L_A\mathcal{M} \\ M \mapsto {}^{\operatorname{co}L}M & \qquad & V \mapsto A \otimes V \end{array}$$

Proof. It is easy to check that the two functors are well defined. For $M \in {}^L_A\mathcal{M}$ we have a map

$$\mu : A \otimes {}^{\mathrm{co}\,L}M \ni a \otimes m \mapsto am \in M.$$

We claim that an inverse for μ is given explicitly by

$$\overline{\mu} : M \ni m \mapsto {m_{(-1)}}^{(1)} \otimes {m_{(-1)}}^{(2)} m_{(0)} \in A \otimes {}^{\mathrm{co}\,L}M.$$

First, we check that $\overline{\mu}$ is well defined. For $m \in M \in {}^L_A\mathcal{M}$, we have ${m_{(-1)}}^{(1)} \otimes {m_{(-1)}}^{(2)} m_{(0)} \in A \otimes {}^{\mathrm{co}\,L}M$, because

$$\begin{aligned}
&{m_{(-1)}}^{(1)} \otimes \lambda({m_{(-1)}}^{(2)} m_{(0)}) \\
&\qquad = {m_{(-2)}}^{(1)} \otimes {{m_{(-2)}}^{(2)}}_{(-1)} m_{(-1)} \otimes {{m_{(-2)}}^{(2)}}_{(0)} m_{(0)} \\
&\qquad = {{m_{(-1)}}^{(1)}}_{(0)} \otimes {{{{m_{(-1)}}^{(1)}}_{(1)}}^{[1]}}_{(-1)} ({{{m_{(-1)}}^{(1)}}_{(1)}}^{[2]} \otimes {m_{(-1)}}^{(2)}) \otimes {{{{m_{(-1)}}^{(1)}}_{(1)}}^{[1]}}_{(0)} m_{(0)} \\
&\qquad = {{m_{(-1)}}^{(1)}}_{(0)} \otimes ({{{{m_{(-1)}}^{(1)}}_{(1)}}^{[1]}}_{(0)} \otimes {{{{{m_{(-1)}}^{(1)}}_{(1)}}^{[1]}}_{(1)}}^{[1]}) ({{{m_{(-1)}}^{(1)}}_{(1)}}^{[2]} \otimes {m_{(-1)}}^{(2)}) \\
&\qquad\qquad \otimes {{{{{m_{(-1)}}^{(1)}}_{(1)}}^{[1]}}_{(1)}}^{[2]} m_{(0)} \\
&\qquad = {{m_{(-1)}}^{(1)}}_{(0)} \otimes {{{{m_{(-1)}}^{(1)}}_{(1)}}^{[1]}}_{(0)} {{{m_{(-1)}}^{(1)}}_{(1)}}^{[2]} \otimes {m_{(-1)}}^{(2)} {{{{{m_{(-1)}}^{(1)}}_{(1)}}^{[1]}}_{(1)}}^{[1]} \\
&\qquad\qquad \otimes {{{{{m_{(-1)}}^{(1)}}_{(1)}}^{[1]}}_{(1)}}^{[2]} m_{(0)} \\
&\qquad = {{m_{(-1)}}^{(1)}}_{(0)} \otimes 1 \otimes {m_{(-1)}}^{(2)} S({{m_{(-1)}}^{(1)}}_{(1)})^{[1]} \otimes S({{m_{(-1)}}^{(1)}}_{(1)})^{[2]} m_{(0)} \\
&\qquad = {m_{(-1)}}^{(1)} \otimes 1 \otimes {{m_{(-1)}}^{(2)}}_{(0)} {{{m_{(-1)}}^{(2)}}_{(1)}}^{[1]} \otimes {{{m_{(-1)}}^{(2)}}_{(1)}}^{[2]} m_{(0)} \\
&\qquad = {m_{(-1)}}^{(1)} \otimes 1 \otimes 1 \otimes {m_{(-1)}}^{(2)} m_{(0)}.
\end{aligned}$$

Here the first equality uses that λ is an algebra map, the second is the definition of comultiplication on L, the third the definition of λ. For the fifth equality we used the formula ${h^{[1]}}_{(0)} h^{[2]} \otimes {h^{[1]}}_{(1)} = 1 \otimes S(h)$ for $h \in H$, which follows from (2.2) and (2.3). The sixth and last equality are applications of (3.1) and (2.6), respectively.

For a vector space V, we only have to tensor the equalizer expressing the isomorphism $k \cong {}^{\mathrm{co}\,L}A$ with V to obtain $V \cong {}^{\mathrm{co}\,L}(A \otimes V)$, completing the proof. □

Using the structure theorem 6.13 for left (A, L)-Hopf modules, we can prove one for relative Hopf bimodules (namely, (6.1) below) which is a counterpart to Theorem 3.3 in the case $R = A$, interchanging the roles of H and L.

Theorem 6.14. *Let k be a field, H a Hopf algebra, A a left faithfully flat H-Galois extension of B, and put $L := L(A, H)$. Then we have category equivalences*

$$ {}^L_A\mathcal{M}_A \cong \mathcal{M}_H \tag{6.1}$$

$$ {}^L_A\mathcal{M}^H \cong \mathcal{M}^H \tag{6.2}$$

$$ {}^L_A\mathcal{M}^H_A \cong \mathcal{YD}^H_H \tag{6.3}$$

induced by the category equivalence from Theorem 6.13.

Proof. We begin with (6.1). Let $(V, \leftharpoonup) \in \mathcal{M}_H$. Then $A \otimes V \in {}^L_A\mathcal{M}_A$ with $(a \otimes v)x = ax_{(0)} \otimes v \leftharpoonup x_{(1)}$ for $a, x \in A$ and $v \in V$. Now let $M \in {}^L_A\mathcal{M}_A$. Then M^B is a right H-module with the Miyashita-Ulbrich action $m \leftharpoonup h = h^{[1]}mh^{[2]}$. We claim that ${}^{\mathrm{co}\,L}M$, which is contained in M^B by 6.11, is actually an H-submodule of M^B. For $m \in {}^{\mathrm{co}\,L}M$ and $h \in H$ we compute

$$\begin{aligned}
\lambda(h^{[1]}mh^{[2]}) &= {h^{[1]}}_{(-1)}m_{(-1)}{h^{[2]}}_{(-1)} \otimes {h^{[1]}}_{(0)}m_{(0)}{h^{[2]}}_{(0)} \\
&= {h^{[1]}}_{(-1)}{h^{[2]}}_{(-1)} \otimes {h^{[1]}}_{(0)}m{h^{[2]}}_{(0)} \\
&= {h^{[1]}}_{(0)}{h^{[2]}}_{(0)} \otimes {{h^{[2]}}_{(1)}}^{[1]}{{h^{[1]}}_{(1)}}^{[1]} \otimes {{h^{[1]}}_{(1)}}^{[2]}m{{h^{[2]}}_{(1)}}^{[2]} \\
&= {h_{(2)}}^{[1]}{h_{(2)}}^{[2]} \otimes {h_{(3)}}^{[1]}S(h_{(1)})^{[1]} \otimes S(h_{(1)})^{[2]}m{h_{(3)}}^{[2]} \\
&= 1 \otimes {h_{(2)}}^{[1]}S(h_{(1)})^{[1]} \otimes S(h_{(1)})^{[2]}m{h_{(2)}}^{[2]} \\
&= \overline{{h_{(2)}}^{[1]}S(h_{(1)})^{[1]}} \cdot 1_L \otimes S(h_{(1)})^{[2]}m{h_{(2)}}^{[2]} \\
&= 1_L \otimes {h_{(2)}}^{[1]}S(h_{(1)})^{[1]}S(h_{(1)})^{[2]}m{h_{(2)}}^{[2]} \\
&= 1_L \otimes h^{[1]}mh^{[2]},
\end{aligned}$$

so that $h^{[1]}mh^{[2]} \in {}^{\mathrm{co}\,L}M$. The key fact we have used here (besides (2.1) and (2.2) in the fourth and (2.3) in the fifth and seventh equality) is that the element ${h_{(2)}}^{[1]}S(h_{(1)})^{[1]} \otimes S(h_{(1)})^{[2]} \otimes {h_{(2)}}^{[2]} \in A \underset{B}{\otimes} A \underset{B}{\otimes} A$ is contained in $B \underset{B}{\otimes} A \underset{B}{\otimes} A$. In view of faithful flatness of ${}_BA$, this follows from (2.2) by the calculation

$$\begin{aligned}
&\delta\left({h_{(2)}}^{[1]}S(h_{(1)})^{[1]}\right) \otimes S(h_{(1)})^{[2]} \otimes {h_{(2)}}^{[2]} \\
&\qquad = {h_{(4)}}^{[1]}S(h_{(1)})^{[1]} \otimes S(h_{(3)})S^2(h_{(2)}) \otimes S(h_{(1)})^{[2]} \otimes {h_{(4)}}^{[2]} \\
&\qquad = {h_{(2)}}^{[1]}S(h_{(1)})^{[1]} \otimes 1 \otimes S(h_{(1)})^{[2]} \otimes {h_{(2)}}^{[2]}.
\end{aligned}$$

We have exhibited well defined functors ${}^L_A\mathcal{M}_A \rightleftarrows \mathcal{M}_H$. That the isomorphisms $V \cong {}^{\mathrm{co}\,L}(A \otimes V)$ for $V \in \mathcal{M}_H$ and $A \otimes {}^{\mathrm{co}\,L}M \cong M$ for $M \in {}^L_A\mathcal{M}_A$ from Theorem 6.13 are in $\mathcal{M}_H$ and ${}^L_A\mathcal{M}_A$, respectively, is not hard to check.

If V is a right H-comodule, then $A \otimes V$ is a right H-comodule with the codiagonal comodule structure. If $M \in {}^L_A\mathcal{M}^H$, then ${}^{\mathrm{co}\,L}M$ is an H-subcomodule of M. We omit checking the remaining details for (6.2).

Now assume that $V \in \mathcal{YD}^H_H$. We have checked in [10] that then $A \otimes V \in {}_A\mathcal{M}^H_A$ with the structures above, so that now $A \otimes V \in {}^L_A\mathcal{M}^H_A$. Conversely, assume $M \in {}^L_A\mathcal{M}^H_A$. Since $M^B \in \mathcal{YD}^H_H$ with the Miyashita-Ulbrich action, see [10], and ${}^{\mathrm{co}\,L}M$ is a submodule and subcomodule of M^B, we conclude ${}^{\mathrm{co}\,L}M \in \mathcal{YD}^H_H$. □

In the case where $B = k$ we have seen in [9] that the roles of L and H for the extension A/k are completely symmetric because L is uniquely determined by A being an L-H-biGalois extension. While this does not directly make sense in the present more general situation, we still have a category equivalence ${}^L_A\mathcal{M}_A \cong \mathcal{M}_H$ by the last theorem, which looks completely symmetric to the equivalence ${}_A\mathcal{M}^H_A \cong {}_L\mathcal{M}$. There is another more direct symmetry between the coactions of H and L on A: The Galois map $A \underset{B}{\otimes} A \cong A \otimes H$ is easily checked to be left L-colinear, where the structure on the left hand side is the codiagonal one, and the structure on the right is induced by that of A. It follows that $H \cong {}^{\mathrm{co}\,L}(A \underset{B}{\otimes} A)$ via $h \mapsto h^{[1]} \otimes h^{[2]}$, which parallels the construction of L as $L = (A \otimes A)^{\mathrm{co}\,H}$.

In [9] we have also found that, in the case $B = k$, we have a category equivalence ${}^H\mathcal{M} \cong {}^L\mathcal{M}$ given by cotensor product with A. This still holds true in our present situation, if we assume k is a field:

Theorem 6.15. *Let k be a field, H a Hopf algebra with bijective antipode and A a faithfully flat H-Galois extension of B. Put $L := L(A, H)$. We have an equivalence of monoidal categories*

$$^H\mathcal{M} \ni V \mapsto A \,\square_H\, V \in {}^L\mathcal{M}$$

Proof. We obviously have a well defined functor ${}^H\mathcal{M} \to {}^L\mathcal{M}$ as stated. For $V, W \in {}^H\mathcal{M}$ we have a map

$$\beta : (A \,\square_H\, V) \underset{B}{\otimes} (A \,\square_H\, W) \to A \,\square_H\, (V \otimes W)$$
$$(\sum x_i \otimes v_i) \otimes \sum (y_j \otimes w_j) \mapsto \sum x_i y_j \otimes v_i \otimes w_j$$

which is left L-colinear since A is a left L-comodule algebra. Arguing as in [18, 19] we see that β is an isomorphism: A_B being faithfully flat, we have an isomorphism

$$\gamma_V : A \underset{B}{\otimes} (A \,\square_H\, V) \cong (A \underset{B}{\otimes} A) \,\square_H\, V \cong (A \otimes H) \,\square_H\, V \cong A \otimes V$$

with $\gamma_V(a \otimes \sum x_i \otimes v_i) = \sum a x_i \otimes v_i$ for $a \in A$ and $\sum x_i \otimes v_i \in A \,\square_H\, V$, and the diagram

$$\begin{tikzcd}
A \underset{B}{\otimes} (A \,\square_H\, V) \underset{B}{\otimes} (A \,\square_H\, W) \arrow[r, "\gamma_V \otimes \mathrm{id}"] \arrow[d, "A \underset{B}{\otimes} \beta"'] & A \otimes V \underset{B}{\otimes} (A \,\square_H\, W) \arrow[d, "V \otimes \gamma_W"] \\
A \underset{B}{\otimes} (A \,\square_H\, (V \otimes W)) \arrow[r, "\gamma_{V \otimes W}"'] & A \otimes V \otimes W
\end{tikzcd}$$

is easily checked to commute, whence β is an isomorphism. That β fulfills the coherence conditions for making $V \mapsto A \,\square_H\, V$ a monoidal functor is a direct consequence of the multiplication in A being associative. It remains to show that the functor is an equivalence, which we will do by a detour through the theorems above. First, we have an equivalence ${}^H\mathcal{M} \cong \mathcal{M}^H$ which maps V to V^S, the right H-comodule with the comodule structure induced by that of V via the antipode. Composing this with the equivalences obtained above we get ${}^H\mathcal{M} \cong \mathcal{M}^H \cong {}^L_A\mathcal{M}^H \cong {}^L\mathcal{M}$, where the second equivalence is Proposition 6.8, and the third is (6.2). The composition coincides with the functor in consideration since we have $A \,\square_H\, V = (A \otimes V^S)^{\mathrm{co}\,H}$ as left L-comodules by [11, Lem.3.1.]. $\square$

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