Braided bi-Galois theory

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1 Introduction

To keep technicalities to a minimum for the beginning of the introduction, we start by letting H be a Hopf algebra with bijective antipode over a field k. A right H-Galois object is a right H-Galois extension of the base field k, that is, a right H-comodule algebra A with $A^{\operatorname{co} H} = k$ for which a certain canonical map $\beta\colon A \otimes A \to A \otimes H$ is bijective. Hopf-Galois extensions are an important application and tool for the study of Hopf algebras. We refer to [9] and [6] for more background and references.

Grunspan [3] has defined a quantum torsor to be an algebra T equipped with an algebra map $\Theta\colon T \to T \otimes T^{\mathrm{op}} \otimes T$ and an algebra map $\vartheta\colon T \to T$ subject to a list of axioms; we will call Θ the torsor structure and have called ϑ the Grunspan map in [8]. By and large, the idea of a quantum torsor allows us to define Hopf-Galois objects without mentioning a Hopf algebra. The torsor structure is a sort of (triple) comultiplication, and the Grunspan map generalizes the square of the antipode of a Hopf algebra. Grunspan shows that every quantum torsor in his sense is a Hopf-Galois object over a suitably constructed Hopf algebra H. The paper [7] makes this an equivalent characterization by showing that every H-Galois extension is also a torsor. While the torsor structure is quite easy to find, the Grunspan map, written down explicitly in [7], is perhaps less obvious. In [8] it was then shown that the Grunspan map can be eliminated altogether from the axioms of a quantum torsor. Its uniqueness was already observed by Grunspan, and it can be proved to exist via constructing a Hopf-Galois extension from a torsor (without Grunspan map) and then a Grunspan map from the Hopf-Galois extension.

In [7] the Grunspan map for an H-Galois object A was simply written down explicitly. Here now is a conceptual reason for its existence: By [5], A is an L-H-bi-Galois object (i.e. a left L-Galois object and a bicomodule) for a Hopf algebra L with bijective antipode. Also by [5], bi-Galois objects are the morphisms of a groupoid; the composition in the groupoid is the cotensor product, and the inverse of A can be constructed as the algebra A^{op} endowed with the H-L-bicomodule structure(s) pulled back along the inverse S^{-1} of the antipode. But surely the inverse of the inverse of A is A again, so there is an algebra automorphism of $A = (A^{\mathrm{op}})^{\mathrm{op}}$. So far the conclusion is trivial, but observe that the automorphism is a bicomodule algebra isomorphism from A to $(A^{\mathrm{op}})^{\mathrm{op}}$, and the latter has the bicomodule structure(s) pulled back along S^{-2}. In fact one can plug the explicit knowledge of the groupoid structure on the set of isomorphism classes of bi-Galois objects into the standard proof that $(x^{-1})^{-1} = x$ in a group(oid) to obtain the isomorphism explicitly, and it turns out to be the Grunspan map.

Now assume that we set all of the above in a braided category $\mathcal{B}$. So H is a braided Hopf algebra, and A is a braided Hopf-Galois extension of the unit object I. Then once again, taking the double inverse of A in the groupoid of bi-Galois extensions will give A, so again there is an algebra isomorphism $\vartheta\colon A \to (A^{\mathrm{op}})^{\mathrm{op}}$. This time, even this statement is not obvious: In the braided setting, taking the opposite algebra is usually not involutive even up to isomorphism. Akrami and Majid [1] recently introduced the term "ribbon algebra" for an algebra with an isomorphism $\vartheta\colon A \to (A^{\mathrm{op}})^{\mathrm{op}}$. Our result shows that also in the braided setting the square of the antipode of a Hopf algebra (an isomorphism $H \cong (H^{\mathrm{op}})^{\mathrm{op}}$) generalizes to a ribbon structure on each H-Galois object. Again, plugging the explicit description of the groupoid structure into the proof that taking inverses is an involution yields ϑ explicitly, and it is a Grunspan morphism for the braided torsor A in a suitable sense.

Now saying all this we have tacitly assumed that bi-Galois theory in a braided category works just the same as in the category of vector spaces — and this and the above is of course only true under suitable assumptions of the existence of equalizers and their preservation under certain tensor products. In particular, we have to be able to construct a bi-Galois object out of a right Hopf-Galois object, and bi-Galois objects should form a groupoid, with the inverse of A built on the algebra A^{op}. And in fact most of the paper we will be occupied with providing just this necessary background, making sure that there are no "truly braided" obstacles to braided bi-Galois theory. We feel that this is justified at least by the "braidedly nontrivial" conclusion that braided Hopf-Galois objects are ribbon algebras. Also, we take care on the way to improve some previously published statements and proofs also for the unbraided case.

For example, we construct the Hopf algebra $L = (A \otimes A)^{\mathrm{co}\,H}$ that makes A an L-H-bi-Galois object not only under the assumption that A is faithfully flat, but under the assumption that both A and L are flat. This may well prove useless when we think of $\mathcal{B}$ the category of k-modules over a commutative ring k. But for example in the opposite category of this, all objects are flat, so the sharper statement is suitable for the study of the formal dual of bi-Galois objects over a base ring (for which objects we hesitate to spell out a name). We also give a better proof that faithfully flat bi-Galois objects form a groupoid even if the Hopf algebras involved do not have bijective antipodes. The inverses in this groupoid were already given in [5], but the proof there is rather clumsy.

After providing some general tools in Section 2 (mainly generalities on braided Hopf algebras as found in [4] and some generalizations of well-known facts from the unbraided case) we deal with the basic properties of Hopf-Galois objects (notably the restricted inverse $\gamma\colon H \to A \otimes A$ of the Galois map β and the Miyashita-Ulbrich action) in Section 3. We construct the Hopf algebra L making an H-Galois object L-H-bi-Galois in Section 4, deal with the groupoid structure on the set of isomorphism classes of bi-Galois objects in Section 5, and specialize some of the results to the case where the antipodes are bijective in Section 6. The material so far is a braided variant, with some improvements, of that found (with references) in [9] or [6]. The conclusion that Galois objects have a Grunspan morphism making them ribbon algebras is then reached in the way sketched above in Section 7.

Acknowledgements

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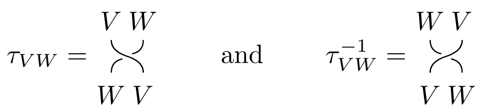
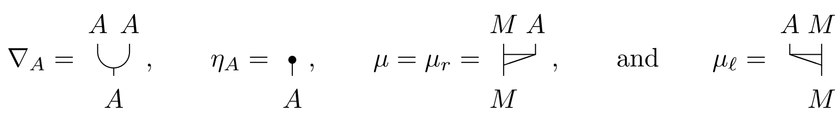
2 Preliminaries

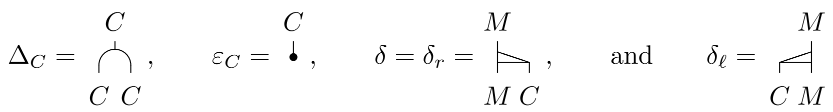
Throughout the paper, $\mathcal{B}$ denotes a monoidal category with equalizers and a braiding τ. We say that an object $X \in \mathcal{B}$ is flat if tensoring with X preserves equalizers. A flat object X is called faithfully flat if tensoring with X reflects isomorphisms. We make free use of the notions of algebras, bialgebras, Hopf algebras, module algebras and comodule algebras in $\mathcal{B}$. We will assume that $\mathcal{B}$ is strict monoidal. We will use graphical calculus to do computations in $\mathcal{B}$, using the notations



for the braiding,

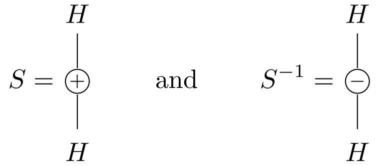


for the multiplication and unit of an algebra A in $\mathcal{B}$, the module structure of a right A-module $M \in \mathcal{B}_A$, and the module structure of a left module $M \in {}_A\mathcal{B}$,

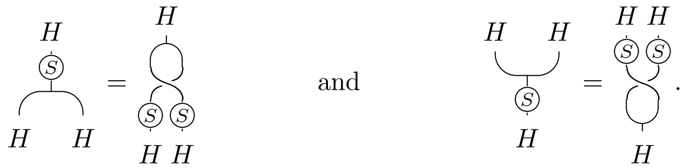


for the comultiplication and counit of a coalgebra C in $\mathcal{B}$, the comodule structure of a right C-comodule $M \in \mathcal{B}^C$, and that of a left C-comodule $M \in {}^C\mathcal{B}$.

We will use



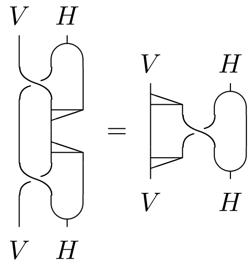
for the antipode of a Hopf algebra in H and, if S is an isomorphism, its inverse. We note that the antipode is an anti-coalgebra map and anti-algebra map in the sense that



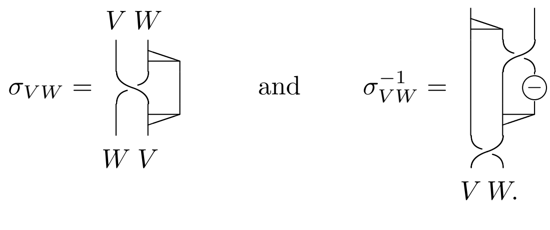
In particular, every left H-comodule V is a right H-comodule, and, if S is an isomorphism every left comodule W is a right comodule with comodule structures

(2.1)

A Yetter-Drinfeld module over a Hopf algebra H in $\mathcal{B}$, as defined in the braided case by Bespalov [2] is a right H-comodule and right H-module V satisfying



In [2] it is shown in particular that if the antipode of H is an isomorphism, then the category of Yetter-Drinfeld modules is braided, with the braiding and its inverse defined by

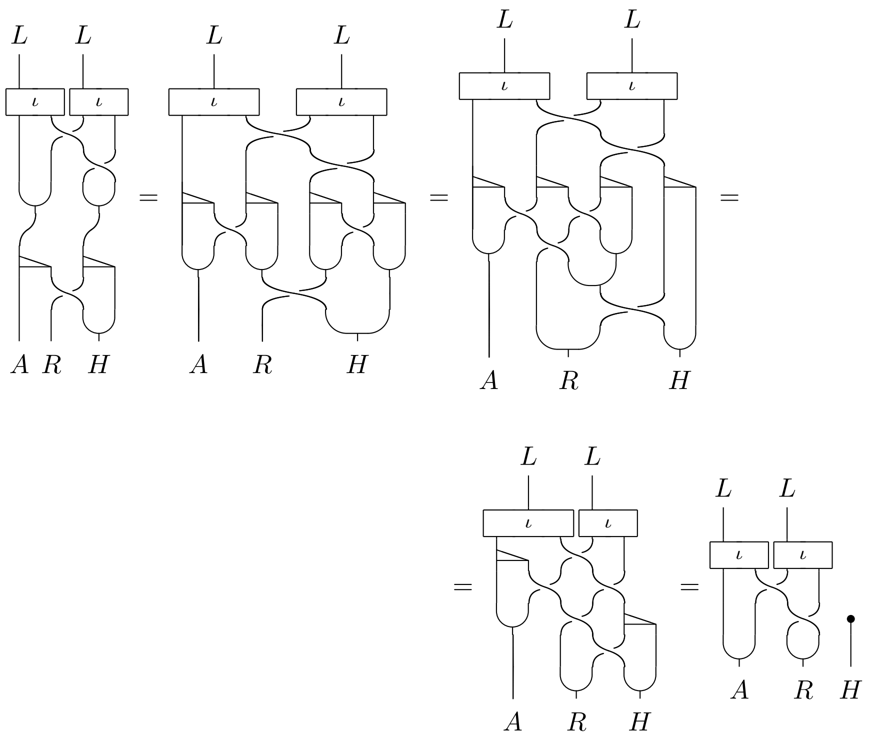


In particular, $\sigma^{\pm 1}$ are H-colinear isomorphisms.

Definition 2.1. Let H be a bialgebra in $\mathcal{B}$, and V a right H-comodule. The coinvariant subobject $V^{\operatorname{co} H} \subset V$ is the equalizer of $\delta, V \otimes \eta \colon V \to V \otimes H$. The notation ${}^{\operatorname{co} H}W$ is used for the left coinvariant subobject of a left H-comodule W.

Proposition 2.2. *Let A, R be right H-comodule algebras for a bialgebra H in $\mathcal{B}$. Then the right coinvariant subobject $(A \otimes R)^{\operatorname{co} H}$ taken with respect to the codiagonal comodule structure is a subalgebra of $A \otimes R^{\mathrm{op}}$.*

Proof. Put $L := (A \otimes R)^{\operatorname{co} H}$ and consider the inclusion $\iota \colon L \to A \otimes R$. By the calculation



we see that $\nabla_{A \otimes R^{\mathrm{op}}}(\iota \otimes \iota)$ factors over ι, so that L is a subalgebra of $A \otimes R^{\mathrm{op}}$. $\square$

The cotensor product $V \,\square_H\, W$ of a right H-comodule V and a left H-comodule W in $\mathcal{B}$ is defined as the equalizer of

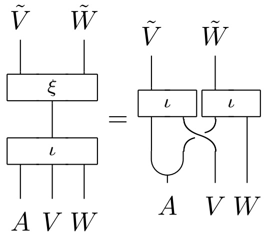
$$\delta_V \otimes W, V \otimes \delta_W \colon V \otimes W \to V \otimes H \otimes W.$$

Lemma 2.3. *Let A be a right H-comodule algebra in $\mathcal{B}$.*

(1) For two left H-comodules V, W, the composition

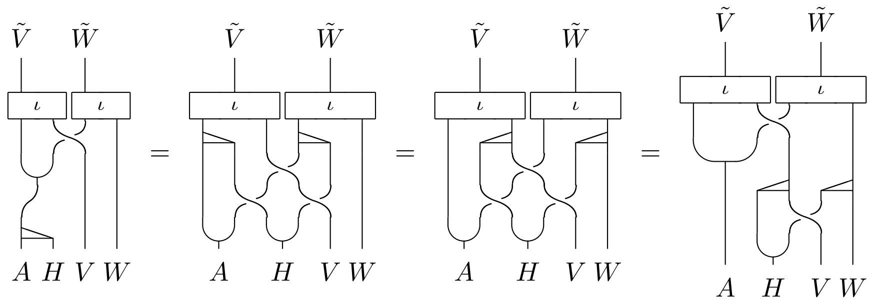
$$A \otimes V \otimes A \otimes W \xrightarrow{A \otimes \tau \otimes W} A \otimes A \otimes V \otimes W \xrightarrow{\nabla \otimes V \otimes W} A \otimes V \otimes W$$

induces a morphism $\xi\colon (A \,\square_H\, V) \otimes (A \,\square_H\, W) \to A \,\square_H\, (V \otimes W)$. Pictorially, writing $\tilde{X} := A \,\square_H\, X$:



(2) For any left H-comodule algebra B, the subobject $A \,\square_H\, B$ is a subalgebra of $A \otimes B$.

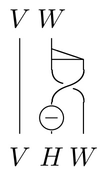
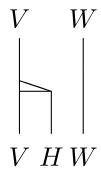
Proof. The second part can be proved using the first, but also simply by observing that $A \,\square_H\, B$ is the equalizer of two algebra morphisms. For the first part, the calculation

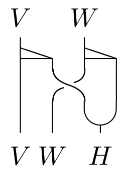


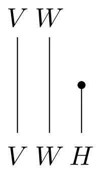
shows that ξ is well-defined. $\square$

Lemma 2.4. *Let H be a Hopf algebra whose antipode is an isomorphism, and let $V, W \in \mathcal{B}^H$. Then $(V \otimes W)^{\mathrm{co}\,H} = V \,\square_H\, W$, where W has the left H-comodule structure in* (2.1)*.*

Proof. $V \,\square_H\, W$ is the equalizer of

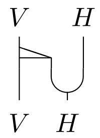
$F =$  and $G =$  , while $(V \otimes W)^{\mathrm{co}\,H}$ is the equalizer of

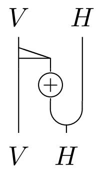
$F' =$ 

and $G' =$ 

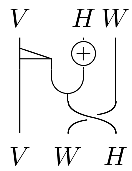
.

Since the following are mutually inverse automorphisms of $V \otimes H$:

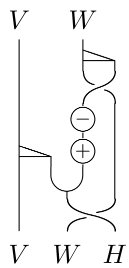


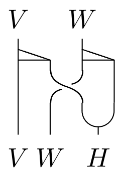


we obtain an isomorphism

$T =$ 

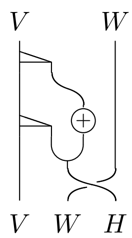
and the calculations

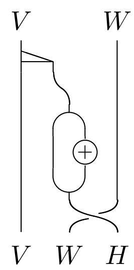
$TF =$ 

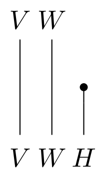
$=$ 

$= F'$

and

$TG =$ 

$=$ 

$=$ 

$= G'$

complete the proof. □

3 Galois objects

We denote the unit object of $\mathcal{B}$ by I, and the unit of an algebra A in $\mathcal{B}$ by $\eta\colon I \to A$.

Definition 3.1. Let A be a right H-comodule algebra in $\mathcal{B}$. We say A is a right H-Galois object, if $\eta\colon I \to A$ is the equalizer of $\delta, A \otimes \eta\colon A \to A \otimes H$, and the morphism

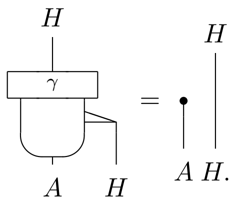
$$\beta = \left(A \otimes A \xrightarrow{A \otimes \delta} A \otimes A \otimes H \xrightarrow{\nabla \otimes H} A \otimes H \right)$$

is an isomorphism.

If A is an H-Galois object, we write

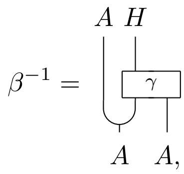
$$\gamma = \left(H \xrightarrow{\eta \otimes H} A \otimes H \xrightarrow{\beta^{-1}} A \otimes A \right).$$

Remark 3.2. As a consequence of the definition



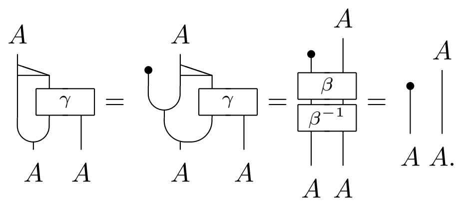
(3.1)

Since β is a left A-module morphism, we have



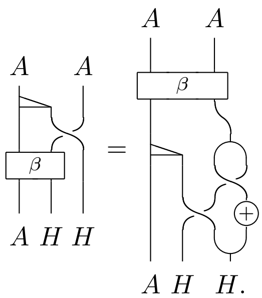
(3.2)

and we also note

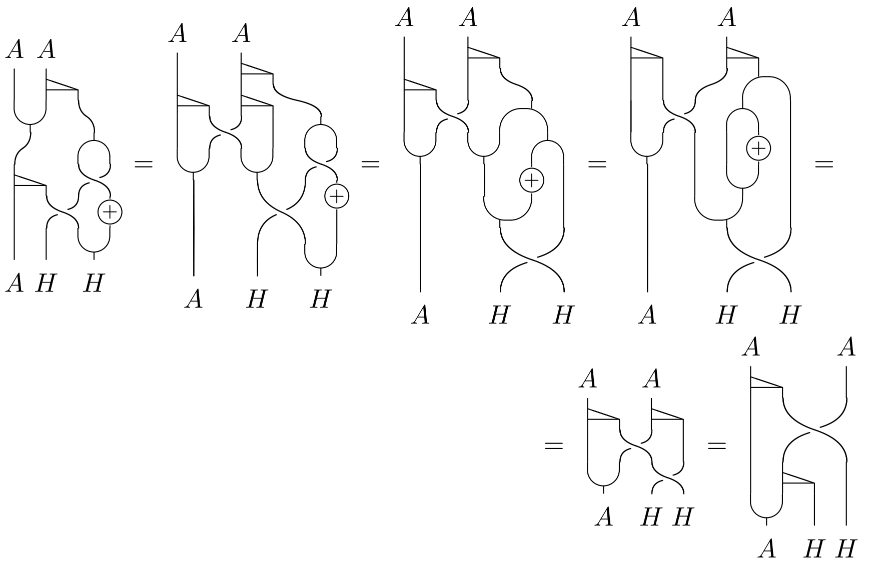


(3.3)

Lemma 3.3. *Let A be a right H-comodule algebra in $\mathcal{B}$. Then*

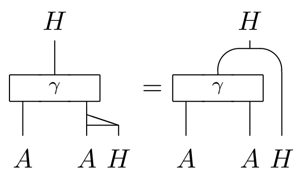


Proof.

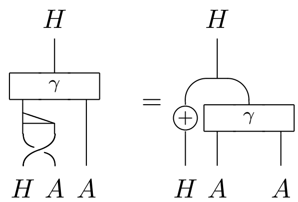


□

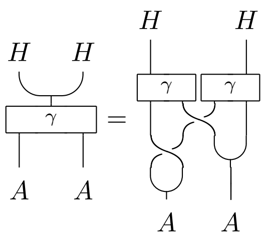
Lemma 3.4. *Let A be a right H-Galois object in $\mathcal{B}$. Then*



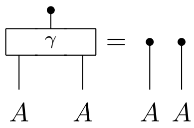
(3.4)



(3.5)

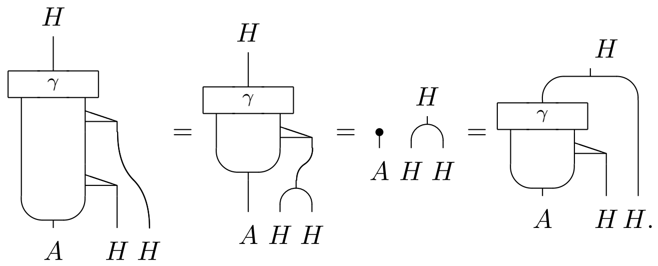


(3.6)

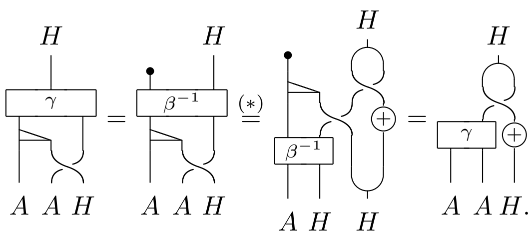


(3.7)

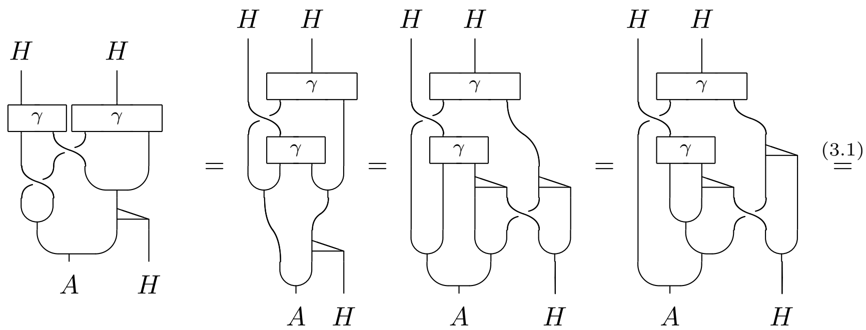
Proof. We prove (3.4) after applying β to the first two legs:

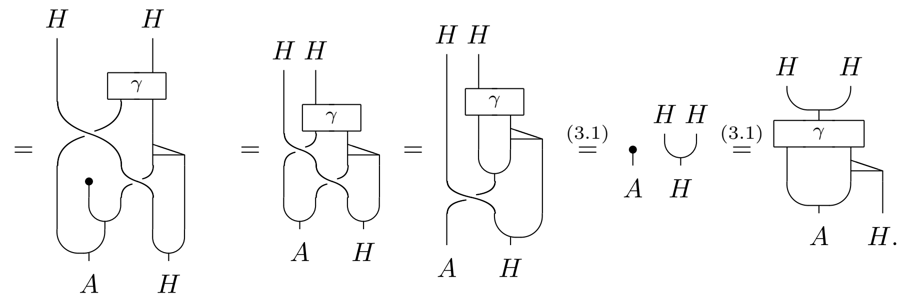


(3.5) is a consequence of Lemma 3.3, which is used in the step marked $(*)$ in the following calculation:



We show (3.6) by applying β to the bottom and calculating

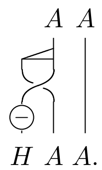




(3.7) is easy to check. □

Remark 3.5. (1) Equations (3.6) and (3.7) say that $\gamma\colon H \to A^{\mathrm{op}} \otimes A$ is an algebra morphism. Assume that A has an additional left L-comodule structure over a bialgebra L such that it is an L-H-bicomodule and a left L-comodule algebra. Then γ induces an algebra homomorphism to the subalgebra ${}^{\mathrm{co}\,L}(A \otimes A) \subset A^{\mathrm{op}} \otimes A$.

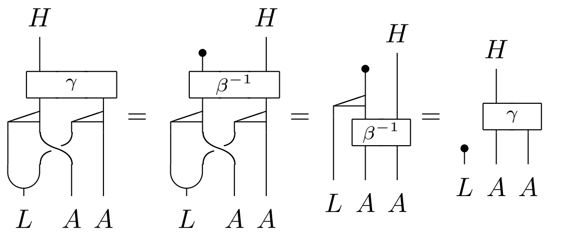
(2) Equation (3.4) says that γ is a right H-comodule map in an obvious sense. If the antipode of H is an isomorphism, then (3.5) says that γ is also a left H-comodule map, where the left comodule structure on $A \otimes A$ should be defined by



In the situation in (1), the two comodule structures of $A \otimes A$ induce an H-bicomodule structure on ${}^{\mathrm{co}\,L}(A \otimes A)$ for which $\overline{\gamma}$ is a bicomodule morphism.

Proof. (1) Note that the coinvariant subobject is in fact a subalgebra by the left comodule version of Proposition 2.2.

Clearly $\beta\colon A \otimes A \to A \otimes H$ is left L-colinear if we endow the source with the codiagonal L-comodule structure. Thus β^{-1} is also left L-colinear, and



so that γ indeed factors through the left coinvariant subobject.

(2) is clear.

□

Lemma 3.6. *Let A be a faithfully flat right H-Galois object in $\mathcal{B}$. Then the morphisms*

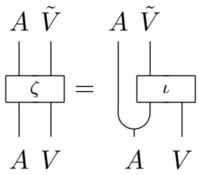
$$\xi\colon (A \underset{H}{\square} V) \otimes (A \underset{H}{\square} W) \to A \underset{H}{\square} (V \otimes W)$$

from Lemma 2.3 are isomorphisms.

Proof. We use the isomorphism

$$\zeta := \left(A \otimes (A \underset{H}{\square} V) \xrightarrow{\beta \square_H V} A \otimes H \underset{H}{\square} V \cong A \otimes V \right)$$

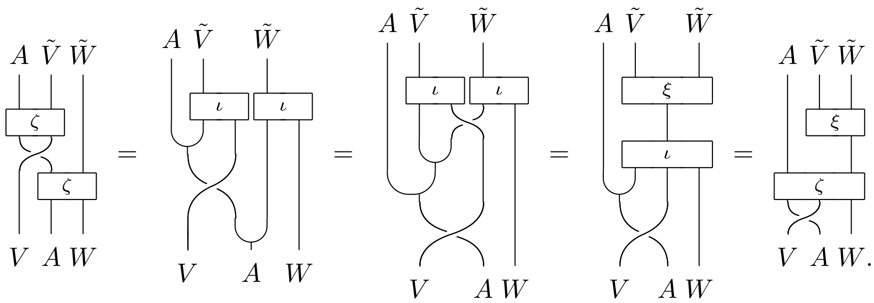
described graphically by



and show that

$$\begin{tikzcd}
A \otimes (A \square_H V) \otimes (A \square_H W) \arrow[rr, "A\otimes\xi"] \arrow[d, "\zeta\otimes(A\square_H W)"'] & & A \otimes A \square_H (V \otimes W) \arrow[d, "\zeta"] \\
A \otimes V \otimes (A \square_H W) \arrow[d, "\tau^{-1}\otimes(A\square_H W)"'] & & A \otimes V \otimes W \arrow[d, "\tau^{-1}\otimes W"] \\
V \otimes A \otimes (A \square_H W) \arrow[rr, "V\otimes\zeta"] & & V \otimes A \otimes W
\end{tikzcd}$$

commutes:



By faithful flatness of A, this proves that ξ is an isomorphism. ☐

The usual definition of a Hopf module can also be given in the braided setting:

Definition 3.7. A Hopf module $M \in \mathcal{B}_A^H$ for a right H-comodule algebra A is a right H-comodule and right A-module such that the module structure on M is an H-comodule morphism with respect to the codiagonal comodule structure on $M \otimes A$.

We will need the structure theorem for Hopf modules:

Proposition 3.8. *Assume that A is a flat H-Galois object in $\mathcal{B}$. Then for every Hopf module in $\mathcal{B}_A^H$ the morphism*

$$\mu_0 \colon M^{\mathrm{co}\,H} \otimes A \to M$$

is an isomorphism. The inverse is determined by commutativity of

$$\begin{tikzcd}
M \arrow[r, "\mu_0^{-1}"] \arrow[d, "\delta"'] & M^{\mathrm{co}\,H} \otimes A \arrow[dd, "\iota\otimes A"] \\
M \otimes H \arrow[d, "M\otimes\gamma"'] & \\
M \otimes A \otimes A \arrow[r, "\mu\otimes A"] & M \otimes A.
\end{tikzcd}$$

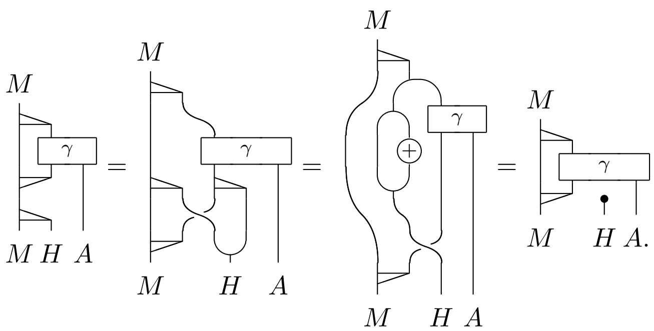
Proof. Let us show first that the morphism

$$F = \left(M \xrightarrow{\delta} M \otimes H \xrightarrow{M\otimes\gamma} M \otimes A \otimes A \xrightarrow{\mu\otimes A} M \otimes A\right)$$

factors through $\iota \otimes A\colon M^{\mathrm{co}\,H} \otimes A \to M \otimes A$. By flatness of A, the latter is the equalizer of

$$\delta \otimes A, M \otimes \eta \otimes A \colon M \otimes A \to M \otimes H \otimes A,$$

so we have to check $(\delta \otimes A)F = (M \otimes \eta \otimes A)F$. This is done in the following calculation:



Next, we need to check that the morphism $F_0\colon M \to M^{\mathrm{co}\,H} \otimes A$ induced by F is the inverse to μ_0. Now

$$\begin{aligned}
\mu_0 F_0 &= \mu F \\
&= \left(M \xrightarrow{\delta} M \otimes H \xrightarrow{M\otimes\gamma} M \otimes A \otimes A \xrightarrow{\mu\otimes A} M \otimes A \xrightarrow{\mu} M\right) \\
&= \left(M \xrightarrow{\delta} M \otimes H \xrightarrow{M\otimes\gamma} M \otimes A \otimes A \xrightarrow{M\otimes\nabla} M \otimes A \xrightarrow{\mu} M\right) \\
&= \left(M \xrightarrow{\delta} M \otimes H \xrightarrow{M\otimes\varepsilon} M \xrightarrow{M\otimes\eta} M \otimes A \xrightarrow{\mu} M\right) \\
&= \mathrm{id}_M \,.
\end{aligned}$$

On the other hand $F = F_M$ is clearly natural in $M \in \mathcal{B}_A^H$, and $F_{V \otimes A} = V \otimes F_A$ for $V \in \mathcal{B}$. We can read (3.3) as saying $F_A = \eta \otimes A$, so that

$$\begin{aligned} F_M \mu_0 = (\mu_0 \otimes A) F_{M^{\mathrm{co}\,H} \otimes A} &= (\mu_0 \otimes A)(M^{\mathrm{co}\,H} \otimes F_A) \\ &= (\mu_0 \otimes A)(M^{\mathrm{co}\,H} \otimes \eta \otimes A) = \iota \otimes A, \end{aligned}$$

where $\iota \colon M^{\mathrm{co}\,H} \to M$. Thus $F_0 \mu_0$ is the identity. □

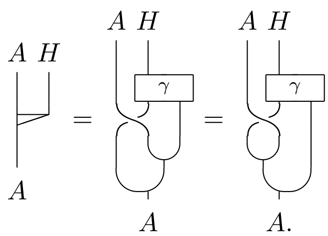
Lemma 3.9. *If A is a faithfully flat H-Galois object, then for each $V \in \mathcal{B}$ the morphism $V \otimes \eta \colon V \to V \otimes A$ induces an isomorphism $f \colon V \to (V \otimes A)^{\mathrm{co}\,H}$.*

Proof. It is easy to check that the composition

$$V \otimes A \xrightarrow{f \otimes A} (V \otimes A)^{\mathrm{co}\,H} \otimes A \xrightarrow{\mu} V \otimes A$$

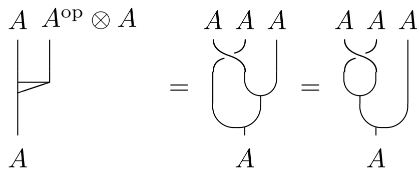
with the isomorphism from Proposition 3.8 is the identity, hence $f \otimes A$ is an isomorphism, and by faithful flatness so is f. □

Corollary 3.10. *Let A be an H-Galois object in $\mathcal{B}$. Then a right H-module algebra structure on A is defined by*

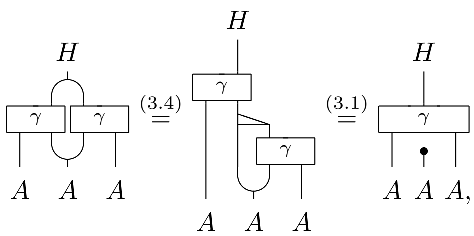


We call this the Miyashita-Ulbrich action of H on A. With this action, A is a Yetter-Drinfeld module over H, and it is a commutative algebra in the category of Yetter-Drinfeld modules over H.

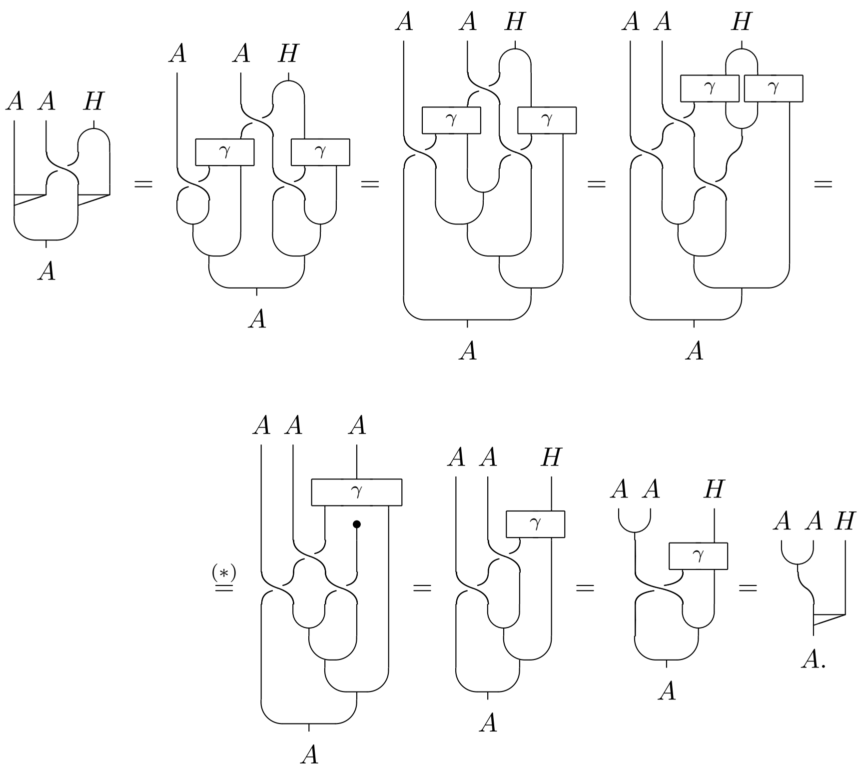
Proof. It should be a standard fact that A is a right $A^{\mathrm{op}} \otimes A$-module by



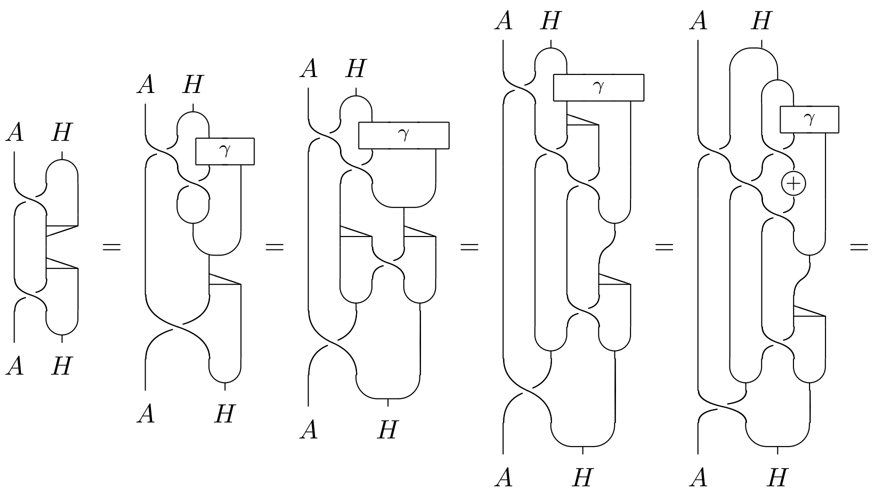
But by (3.6) and (3.7), $\gamma \colon H \to A^{\mathrm{op}} \otimes A$ is an algebra morphism, so A is a right H-module as claimed. To see that it is a module algebra, we first observe

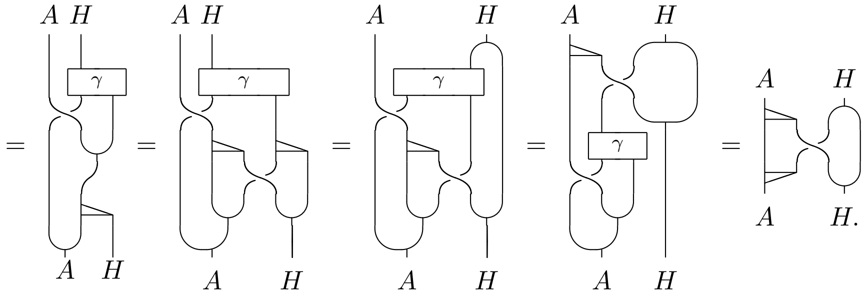


which we use in the following calculation at the step marked $(*)$:

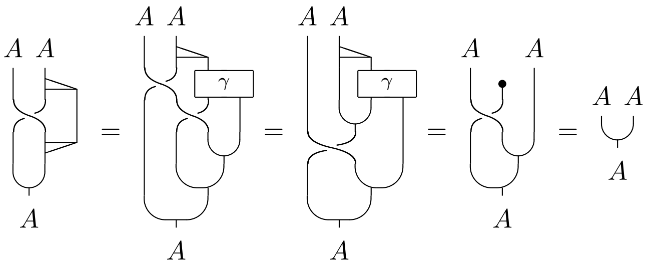


To see that A is a Yetter-Drinfeld module, we compute





Finally the calculation



shows that A is commutative in the category of Yetter-Drinfeld modules. $\square$

So far we have only been dealing with one right H-Galois object A, with comodule structure δ, Galois morphism β, and its partial inverse γ. When working with left Galois objects, defined in the obvious way, we will use the notations δ_ℓ, β_ℓ and γ_ℓ for the left sided analogs of these morphisms. When there is need to emphasize the comodule algebra A in question, we use δ^A, β^A and γ^A, in the left versions δ^A_ℓ etc.

4 Galois objects are bi-Galois objects

Definition 4.1. An L-H-Bi-Galois object A in $\mathcal{B}$ is an algebra A with structures of a left L-Galois object and a right H-Galois object making it an L-H-bicomodule algebra.

We denote by $\mathrm{BiGal}(L, H)$ the set of isomorphism classes of faithfully flat L-H-bi-Galois objects.

Lemma 4.2. *If A is a faithfully flat L-H-bi-Galois object, then the morphism $\overline{\gamma}\colon H \to {}^{\mathrm{co}\,L}(A \otimes A)$ from Remark 3.5 is an isomorphism.*

Proof. The composition

$$A \otimes H \xrightarrow{A\otimes\overline{\gamma}} A \otimes {}^{\mathrm{co}\,L}(A \otimes A) \xrightarrow{\mu} A \otimes A$$

of $A \otimes \overline{\gamma}$ with the left version of the isomorphism μ from Proposition 3.8 is β^{-1}, according to (3.2). In particular, the inverse of $A \otimes \overline{\gamma}$ is given by $\beta\mu$. For later use we note

$$\eta_A \otimes \overline{\gamma}^{-1} = (A \otimes \overline{\gamma})^{-1}(\eta_A \otimes {}^{\mathrm{co}\,L}(A \otimes A)) = \beta\mu(\eta_A \otimes {}^{\mathrm{co}\,L}(A \otimes A)) = \beta\iota \quad (4.1)$$

where $\iota\colon {}^{\mathrm{co}\,L}(A \otimes A) \to A \otimes A$ is the inclusion $\square$

The lemma says that, as an algebra, H is uniquely determined by A as a left L-Galois object. Vice versa, L is determined by A as a right H-Galois object, and the lemma recommends $L := (A \otimes A)^{\mathrm{co}\,H}$ as a candidate for the construction of a left L-Galois structure on a given right H-Galois object A.

Theorem 4.3. *Let A be a right H-Galois object. Assume that A and $L := (A \otimes A)^{\mathrm{co}\,H}$ are flat (e.g. assume that A is faithfully flat).*

Then L is a Hopf algebra, and A is an L-H-bi-Galois object.

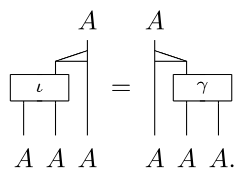
Proof. As a special case of Proposition 2.2, L is a subalgebra of $A \otimes A^{\mathrm{op}}$. From Proposition 3.8 we know that multiplication on the right tensor factors induces an isomorphism $\mu\colon L \otimes A \to A \otimes A$, whose inverse we denote by $\beta_\ell = \mu^{-1}$. By this isomorphism, A faithfully flat implies L flat. Our candidate for a left L-comodule structure is $\delta_\ell = \beta_\ell(A \otimes \eta)$, for which indeed β_ℓ is the corresponding Galois morphism, so that A is L-Galois. Also, β_ℓ and hence δ_ℓ are automatically H-colinear, and A will be an L-H-bicomodule. Let $\iota\colon L \to A \otimes A$ denote the inclusion. From Proposition 3.8 we know that β_ℓ is determined by commutativity of the diagram

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\beta_\ell"] \arrow[d, "\delta_{A \otimes A}"'] & & L \otimes A \arrow[dd, "\iota \otimes A"] \\
A \otimes A \otimes H \arrow[d, "A \otimes A \otimes \gamma"'] & & \\
A \otimes A \otimes A \otimes A \arrow[rr, "A \otimes \nabla \otimes A"] & & A \otimes A \otimes A.
\end{tikzcd}$$

From this it is straightforward to deduce

$$(\iota \otimes A)\delta_\ell = \left(A \xrightarrow{\delta} A \otimes H \xrightarrow{A \otimes \gamma} A \otimes A \otimes A\right).$$

In particular δ_ℓ is an algebra map since δ and γ are, and A, L are flat; pictorially



The bialgebra structure of L will be particularly easy to construct using the following universal property of L:

Proposition 4.4. *Let X be an algebra in $\mathcal{B}$ and $\phi\colon A \to X \otimes A$ an algebra morphism. If X is flat or A is faithfully flat, then there is a unique algebra morphism $f\colon L \to X$ with $\phi = (f \otimes A)\delta_\ell$.*

Proof. Note first that X flat implies $(X \otimes A)^{\mathrm{co}\,H} \cong X$. The same is true if A is faithfully flat, by Lemma 3.9.

We prove uniqueness first: f with the desired properties will make the diagram

$$\begin{tikzcd} A \otimes A \arrow[r, "\beta_\ell"] \arrow[d, "\phi\otimes A"'] & L \otimes A \arrow[d, "f\otimes A"] \\ X \otimes A \otimes A \arrow[r, "X\otimes\nabla"] & X \otimes A \end{tikzcd}$$

commute. This determines f, since the tensor factor A can be cancelled by taking coinvariants.

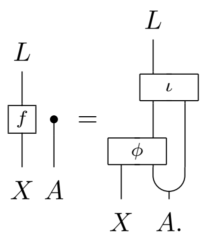
To prove existence, consider the composition

$$g := \left(A \otimes A \xrightarrow{\phi\otimes A} X \otimes A \otimes A \xrightarrow{X\otimes\nabla} X \otimes A \right).$$

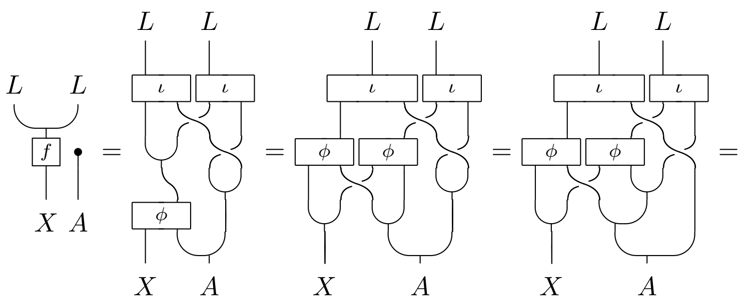
It is a right A-comodule morphism, thus

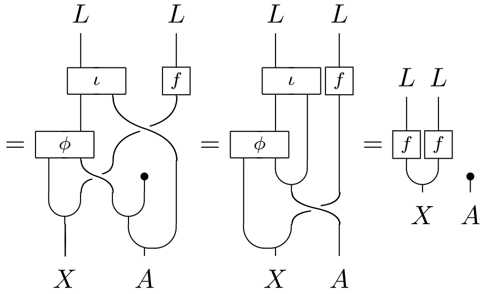
$$f := \left((A \otimes A)^{\mathrm{co}\,H} \xrightarrow{g^{\mathrm{co}\,H}} (X \otimes A)^{\mathrm{co}\,H} \cong X \right)$$

is well-defined. Note that f is determined by

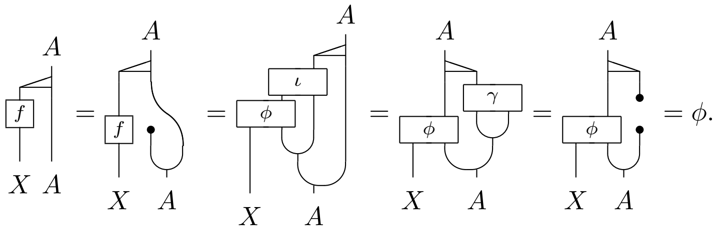


It is multiplicative by the calculation





and we omit checking that it is unital. Finally



□

Now we continue the proof of Theorem 4.3. By Proposition 4.4 there is a unique algebra morphism $\Delta\colon L \to L \otimes L$ with $(\Delta \otimes A)\delta_\ell = (L \otimes \delta_\ell)\delta_\ell$, and a unique algebra morphism $\varepsilon\colon L \to k$ with $(\varepsilon \otimes A)\delta_\ell = \mathrm{id}_A$. Coassociativity can be deduced from Proposition 4.4 as well, by the somewhat standard calculation

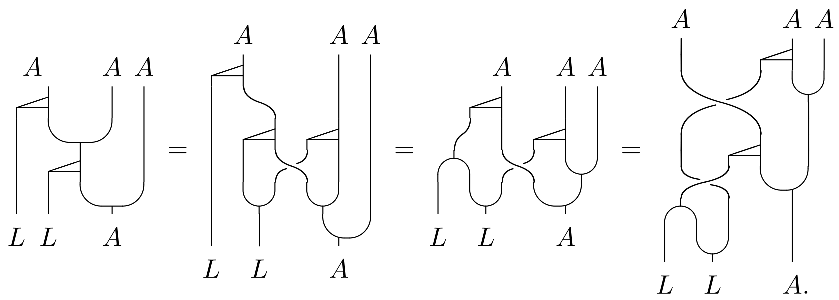
$$\begin{aligned}((\Delta \otimes L)\Delta \otimes A)\delta_\ell &= (\Delta \otimes L \otimes A)(L \otimes \delta_\ell)\delta_\ell = (L \otimes L \otimes \delta_\ell)(\Delta \otimes A)\delta_\ell \\ &= (L \otimes L \otimes \delta_\ell)(L \otimes \delta_\ell)\delta_\ell = (L \otimes \Delta)(L \otimes \delta_\ell)\delta_\ell = ((L \otimes \Delta)\Delta \otimes A)\delta_\ell.\end{aligned}$$

We omit proving the counit property.

To show that L is a Hopf algebra, we vary an unpublished trick of Takeuchi [10] and show that the diagram

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[rr, "\beta_\ell \otimes A"] \arrow[d, "A \otimes \beta_\ell"'] & & L \otimes A \otimes A \arrow[d, "L \otimes \beta_\ell"] \\
A \otimes L \otimes A \arrow[d, "\tau \otimes A"'] & & L \otimes L \otimes A \\
L \otimes A \otimes A \arrow[r, "L \otimes \beta_\ell"'] & L \otimes L \otimes A \arrow[r, "\tau^{-1} \otimes A"'] & L \otimes L \otimes A \arrow[u, "\beta_\ell^L \otimes A"']
\end{tikzcd} \tag{4.2}$$

commutes, by the calculation



It is well-known that β_ℓ^L an isomorphism implies that L is a Hopf algebra with antipode $S_L = (\varepsilon \otimes L)(\beta_\ell^L)^{-1}(L \otimes \eta)$. But by the diagram, $\beta_\ell^L \otimes A$ is an isomorphism, and since $L \otimes L$ is flat, taking coinvariants removes the tensor factor A. $\square$

5 The groupoid of bi-Galois objects

Proposition 5.1. *Let H be a flat Hopf algebra and A a flat right H-Galois object in $\mathcal{B}$. Consider $H \otimes A$ as an H-H-bicomodule with the codiagonal right H-comodule structure, and the left comodule structure induced by the left tensor factor. Put*

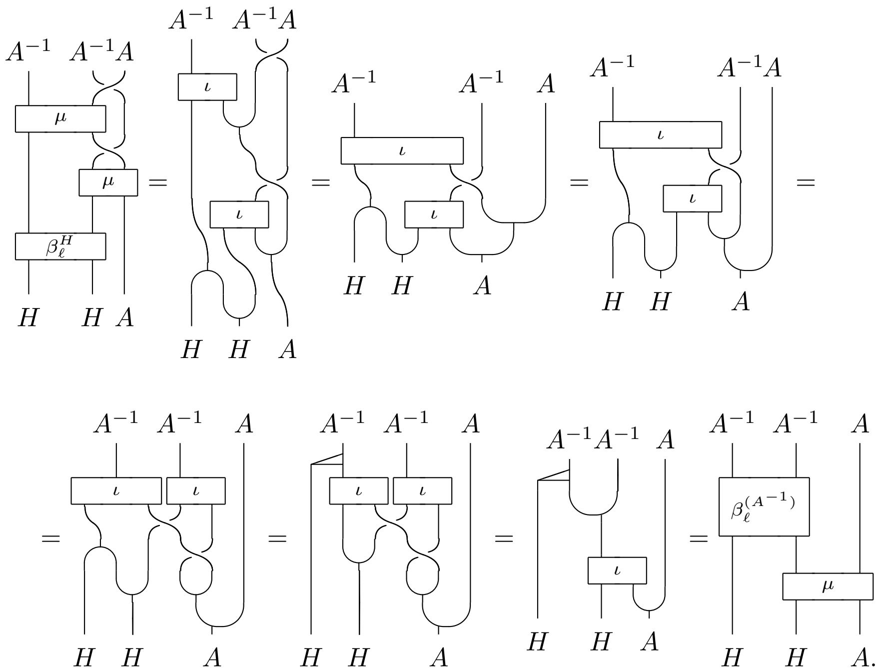
$$A^{-1} := (H \otimes A)^{\mathrm{co}\,H} \subset H \otimes A^{\mathrm{op}},$$

which is a left H-subcomodule and a subalgebra of $H \otimes A^{\mathrm{op}}$. If A is faithfully flat, then so is A^{-1}. If A^{-1} is flat, then it is a left H-Galois object in $\mathcal{B}$.

Proof. A^{-1} is a subalgebra according to Proposition 2.2, and obviously a left H-subcomodule. To show that A^{-1} is left H-Galois, we note first that ${}^{\mathrm{co}\,H}\big(A^{-1}\big) \cong \big({}^{\mathrm{co}\,H}(H \otimes A)\big)^{\mathrm{co}\,H} \cong A^{\mathrm{co}\,H} \cong I$ using flatness of H to switch the order of taking left and right coinvariants. To see that the left Galois morphism $\beta_\ell^{(A^{-1})} \colon A^{-1} \otimes A^{-1} \to H \otimes A^{-1}$ is an isomorphism, we use the isomorphism $\mu \colon A^{-1} \otimes A \to H \otimes A$ from Proposition 3.8 and show that the diagram

$$\begin{tikzcd}
A^{-1} \otimes A^{-1} \otimes A \arrow[r, "\beta_\ell^{(A^{-1})} \otimes A"] \arrow[d, "A \otimes \tau^{-1}"'] & H \otimes A^{-1} \otimes A \arrow[dd, "H \otimes \mu"] \\
A^{-1} \otimes A \otimes A^{-1} \arrow[d, "\mu \otimes A^{-1}"'] & \\
H \otimes A \otimes A^{-1} \arrow[d, "H \otimes \tau"'] & H \otimes H \otimes A \\
H \otimes A^{-1} \otimes A \arrow[r, "H \otimes \mu"] & H \otimes H \otimes A \arrow[u, "\beta_\ell^H \otimes A"']
\end{tikzcd}$$

commutes. This is done by the following calculation:



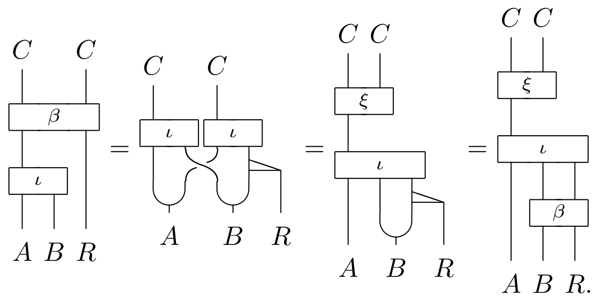
Thus $\beta_\ell^{(A^{-1})} \otimes A$ is an isomorphism. But by flatness of A^{-1} and H, taking coinvariants will remove the tensor factor A. Note also that faithful flatness of A implies that of A^{-1} by the isomorphism $\mu\colon A^{-1} \otimes A \cong H \otimes A$. $\square$

Theorem 5.2. *The flat Hopf algebras in $\mathcal{B}$ are the objects of a groupoid. Morphisms from L to H are the elements of the set* $\mathrm{BiGal}(L, H)$ *of isomorphism classes of faithfully flat L-H-bi-Galois objects. Composition is given by the cotensor product.*

Proof. Let $A \in \mathrm{BiGal}(L, H)$ and $B \in \mathrm{BiGal}(H, R)$ for flat Hopf algebras L, H, R. Clearly $A \,\square_H\, B$ is an L-R-bicomodule algebra. We will prove that it is right R-Galois, by factoring the Galois morphism as

$$\begin{tikzcd}
(A \,\square_H\, B) \otimes (A \,\square_H\, B) \arrow[r, "\beta^{A\square_H B}"] \arrow[d, "\xi"'] & (A \,\square_H\, B) \otimes R \\
A \,\square_H\, (B \otimes B) \arrow[ur, "A\square_H \beta^B"'] &
\end{tikzcd}$$

by the following calculation (in which we write $C = A \,\square_H\, B$):



Similarly $A \,\square_H\, B$ is left L-Galois.

$A \,\square_H\, B$ is faithfully flat by the isomorphism $\zeta\colon A \otimes (A \,\square_H\, B) \cong A \otimes B$ from the proof of Lemma 3.6

Coassociativity follows from the fact that the objects involved are flat and equalizers commute with equalizers; thus we have a category of bi-Galois objects under cotensor product.

Now let $A \in \mathrm{BiGal}(L, H)$, and consider the left H-Galois object $A^{-1} := (H \otimes A)^{\mathrm{co}\,H}$ as in Proposition 5.1. By the left version of Theorem 4.3, A^{-1} is an H-R-bi-Galois object for some Hopf algebra R in $\mathcal{B}$. We have a composed isomorphism

$$j = \left(A \underset{H}{\square} A^{-1} = A \underset{H}{\square} (H \otimes A)^{\mathrm{co}\,H} \cong (A \underset{H}{\square} H \otimes A)^{\mathrm{co}\,H} \cong (A \otimes A)^{\mathrm{co}\,H} \right)$$

which is given by the restriction of $A \otimes \varepsilon \otimes A\colon A \otimes H \otimes A \to A \otimes A$ and hence a left L-colinear algebra map. We can compose this with the inverse of the left version of the isomorphism $\overline{\gamma}$ in Lemma 4.2, to obtain an isomorphism $\alpha\colon A \,\square_H\, A^{-1} \to L$ of left L-comodule algebras.

As a consequence, there is a right R-comodule algebra structure ρ on L making it an L-R-bi-Galois extension. Since ρ is left L-colinear, it has the form $\rho = (L \otimes f)\Delta$ for $f = (\varepsilon \otimes R)\rho\colon L \to R$. Since both ε and ρ are algebra maps, so is f. Moreover, ρ is a right R-comodule map, hence so is f, whence

$$\Delta f = (f \otimes R)\rho = (f \otimes R)(L \otimes f)\Delta = (f \otimes f)\Delta.$$

Also $\varepsilon f = (\varepsilon \otimes \varepsilon)\rho = \varepsilon$, so $f\colon L \to R$ is a bialgebra map. Since L is R-Galois, f is an isomorphism. Thus we can twist the right R-comodule structure of A^{-1} back along f^{-1} to give an H-L-bi-Galois structure on A^{-1}; with this right L-comodule structure, $A \,\square_H\, A^{-1} \cong L$ as bi-Galois objects.

This shows that every bi-Galois object has a right inverse. One may either prove the existence of left inverses in a similar fashion, or rely on the well-known fact that proving the existence of all inverses on one side is sufficient. $\square$

Corollary 5.3. *If all objects of $\mathcal{B}$ are flat, then the Hopf algebras in $\mathcal{B}$ are the objects of a groupoid whose morphisms are all bi-Galois objects.*

Proof. We have to make amends at each step of the proof of Theorem 5.2 that used faithful flatness. The Galois property of $A \,\square_H\, B$ was shown using the isomorphism ξ, which was proved in Lemma 3.6 under the assumption that A is faithfully flat. Without that assumption, the proof of Lemma 3.6 shows that at least $A \otimes \xi$ is an isomorphism. But if all objects are flat, the tensor factor A can be cancelled by taking, say, left L-coinvariants. We certainly do not need faithful flatness of A to show $A \,\square_H\, B$ is flat, if we assume that *all* objects are flat. Also Proposition 5.1 can be applied without faithful flatness of A, since A^{-1} is flat by assumption. $\square$

6 The case of bijective antipodes

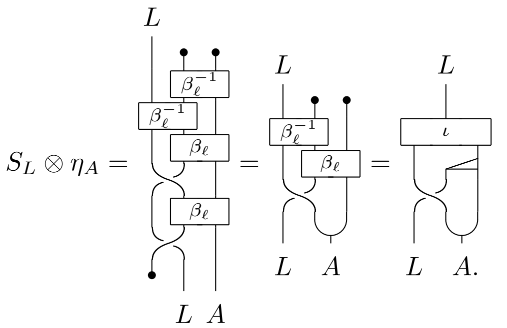
Proposition 6.1. *Let H be a flat Hopf algebra in $\mathcal{B}$, and A a flat right H-Galois object such that $L := (A \otimes A)^{\mathrm{co}\,H}$ is flat.*

If the antipode of H is an isomorphism, then so is the antipode of the Hopf algebra L constructed in Theorem 4.3.

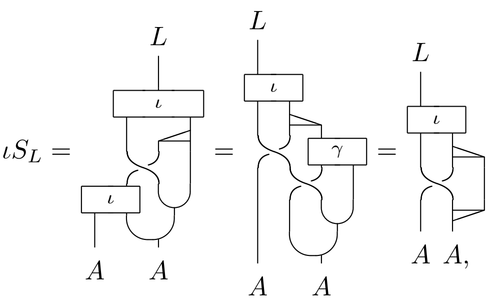
Proof. We return to the proof of Theorem 4.3 and first compute

$$S_L \otimes \eta_A = (\varepsilon \otimes A \otimes A)\left(\left(\beta_\ell^L\right)^{-1} \otimes A\right)(L \otimes \eta_L \otimes \eta_A)$$

by a chase around the diagram (4.2):



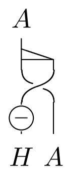
From this we conclude



where the right module structure on A is the Miyashita-Ulbrich action. Thus S_L is induced by the braiding on the Yetter-Drinfeld-module A with the Miyashita-Ulbrich action. Hence S_L is an isomorphism, with inverse induced by the inverse of the braiding; recall that the braiding is indeed invertible since the antipode of H is assumed to be an isomorphism. □

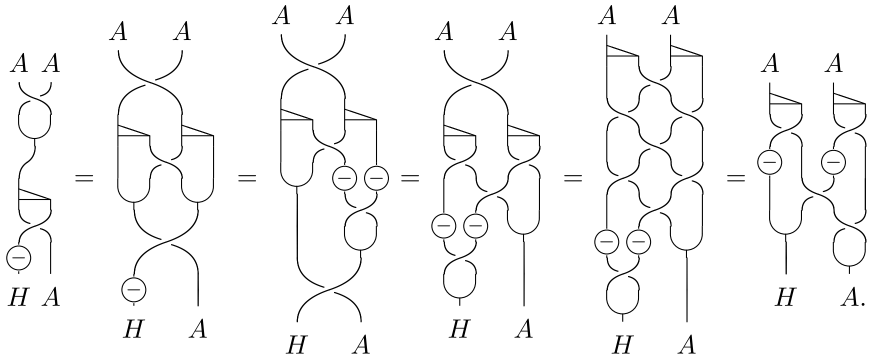
Proposition 6.2. *Let A be a right H-Galois object, and assume the antipode of H is an isomorphism. Then A^{op} with the left coaction*

$$\delta_\ell^{\mathrm{op}} := \delta_\ell^{(A^{\mathrm{op}})} :=$$



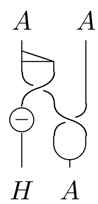
is a left H-Galois object.

Proof. A^{op} is a comodule algebra by

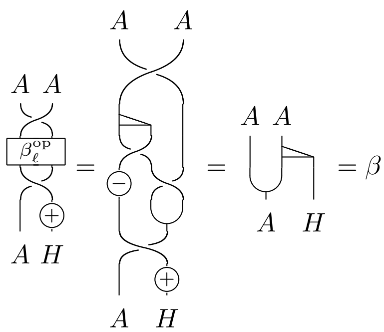


The Galois morphism

$$\beta_\ell^{\mathrm{op}} := \beta_\ell^{(A^{\mathrm{op}})} =$$



for the left H-comodule algebra A^{op} is an isomorphism since



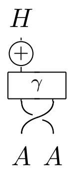
is the Galois morphism for A as a right H-comodule algebra. □

Corollary 6.3 (to the proof). *The morphism*

$$\gamma_\ell^{\mathrm{op}} = (\beta_\ell^{\mathrm{op}})^{-1}(H \otimes \eta)$$

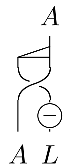
for the left Galois object A^{op} *is given by*

$$\gamma_\ell^{\mathrm{op}} =$$



Corollary 6.4. *Similarly, if* A *is a left* L*-Galois object for a Hopf algebra* L *whose antipode is an isomorphism, then* A^{op} *is a right* L*-Galois object with coaction*

$$\delta^{\mathrm{op}} := \delta^{(A^{\mathrm{op}})} :=$$



Lemma 6.5. *Let* H *be a Hopf algebra whose antipode is an isomorphism, and* A *a right* H*-Galois object. Then*

$$A \xrightarrow{\delta_r} A \otimes H \xrightarrow{\tau^{-1}} H \otimes A \xrightarrow{S^{-1}\otimes A} H \otimes A$$

induces a left H*-comodule algebra isomorphism* $A^{\mathrm{op}} \to A^{-1} = (H \otimes A)^{\mathrm{co}\,H}$*, whose inverse is induced by* $\varepsilon \otimes A\colon H \otimes A \to A$.

Proof. By Lemma 2.4 we have

$$(H \otimes A)^{\mathrm{co}\,H} = H \underset{H}{\square} A^{\mathrm{op}} \cong A^{\mathrm{op}},$$

and the isomorphism of $H \square_H A^{\mathrm{op}}$ with A^{op} obviously has the claimed form. $\square$

Theorem 6.6. *Let* H *be a Hopf algebra with bijective antipode, and* $A \in \mathrm{BiGal}(L, H)$*. Then* $A^{-1} \cong A^{\mathrm{op}}$ *as* H*-*L*-bicomodule algebras, with the left and right comodule structures on* A^{op} *given as in Proposition 6.2 and Corollary 6.4.*

Proof. We prove that A^{op} is a right inverse for A. From Lemma 2.4 we know that $A \square_H A^{\mathrm{op}} = (A \otimes A)^{\mathrm{co}\,H}$ as an L-L-bicomodule subalgebra of $A^{\mathrm{op}} \otimes A$. And by the left comodule versions of Remark 3.5 and Lemma 4.2, we have an isomorphism $\overline{\gamma_\ell}\colon L \to (A \otimes A)^{\mathrm{co}\,H}$. $\square$

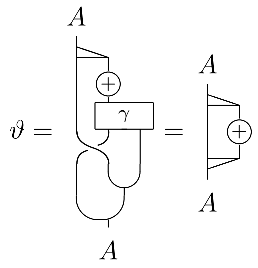
7 Ribbon algebras and the Grunspan morphism

The following terminology was recently introduced by Akrami and Majid [1]

Definition 7.1. A ribbon algebra in $\mathcal{B}$ is an algebra A equipped with an algebra isomorphism $A \cong (A^{\mathrm{op}})^{\mathrm{op}}$.

Corollary 7.2. *Let $A \in \mathrm{BiGal}(L, H)$ for flat Hopf algebras $L, H \in \mathcal{B}$ whose antipodes are isomorphisms.*

Then A is a ribbon algebra. A specific isomorphism $\vartheta \colon A \to (A^{\mathrm{op}})^{\mathrm{op}}$, which we will call the Grunspan morphism of A, can be given by

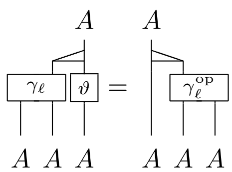


where the right module structure on A is the Miyashita-Ulbrich action.

Proof. As in any groupoid the inverse of the inverse of the isomorphism class of A is again the isomorphism class of A. The explicit form of the isomorphism can be harvested from the standard proof of this fact: We consider the chain of isomorphisms

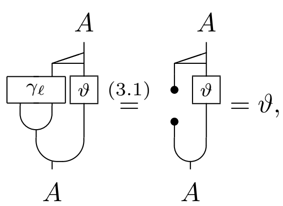
$$\vartheta = \left(A \cong A \underset{H}{\square} H \cong A \underset{H}{\square} A^{-1} \underset{L}{\square} \left(A^{-1}\right)^{-1} \cong L \underset{L}{\square} \left(A^{-1}\right)^{-1} \cong \left(A^{-1}\right)^{-1} \right).$$

Since the isomorphisms $H \cong A^{-1} \square_L \left(A^{-1}\right)^{-1}$ and $L \cong A \square_H A^{-1}$ are induced by $\gamma_\ell^{\mathrm{op}}$ and γ_ℓ, respectively, ϑ satisfies

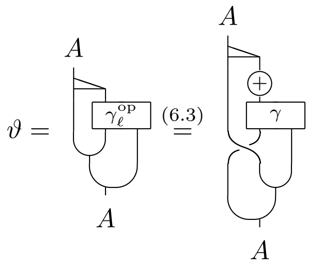


(7.1)

Multiplying the bottom three ends of this equation together, we obtain for the left hand side

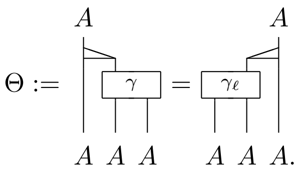


and so



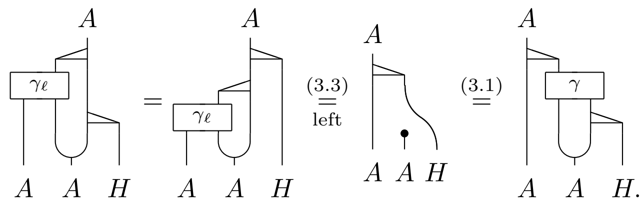
∎

Lemma and Definition 7.3. *Let A be a faithfully flat L-H-bi-Galois object for flat Hopf algebras H,L whose antipodes are isomorphisms. Then*



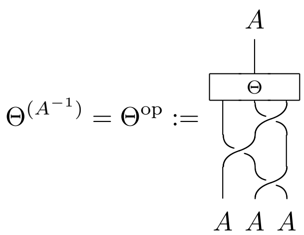
$\Theta\colon A \to A \otimes A^{\mathrm{op}} \otimes A$ is an algebra morphism that we call the torsor structure of A.

Proof. Strictly speaking, the equality occurred already in the proof of Theorem 4.3, where $\iota\colon (A \otimes A)^{\mathrm{co}\,H} \to A \otimes A$ played the role of γ_ℓ. By uniqueness of L this is enough. We will nevertheless give a quick direct proof by applying β to the second and third leg:

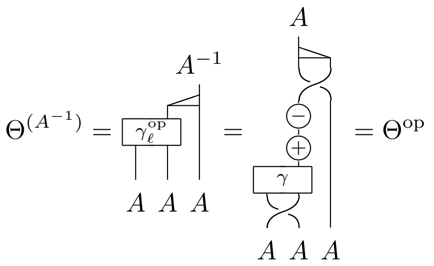


Since δ and γ are algebra maps, so is Θ. ∎

Lemma 7.4. *Let H, L be Hopf algebras whose antipodes are isomorphisms, and $A \in \mathrm{BiGal}(L, H)$. The torsor structure of $A^{\mathrm{op}} \in \mathrm{BiGal}(H, L)$ is*



Proof.

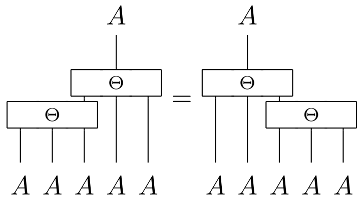


∎

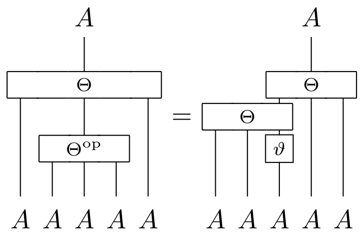
Theorem 7.5. *Let H and L be flat Hopf algebras whose antipodes are isomorphisms.*

The torsor structure Θ and the Grunspan morphism ϑ of $A \in \mathrm{BiGal}(L, H)$ satisfy:

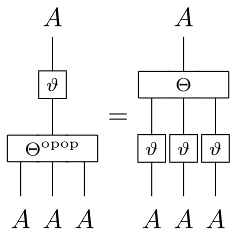
(7.2)



(7.3)

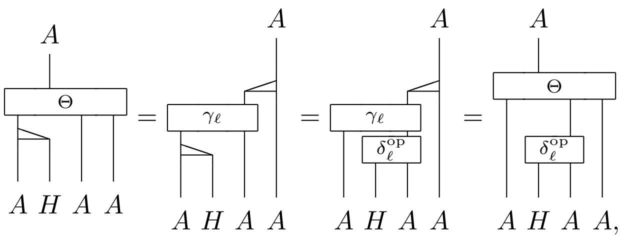


(7.4)

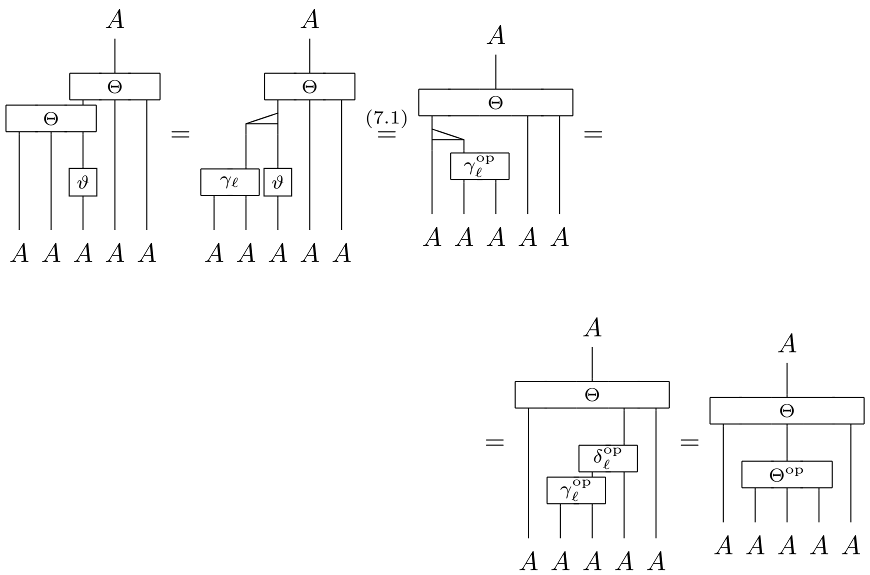


(7.5)

Proof. Note that (7.4) and (7.5) involve a great deal of braiding hidden in the definition of Θ^{op}. We will only prove these two, (7.2) and (7.3) being relatively easy. As for (7.5), this follows from the fact that ϑ is a morphism of bi-Galois objects, and so is compatible with the torsor structures of A and $(A^{-1})^{-1}$. But the latter is Θ^{opop} by Lemma 7.4. For (7.4), we note first that



where the second equality uses that γ_ℓ induces a morphism to $A \,\square_H\, A^{\mathrm{op}}$, and then compute



$\square$

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