ACTIONS OF MONOIDAL CATEGORIES, AND GENERALIZED HOPF SMASH PRODUCTS

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ABSTRACT. Let R be a k-algebra, and $\mathcal{C}$ a monoidal category. Assume given the structure of a $\mathcal{C}$-category on the category ${}_R\mathcal{M}$ of left R-modules; that is, the monoidal category $\mathcal{C}$ is assumed to act on the category ${}_R\mathcal{M}$ by a coherently associative bifunctor $\diamondsuit\colon \mathcal{C} \times {}_R\mathcal{M} \to {}_R\mathcal{M}$. We assume that this bifunctor is right exact in its right argument.

In this setup we show that every algebra A (resp. coalgebra C) in $\mathcal{C}$ gives rise to an R-ring $A\diamondsuit R$ (resp. an R-coring $C\diamondsuit R$) whose modules (resp. comodules) are the A-modules (resp. C-comodules) within the category ${}_R\mathcal{M}$.

We show that this very general scheme for constructing (co)associative (co)rings gives conceptual explanations for the double of a quasi-Hopf algebra as well as certain doubles of Hopf algebras in braided categories, each time avoiding ad hoc computations showing associativity.

1. Introduction

In the theory of Hopf algebras and quantum groups, there is a large variety of constructions of new algebras from old ones by a modification of the tensor product of algebras (and a similar variety of coalgebra constructions). The oldest such constructions are smash products and crossed products (see for example [19]). More recent constructions are double crossproducts, including the Drinfeld double. Since there are interesting generalizations of the notion of a Hopf algebra like quasi-Hopf algebras or Hopf algebras in braided monoidal categories, one is also led to consider generalized crossed products involving these. Needless to say that constructing increasingly complicated versions of crossed products may involve increasingly complicated, unpleasant, and technical computations.

Often a better understanding of a generalized crossed product construction can be gained through a description of the corresponding category of modules. For example, the category of modules over a Hopf smash product $A\#H$ is the category of A-modules within the monoidal category of H-modules; the category of modules over the Drinfeld double $D(H)$ is the category of Yetter-Drinfeld modules over H, or, better yet, the center of the monoidal category of H-modules. A good description of the module category will also help to understand (or construct) additional structures on a generalized crossed product (such as a Hopf algebra or quasi-Hopf algebra structure). Moreover, the idea of Tannaka reconstruction is that, given an abstract category and a functor to the base category of k-modules, one can (under favorable conditions) find a suitable algebra (or coalgebra) such that its representations are closely related to the abstract category; in a sense, it is not an essential problem to find the (co)algebra once we have the category.

2000 *Mathematics Subject Classification.* 16W30,16S40,18D10,18D35.

However, in many situations reconstruction cannot be applied as a mathematical machinery (the conditions may not be sufficiently favorable) but merely as a guiding idea. And even when a reconstruction theorem can be applied, it may be hard to work out the resulting (co)algebra explicitly. Sometimes one may guess a candidate for an algebra (in the case of the Drinfeld double $D(H)$, the modules are supposed to be Yetter-Drinfeld modules, that is H-modules and H-comodules with a compatibility condition, which lets us guess $D(H) = H \otimes H^*$ with a multiplication reflecting the compatibility condition; this guess works just as well in the quasi-Hopf algebra case) but the task remains to show that this candidate is what we are looking for, in particular, that it is an associative algebra at all.

In this paper we will deal with a type of categories where reconstruction is always possible, and, once an example is fitted into our general scheme, can be carried out mechanically without doing any further ad hoc calculations. The machinery will be sufficiently general to allow rather easy approaches to the Drinfeld double of a quasi-Hopf algebra as well as certain versions of the Drinfeld double of a braided Hopf algebra — two cases where explicit calculations are rather unpleasant.

Our setup is the following: Assume we have a k-algebra R, an abstract monoidal category $\mathcal{C}$, and an algebra A in $\mathcal{C}$. To construct a crossed product of A and R, all we need is an action of the category $\mathcal{C}$ on the category ${}_R\mathcal{M}$ of left R-modules. By such an action we mean a functor $\diamondsuit\colon \mathcal{C} \times {}_R\mathcal{M} \to {}_R\mathcal{M}$ and an associativity isomorphism $\Psi\colon (X \otimes Y)\diamondsuit V \to X\diamondsuit(Y\diamondsuit V)$ for $X, Y \in \mathcal{C}$ and $V \in {}_R\mathcal{M}$ which is coherent with the associativity constraint given within $\mathcal{C}$. In short, we need a $\mathcal{C}$-category structure on ${}_R\mathcal{M}$ as defined by Pareigis [20]. We need the additional assumption that $X\diamondsuit -$ is a right exact functor for each $X \in \mathcal{C}$ (actually, we will treat an even more categorical setup below, in which R is allowed to be an algebra in another abstract monoidal category $\mathcal{B}$, and additional technical conditions are needed). Then $A\diamondsuit R$ is an algebra (actually even an R-ring), and its modules are the A-modules within the category ${}_R\mathcal{M}$. Dually, if C is a coalgebra in $\mathcal{C}$, then $C\diamondsuit R$ is an R-coring (which, incidentally, has a dual algebra), and its comodules are the C-comodules within ${}_R\mathcal{M}$.

We should quickly acknowledge that it is far from evident how to come up with monoidal categories acting on ${}_R\mathcal{M}$. For several of our examples it will be at least as hard to provide $\diamondsuit$ and Ψ as to prove associativity of multiplication in $A\diamondsuit R$ ad hoc. In these cases our results merely provide a unifying additional viewpoint on known constructions, and an alternative interpretation of the data necessary for those constructions. However, there will also be situations in which actions of a category $\mathcal{C}$ will arise naturally — for example, if T is an algebra in $\mathcal{C}$, then $\mathcal{C}_T$ is a left $\mathcal{C}$-category, and every category equivalent to a $\mathcal{C}$-category is itself a $\mathcal{C}$-category. In such cases our machinery will be at its best, furnishing associative algebras from conceptual reasons without much calculations.

Due to the categorical nature of our techniques, the paper will have a somewhat strange structure. It will take considerable time to line up all the basic notions of monoidal categories, $\mathcal{C}$-categories, algebras and coalgebras in monoidal categories, and (co)modules over them. However, all the facts we list in Section 2 are quite trivial and intuitively obvious, given that the intricate definitions are by now well-established and understood. An exception in this long list of preliminaries are perhaps Sections 2.4 and 2.5: They fix the details on corings and their duals within monoidal categories. All the facts here are well-known for ordinary corings, and not at all surprising for corings over algebras R in a monoidal category in which tensor products are right exact. However, this last assumption rules out the opposite of the category of k-modules over a commutative ring k. Algebras in this category are of course k-coalgebras, and so this example is certainly relevant. We report how to work around the assumption that tensor products are right exact, though largely without giving complete proofs.

After these tedious preparations, the centerpiece of the paper is quite short: The main Theorem 3.3 and its two main corollaries are hardly more than direct consequences of the preliminaries.

In Section 4 we then explain how to obtain explicit algebras from the main results — giving both general recipes and a number of examples. This will take up the largest part of the paper.

2. Preliminaries and generalities

2.1. **Monoidal categories.** A monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, \Phi)$ is a category equipped with a bifunctor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an isomorphism $\Phi = \Phi_{XYZ} \colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, natural in $X, Y, Z \in \mathcal{C}$; the isomorphism Φ has to satisfy Mac Lane's "pentagon" equation

$$(W \otimes \Phi_{XYZ})\Phi_{W,X\otimes Y,Z}(\Phi_{WXY} \otimes Z) = \Phi_{W,X,Y\otimes Z}\Phi_{W\otimes X,Y,Z},$$

where both sides are morphisms $((W\otimes X)\otimes Y)\otimes Z \to W\otimes(X\otimes(Y\otimes Z))$; moreover, we require the existence of a unit object $I \in \mathcal{C}$ with $X\otimes I = X = I\otimes X$, functorially in $X \in \mathcal{C}$, satisfying $\Phi_{XIY} = \mathrm{id}_{XY}$ (which also implies $\Phi_{XYI} = \Phi_{IXY} = \mathrm{id}_{XY}$) for all $X, Y \in \mathcal{C}$. A monoidal category is strict if the isomorphisms Φ are identities.

With any (strict) monoidal category $\mathcal{C}$ we have the (strict) monoidal categories $\mathcal{C}^{\mathrm{op}}$ (the opposite category in which arrows are reversed) and $\mathcal{C}^{\mathrm{sym}}$, the category in which tensor factors are switched in forming the tensor product. We also note that the cartesian product of (strict) monoidal categories is (strict) monoidal.

Let $\mathcal{C}, \mathcal{D}$ be monoidal categories. An incoherent tensor functor $\mathcal{F} = (\mathcal{F}, \xi) \colon \mathcal{C} \to \mathcal{D}$ is a functor equipped with an isomorphism $\xi = \xi_{XY} : \mathcal{F}(X)\otimes\mathcal{F}(Y) \to \mathcal{F}(X\otimes Y)$, natural in $X, Y \in \mathcal{C}$ and such that $\xi_{XI} = \xi_{IX} = \mathrm{id}_{\mathcal{F}(X)}$ for all $X \in \mathcal{C}$. An incoherent tensor functor is strict if all the isomorphisms ξ are identities. An incoherent tensor functor is coherent, or simply a monoidal functor, if it satisfies

$$\xi_{X,Y\otimes Z}(\mathcal{F}(X) \otimes \xi_{YZ})\mathcal{F}(\Phi_{XYZ}) = \Phi_{\mathcal{F}(X),\mathcal{F}(Y),\mathcal{F}(Z)}\xi_{X\otimes Y,Z}(\xi_{XY} \otimes \mathcal{F}(Z))$$

for all $X, Y, Z \in \mathcal{C}$, where both sides are morphisms $(\mathcal{F}(X) \otimes \mathcal{F}(Y)) \otimes \mathcal{F}(Z) \to \mathcal{F}(X \otimes (Y \otimes Z))$.

We note that incoherent tensor functors $(\mathcal{F}, \xi) \colon \mathcal{C} \to \mathcal{C}'$ and $(\mathcal{F}', \xi') \colon \mathcal{C}' \to \mathcal{C}''$ can be composed by the rule $(\mathcal{F}', \xi')(\mathcal{F}, \xi) = (\mathcal{F}'\mathcal{F}, \mathcal{F}'(\xi)(\xi(\mathcal{F}\times\mathcal{F})))$ i.e. the tensor functor structure of $\mathcal{F}'\mathcal{F}$ is given by

$$\mathcal{F}'\mathcal{F}(X) \otimes \mathcal{F}'\mathcal{F}(Y) \xrightarrow{\xi'} \mathcal{F}'(\mathcal{F}(X) \otimes \mathcal{F}(Y)) \xrightarrow{\mathcal{F}'(\xi)} \mathcal{F}'\mathcal{F}(X \otimes Y).$$

If both $\mathcal{F}$ and $\mathcal{F}'$ are coherent, then so is $\mathcal{F}'\mathcal{F}$.

Let $(\mathcal{F},\xi),(\mathcal{F}',\xi')\colon \mathcal{C}\to\mathcal{C}'$ be two monoidal functors. A natural transformation $\varphi\colon \mathcal{F}\to\mathcal{F}'$ is monoidal if

$$\begin{tikzcd} \mathcal{F}(X)\otimes\mathcal{F}(Y) \arrow[r, "\xi"] \arrow[d, "\varphi\otimes\varphi"'] & \mathcal{F}(X\otimes Y) \arrow[d, "\varphi"] \\ \mathcal{F}'(X)\otimes\mathcal{F}'(Y) \arrow[r, "\xi'"] & \mathcal{F}'(X\otimes Y) \end{tikzcd}$$

commutes for all $X,Y\in\mathcal{C}$.

A monoidal functor $(\mathcal{F},\xi)\colon \mathcal{C}\to\mathcal{C}'$ is a monoidal equivalence if there exist a monoidal functor $(\mathcal{G},\zeta)\colon\mathcal{C}'\to\mathcal{C}$ and monoidal isomorphisms $\mathcal{GF}\cong Id$ and $\mathcal{FG}\cong Id$. If $(\mathcal{F},\xi)$ is a monoidal functor, and $\mathcal{F}$ is a category equivalence, then it is a monoidal equivalence. More precisely, if $\mathcal{F}$ is part of an adjoint equivalence with quasi-inverse functor $\mathcal{G}$ and adjunction (iso)morphisms $u\colon Id_{\mathcal{C}}\to\mathcal{GF}$ and $v\colon\mathcal{FG}\to Id_{\mathcal{C}'}$, then there is a unique monoidal functor structure on $\mathcal{G}$ making u and v monoidal transformations, namely

$$\mathcal{G}(X)\otimes\mathcal{G}(Y)\xrightarrow{u}\mathcal{GF}(\mathcal{G}(X)\otimes\mathcal{G}(Y))\xrightarrow{\mathcal{G}(\xi^{-1})}\mathcal{G}(\mathcal{FG}(X)\otimes\mathcal{FG}(Y))\xrightarrow{\mathcal{G}(v\otimes v)}\mathcal{G}(X\otimes Y).$$

The main examples for monoidal categories relevant to this paper arise from Hopf algebra theory and its generalizations: The category ${}_H\mathcal{M}$ of modules over a k-bialgebra H is a monoidal category with the tensor product over k of H-modules, equipped with the diagonal module structure. In this case, the associativity isomorphisms $(U\otimes V)\otimes W\cong U\otimes(V\otimes W)$ coincide with the canonical isomorphisms for k-modules. More generally, if (H,ϕ) is a quasibialgebra as defined by Drinfeld [6], then ${}_H\mathcal{M}$ is still a monoidal category with the tensor product of H-modules formed over k and equipped with the diagonal module structure, but now the associativity isomorphism is given as left multiplication with $\phi\in H\otimes H\otimes H$. Since we will never need the axioms for a quasibialgebra or quasi-Hopf algebra explicitly, we will not recall them here. The formally dual notion of a coquasibialgebra (H,ϕ) involves a convolution invertible trilinear form $\phi\colon H^{\otimes 3}\to k$ and ${}^H\mathcal{M}$ is a monoidal category, with the tensor product of H-comodules taken again over k, and the associativity isomorphism given as multiplication by ϕ in the obvious sense. Coquasibialgebras were first considered by Majid [15].

Whenever we work out examples involving coalgebras or comodules, we will use the following variations of Sweedler's notation: $c\mapsto c_{(1)}\otimes c_{(2)}$ for the comultiplication of a coalgebra, $v\mapsto v_{(-1)}\otimes v_{(0)}$ for a left comodule structure, $v\mapsto v_{(0)}\otimes v_{(1)}$ for a right comodule structure. We will denote the opposite algebra of an algebra A by A^{op} and the coopposite coalgebra of a coalgebra C by C^{cop}.

2.2. **$\mathcal{C}$-categories.** Assume that $\mathcal{C}$ is a monoidal category. An incoherent left $\mathcal{C}$-category $\mathcal{D}=(\mathcal{D},\diamondsuit,\Psi)$ is a category equipped with a bifunctor $\diamondsuit\colon\mathcal{C}\times\mathcal{D}\to\mathcal{D}$ and an isomorphism $\Psi=\Psi_{XYV}\colon(X\otimes Y)\diamondsuit V\to X\diamondsuit(Y\diamondsuit V)$, natural in $X,Y\in\mathcal{C}$ and $V\in\mathcal{D}$, such that $I\otimes V=V$ functorially in $V\in\mathcal{D}$, and $\Psi_{IXV}=\Psi_{XIV}=\mathrm{id}$. An incoherent left $\mathcal{C}$-category is coherent, or simply a $\mathcal{C}$-category, if

$$(X\diamondsuit\Psi_{YZV})\Psi_{X,Y\otimes Z,V}(\Phi_{XYZ}\diamondsuit V)=\Psi_{X,Y,Z\diamondsuit V}\Psi_{X\otimes Y,Z,V}$$

holds for all $X,Y,Z\in\mathcal{C}$ and $V\in\mathcal{D}$. Given two incoherent $\mathcal{C}$-categories $(\mathcal{D},\diamondsuit,\Psi)$ and $(\mathcal{D}',\diamondsuit',\Psi')$, an incoherent $\mathcal{C}$-functor $(\mathcal{F},\zeta)\colon\mathcal{D}\to\mathcal{D}'$ consists of a functor $\mathcal{F}\colon\mathcal{D}\to\mathcal{D}'$ and an isomorphism $\zeta=\zeta_{XV}\colon X\diamondsuit'\mathcal{F}(V)\to\mathcal{F}(X\diamondsuit V)$ natural in

$X \in \mathcal{C}$ and $V \in \mathcal{D}$ and satisfying $\zeta_{IV} = \mathrm{id}$. It is said to be coherent, or simply a $\mathcal{C}$-functor, if

$$\zeta_{X,Y\diamondsuit V}(X\diamondsuit'\zeta_{YV})\Psi'_{X,Y,\mathcal{F}(V)} = \mathcal{F}(\Psi_{XYV})\zeta_{X\otimes Y,V}$$

for all $X, Y \in \mathcal{C}$ and $V \in \mathcal{D}$.

A natural transformation $\varphi\colon \mathcal{F} \to \mathcal{F}'$ between $\mathcal{C}$-functors $(\mathcal{F}, \zeta), (\mathcal{F}', \zeta')\colon \mathcal{D} \to \mathcal{D}'$ is a $\mathcal{C}$-natural transformation if $\varphi_{X\diamondsuit V}\zeta_{XV} = \zeta'_{XV}(X\diamondsuit'\varphi_V)$ holds for all $X \in \mathcal{C}$ and $V \in \mathcal{D}$.

If $\mathcal{C}$ is a monoidal category, then it is naturally a left $\mathcal{C}$-category, and a right $\mathcal{C}$-category, and, combining the two, a left $\mathcal{C} \times \mathcal{C}^{\mathrm{sym}}$-category.

If k is a commutative ring, and R is a k-algebra, then the category $\mathcal{M}_R$ of right R-modules is a (left) $\mathcal{M}_k$-category; this example generalizes to modules over algebras in general monoidal categories, see the next subsection. It also generalizes as follows: Assume k is a field, then any k-linear additive category is a $\mathcal{M}_k^f$-category for the category $\mathcal{M}_k^f$ of finite dimensional k-vector spaces [24, Lem. 2.2.2]; any k-linear additive functor between examples of this type is a $\mathcal{M}_k^f$-functor. More generally, if k is a commutative ring, then any k-linear additive category in which idempotents split is a $\mathcal{M}_k^f$-category for the category $\mathcal{M}_k^f$ of finitely generated projective k-modules [22, Appendix], and any k-linear additive functor between such $\mathcal{M}_k^f$-categories is a $\mathcal{M}_k^f$-functor. With similar ideas, one can show that any k-linear category $\mathcal{D}$ having colimits is a $\mathcal{M}_k$-category: For a k-module X choose a presentation $k^{(J)} \to k^{(I)} \to X \to 0$ and define $X\diamondsuit V$ to be a corresponding cokernel $V^{(J)} \to V^{(I)} \to X\diamondsuit V \to 0$. Any k-linear colimit-preserving functor between such $\mathcal{M}_k$-categories is a $\mathcal{M}_k$-functor. We shall pretend below that such a $\mathcal{M}_k$-category is strict, and the same for the relevant $\mathcal{M}_k$-functors.

Note that for any category $\mathcal{D}$ the category $\mathfrak{Fun}(\mathcal{D}, \mathcal{D})$ is a strict monoidal category with composition of functors as tensor product (we sloppily set aside set-theoretical scruples coming from the problem that functorial morphisms between two given functors may not form a set). Giving the structure $(\diamondsuit, \Psi)$ of an incoherent $\mathcal{C}$-category for $\mathcal{D}$ is the same as giving an incoherent tensor functor $(\mathcal{L}, \tilde{\Psi})\colon \mathcal{C} \to \mathfrak{Fun}(\mathcal{D}, \mathcal{D})$; a bijection between the two sets of data is established by the equation $X\diamondsuit V = \mathcal{L}(X)(V)$ and by identifying

$$\left(\tilde{\Psi}_{XY}\right)_V : (\mathcal{L}(X) \circ \mathcal{L}(Y))(V) \to \mathcal{L}(X \otimes Y)(V)$$

with

$$\Psi^{-1}_{XYV}\colon X\diamondsuit(Y\diamondsuit V) \to (X \otimes Y)\diamondsuit V.$$

It is easily checked that $(\diamondsuit, \Psi)$ is coherent if and only if $(\mathcal{L}, \tilde{\Psi})$ is.

As an application let $(\mathcal{F}, \xi)\colon \mathcal{C} \to \mathcal{C}'$ be an incoherent tensor functor, and $(\mathcal{D}, \diamondsuit', \Psi')$ an incoherent left $\mathcal{C}'$-category. Then $(\mathcal{D}, \diamondsuit, \Psi)$ is an incoherent left $\mathcal{C}$-category by the composition

$$\mathcal{C} \xrightarrow{(\mathcal{F},\xi)} \mathcal{C}' \xrightarrow{(\mathcal{L},\tilde{\Psi}')} \mathfrak{Fun}(\mathcal{D}, \mathcal{D})$$

of $\mathcal{F}$ with the incoherent tensor functor $\mathcal{L}$ corresponding to the $\mathcal{C}'$-category structure of $\mathcal{D}$. Explicitly this means that $X\diamondsuit V = \mathcal{F}(X)\diamondsuit' V$ for $X \in \mathcal{C}$ and $V \in \mathcal{D}$, and Ψ

is characterized by commutativity of

$$\begin{tikzcd}
(X\otimes Y)\Diamond V \arrow[rr, "\Psi'_{XYV}"] \arrow[d, equal] & & X\Diamond(Y\Diamond V) \arrow[d, equal] \\
\mathcal{F}(X\otimes Y)\Diamond' V \arrow[r, "\xi^{-1}\Diamond' V"] & (\mathcal{F}(X)\otimes\mathcal{F}(Y))\Diamond' V \arrow[r, "\Psi'"] & \mathcal{F}(X)\Diamond'(\mathcal{F}(Y)\Diamond' V).
\end{tikzcd}$$

Note that if both $(\mathcal{D},\Diamond',\Psi')$ and $(\mathcal{F},\xi)$ are coherent, it follows that so is $(\mathcal{D},\Diamond,\Psi)$.

Let $(\mathcal{L},\tilde{\Psi})$ and $(\mathcal{L}',\tilde{\Psi}')$ be the monoidal functors associated to two $\mathcal{C}$-category structures $(\Diamond,\Psi)$ and $(\Diamond',\Psi')$ on a category $\mathcal{D}$. Then isomorphisms $\varphi\colon\mathcal{L}\to\mathcal{L}'$ of monoidal functors are in bijection with isomorphisms ζ making the identity on $\mathcal{D}$ into a $\mathcal{C}$-functor $(Id,\zeta)\colon(\mathcal{D},\Diamond',\Psi')\to(\mathcal{D},\Diamond,\Psi)$; the bijection is given by $\zeta_{XV}=(\varphi_V)_X$.

2.3. **Algebras and modules.** Let $\mathcal{C}$ be a monoidal category. An algebra $A=(A,\nabla)$ in $\mathcal{C}$ consists of an object A of $\mathcal{C}$ and a morphism $\nabla\colon A\otimes A\to A$ which is associative in the sense that

$$\nabla(\nabla\otimes A)=\nabla(A\otimes\nabla)\Phi\colon(A\otimes A)\otimes A\to A$$

and admits a unit morphism $\eta\colon I\to A$ with $\nabla(\eta\otimes A)=\mathrm{id}_A=\nabla(A\otimes\eta)\colon A$.

A coalgebra $C=(C,\Delta)$ in $\mathcal{C}$ is by definition an algebra in the opposite category $\mathcal{C}^{\mathrm{op}}$, that is, an object C with a morphism $\Delta\colon C\to C\otimes C$ such that

$$(C\otimes\Delta)\Delta=\Phi(\Delta\otimes C)\Delta\colon C\to C\otimes(C\otimes C)$$

and there exists a counit morphism $\varepsilon\colon C\to I$ with $(\varepsilon\otimes C)\Delta=\mathrm{id}_C=(C\otimes\varepsilon)\Delta$.

(Co)algebras in the monoidal categories of (co)modules over a bialgebra H are standard objects in Hopf algebra theory; for example an algebra in ${}_H\mathcal{M}$ is called a module algebra, an algebra in $\mathcal{M}^H$ is called a comodule algebra.

Monoidal functors transport algebras and coalgebras: If $(\mathcal{F},\xi)\colon\mathcal{C}\to\mathcal{C}'$ is a monoidal functor, then

(1) If (A,∇) is an algebra in $\mathcal{C}$, then $\mathcal{F}(A)$ is an algebra in $\mathcal{C}'$ with multiplication $\mathcal{F}(A)\otimes\mathcal{F}(A)\xrightarrow{\xi}\mathcal{F}(A\otimes A)\xrightarrow{\mathcal{F}(\nabla)}\mathcal{F}(A)$ and unit $I=\mathcal{F}(I)\xrightarrow{\mathcal{F}(\eta)}\mathcal{F}(A)$.
(2) If (C,Δ) is a coalgebra in $\mathcal{C}$, then $\mathcal{F}(C)$ is a coalgebra in $\mathcal{C}'$ with comultiplication $\mathcal{F}(C)\xrightarrow{\mathcal{F}(\Delta)}\mathcal{F}(C\otimes C)\xrightarrow{\xi^{-1}}\mathcal{F}(C)\otimes\mathcal{F}(C)$ and counit $\mathcal{F}(C)\xrightarrow{\mathcal{F}(\varepsilon)}\mathcal{F}(I)=I$.

Of course, coherence of the monoidal functor is only needed to prove (co)associativity of the transported structures; an incoherent tensor functor will also transport non(co)associative (co)algebras by the same formulas.

Let $\mathcal{D}=(\mathcal{D},\Diamond,\Psi)$ be a left $\mathcal{C}$-category.

An A-module (M,μ) in $\mathcal{D}$ over an algebra A in $\mathcal{C}$ is an object $M\in\mathcal{D}$ with a morphism $\mu\colon A\Diamond M\to M$ in $\mathcal{D}$ which is associative in the sense that

$$\mu(\nabla\Diamond M)=\mu(A\Diamond\mu)\Psi_{AAM}\colon(A\otimes A)\Diamond M\to M,$$

and satisfies $\mu(\eta\Diamond M)=\mathrm{id}_M$. We denote the category of A-modules in $\mathcal{D}$ by ${}_A\mathcal{D}$.

A C-comodule (M,δ) in $\mathcal{D}$ over a coalgebra C in $\mathcal{C}$ is an object $M\in\mathcal{D}$ with a morphism $\delta\colon M\to C\Diamond M$ in $\mathcal{D}$ which is coassociative in the sense that

$$(C\Diamond\delta)\delta=\Psi_{CCM}(\Delta\Diamond M)\delta\colon M\to C\Diamond(C\Diamond M),$$

and satisfies $(\varepsilon\diamondsuit M)\delta = \mathrm{id}_M$. In short, a comodule over C in $\mathcal{D}$ is the same as a module in $\mathcal{D}^{\mathrm{op}}$ over the algebra C in $\mathcal{C}^{\mathrm{op}}$. We denote the category of C-comodules in $\mathcal{D}$ by ${}^C\mathcal{D}$.

$\mathcal{C}$-functors transport modules and comodules: Let $(\mathcal{F},\zeta)\colon \mathcal{D} \to \mathcal{D}'$ be a $\mathcal{C}$-functor between $\mathcal{C}$-categories.

(1) If A is an algebra in $\mathcal{C}$, then $(\mathcal{F},\zeta)$ induces a functor ${}_A\mathcal{D} \to {}_A\mathcal{D}'$ which maps $M = (M,\mu) \in {}_A\mathcal{D}$ to $\mathcal{F}(M)$ with the module structure
$$A\diamondsuit\mathcal{F}(M) \xrightarrow{\zeta} \mathcal{F}(A\diamondsuit M) \xrightarrow{\mathcal{F}(\mu)} \mathcal{F}(M).$$
(2) If C is a coalgebra in $\mathcal{C}$, then $(\mathcal{F},\zeta)$ induces a functor ${}^C\mathcal{D} \to {}^C\mathcal{D}'$ which maps $M = (M,\delta) \in {}^C\mathcal{D}$ to $\mathcal{F}(M)$ with the comodule structure
$$\mathcal{F}(M) \xrightarrow{\mathcal{F}(\delta)} \mathcal{F}(C\diamondsuit M) \xrightarrow{\zeta^{-1}} C\diamondsuit\mathcal{F}(M).$$

In particular, if $\mathcal{D}$ and $\mathcal{D}'$ are equivalent as $\mathcal{C}$-categories, then ${}_A\mathcal{D} \cong {}_A\mathcal{D}'$ for any algebra A in $\mathcal{C}$, and similarly for comodule categories.

We also note the following trivial 'transformation' of modules under functors: Let $(\mathcal{D},\diamondsuit',\Psi')$ be a $\mathcal{C}'$-category, and let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}'$ be a monoidal functor, leading to an induced $\mathcal{C}$-category structure on $\mathcal{D}$. If A is an algebra in $\mathcal{C}$, then we have an equality of categories ${}_A\mathcal{D} = {}_{\mathcal{F}(A)}\mathcal{D}$ for any algebra A in $\mathcal{C}$: An A-module structure on an object $V \in \mathcal{D}$ is by definition a morphism $\mathcal{F}(A)\diamondsuit V \to V$, just like a $\mathcal{F}(A)$-module, and it is straightforward to check that the respective associativity requirements are also the same.

Now let $\mathcal{C}$ be a monoidal category, A an algebra in $\mathcal{C}$, $M \in \mathcal{C}_A$ and $N \in {}_A\mathcal{D}$, where $\mathcal{D}$ is a left $\mathcal{C}$-category. The tensor product $M\diamondsuit_A N \in \mathcal{D}$ is defined to be a coequalizer
$$(M\otimes A)\diamondsuit N \underset{(M\diamondsuit\mu)\Psi}{\overset{\mu\diamondsuit N}{\rightrightarrows}} M\diamondsuit N \longrightarrow M\diamondsuit_A N,$$
if this exists.

2.4. **Bimodules, R-rings and -corings.** Throughout the subsection, we maintain the following general hypothesis: $\mathcal{B}$ is a strict monoidal category, whose neutral object we denote by b. We assume that $\mathcal{B}$ has coequalizers.

If $R \in \mathcal{B}$ is an algebra, then there are natural notions of left and right R-modules in $\mathcal{B}$, as well as of R-S-bimodules whenever S is another algebra in $\mathcal{B}$. However, not all is so nice as one may hope: For example, if $M \in {}_R\mathcal{B}$ and $N \in \mathcal{B}_T$ for algebras R and T, then $M \otimes N \in {}_R\mathcal{B}_T$ in the obvious way. But if $M \in {}_R\mathcal{B}_S$ and $N \in {}_S\mathcal{B}_T$, then we need not have $M \otimes_S N \in {}_R\mathcal{B}_T$ as a quotient of $M \otimes N$. To define the left module structure $R \otimes (M \otimes_S N) \to M \otimes_S N$ we need the additional assumption that $R \otimes (M \otimes_S N)$ is itself a coequalizer of two morphisms $R \otimes M \otimes S \otimes N \to R \otimes M \otimes N$, so that we can make use of its universal property. If we assume generally that tensor product in $\mathcal{B}$ preserves coequalizers, all is well, in particular ${}_R\mathcal{B}_R$ is again a monoidal category, and ${}_R\mathcal{B}$ is a left ${}_R\mathcal{B}_R$-category. However, while this general assumption is well-suited to the case where $\mathcal{B}$ is a category of (bi-)modules over some ring, it rules out the case where $\mathcal{B}$ is opposite to a module category. This is of course the case when the theory of modules in $\mathcal{B}$ is the theory of comodules over a k-coalgebra, and tensor product over an algebra in base is cotensor product over a k-coalgebra, which is in fact known to pose problems when some of the k-modules involved are not flat. Most of this section deals with notions and facts that allow us to carry over well known or obvious results on ordinary (co)rings to such a situation.

Definition 2.1. An object $V \in \mathcal{B}$ is right coflat if the functor $V \otimes (-) \colon \mathcal{B} \to \mathcal{B}$ preserves coequalizers. If an object is both left and right coflat, we shall say it is coflat.

Let R be a right coflat algebra in $\mathcal{B}$. Then for any algebra S in $\mathcal{B}$, $M \in {}_R\mathcal{B}_S$ and $N \in {}_S\mathcal{B}$ we do have $M \otimes_R N \in {}_R\mathcal{B}$. In particular, if R is coflat then there is a natural tensor product functor ${}_R\mathcal{B}_R \times {}_R\mathcal{B}_R \to {}_R\mathcal{B}_R$. However, assuming R coflat still does not guarantee that ${}_R\mathcal{B}_R$ is monoidal, since we do not have any obvious morphisms between $M \otimes_R (N \otimes_R P)$ and $(M \otimes_R N) \otimes_R P$ for $M, N, P \in {}_R\mathcal{B}_R$. After this warning (and the following remark) we shall set out to make proper amendments for this failure, following Pareigis [21], though with slightly altered terminology.

Remark 2.2. Let R be a right coflat algebra in $\mathcal{B}$ and $M \in \mathcal{B}_R$ a right module. The following are equivalent:

(1) M is right coflat in $\mathcal{B}$.
(2) The functor $M \otimes_R (-) \colon {}_R\mathcal{B} \to \mathcal{B}$ preserves coequalizers.

In fact (2) follows from (1) since colimits commute with colimits, and (1) follows from (2) since for $V \in \mathcal{B}$ we have $M \otimes V \cong M \otimes_R (R \otimes V)$.

Definition 2.3. Let R be an algebra in $\mathcal{B}$. A right R-module $M \in \mathcal{B}_R$ is (right) robust, if for all $T \in {}_R\mathcal{B}$ and $V \in \mathcal{B}$ the canonical morphism

$$M \underset{R}{\otimes} (T \otimes V) \to (M \underset{R}{\otimes} T) \otimes V$$

induced by the identity on $M \otimes T \otimes V$ is an isomorphism.

We denote the category of robust right R-modules that are also right coflat by $\mathcal{B}^{!}_R$. When S is another algebra in $\mathcal{B}$, we denote the category of S-R-bimodules that are right robust (as R-modules) and right coflat by ${}_S\mathcal{B}^{!}_R$.

Robust modules were introduced by Pareigis [21, Def.1.1], in whose terminology a robust R-module is "R-coflat over b". While this condition *is* a kind of opposite to the ordinary notion of flatness (the latter says that tensor product over R preserves equalizers, the former says that the coequalizer defining tensor product over R is preserved by other tensor products) we shall avoid using the term coflat in this sense; it might prove unfortunate in the context of coalgebra theory where one may run into coflat comodules at any time. Our use of the word "coflat" above seems to be less problematic in the context of comodule theory.

Note that for $M \in \mathcal{B}_R$, the functor $M \otimes_R (-) \colon {}_R\mathcal{B} \to \mathcal{B}$ is a quotient of the strict right $\mathcal{B}$-functor $M \otimes (-) \colon {}_R\mathcal{B} \to \mathcal{B}$. Definition 2.3 can be read as saying that $M \otimes_R (-)$ is also a right $\mathcal{B}$-functor, in such a way that the epimorphism is a strict $\mathcal{B}$-epimorphism. In particular, if S is any algebra in $\mathcal{B}$, and $N \in {}_R\mathcal{B}_S$, then $M \otimes_R N \in \mathcal{B}_S$ if M is robust.

Lemma 2.4 ([21], Lemma 1.4)**.** *Let R be a right coflat algebra in $\mathcal{B}$, and $M \in \mathcal{B}^{!}_R$.*

Then for all algebras S in $\mathcal{B}$, all $T \in {}_R\mathcal{B}_S$, and $V \in {}_S\mathcal{B}$, the identity on $M \otimes T \otimes V$ induces an isomorphism

$$M \underset{R}{\otimes} (T \underset{S}{\otimes} V) \to (M \underset{R}{\otimes} T) \underset{S}{\otimes} V.$$

Sketch of proof: The morphism $M \otimes_R (T \otimes_S V) \to (M \otimes_R T) \otimes_S V$ is well-defined since M is right coflat, while its inverse is well-defined because M is robust.

Lemma 2.5 ([21], Lemma 1.3)**.** *Let R, S be right coflat algebras in $\mathcal{B}$, $M \in \mathcal{B}^!_R$, $N \in {}_R\mathcal{B}^!_S$. Then $M \otimes_R N \in \mathcal{B}^!_S$.*

Sketch of proof:

$$\begin{aligned}
(M \underset{R}{\otimes} T) \underset{S}{\otimes} (V \otimes X) &\cong M \underset{R}{\otimes} (T \underset{S}{\otimes} (V \otimes X)) && \text{Lemma 2.4}\\
&\cong M \underset{R}{\otimes} ((T \underset{S}{\otimes} V) \otimes X) && T \text{ is robust}\\
&\cong (M \underset{R}{\otimes} (T \underset{S}{\otimes} V)) \otimes X && M \text{ is robust}\\
&\cong ((M \underset{R}{\otimes} T) \underset{S}{\otimes} V) \otimes X && \text{Lemma 2.4}
\end{aligned}$$

Corollary 2.6. *Let R be a right coflat algebra in $\mathcal{B}$. The category ${}_R\mathcal{B}^!_R$ is monoidal with tensor product $\otimes_R$ and associator induced by the (identity) associator in $\mathcal{B}$, and ${}_R\mathcal{B}$ is a left ${}_R\mathcal{B}^!_R$-category in the same way.*

Remark 2.7. Contrary to the facts, we will pretend that the associativity isomorphisms in ${}_R\mathcal{B}^!_R$, as well as between ${}_R\mathcal{B}^!_R$ and ${}_R\mathcal{B}$, are identities.

Definition 2.8. Let $\mathcal{B}$ be a strict monoidal category, and R a right coflat algebra in $\mathcal{B}$.

An R-ring Γ in $\mathcal{B}$ is an algebra in ${}_R\mathcal{B}^!_R$. A left Γ-module is a Γ-module in the left ${}_R\mathcal{B}^!_R$-category ${}_R\mathcal{B}$.

An R-coring Λ in $\mathcal{B}$ is a coalgebra in ${}_R\mathcal{B}^!_R$. A left Λ-comodule is a Λ-comodule in the left ${}_R\mathcal{B}^!_R$-category ${}_R\mathcal{B}$.

Obviously an R-ring Γ is the same as an algebra Γ with an algebra morphism $R \to \Gamma$, but with the additional hypothesis that Γ is right robust and coflat. The category of left Γ-modules in the sense of the definition above identifies with the category ${}_\Gamma\mathcal{B}$ of modules over the algebra Γ. Although a coring Λ is not a coalgebra, we will write ${}^\Lambda\mathcal{B} := {}^\Lambda({}_R\mathcal{B})$ for the category of its left comodules.

Let R be a right coflat algebra in $\mathcal{B}$, and let $V \in \mathcal{B}$ be right coflat. The right module $V \otimes R \in \mathcal{B}_R$ is coflat and robust. Any right R-module morphism $\mu_\ell \colon S \otimes V \otimes R \to V \otimes R$ has the form

$$\mu_\ell = (S \otimes V \otimes R \xrightarrow{\sigma \otimes R} V \otimes R \otimes R \xrightarrow{V \otimes \nabla} V \otimes R)$$

with

$$\sigma := (S \otimes V \xrightarrow{S \otimes V \otimes \eta} S \otimes V \otimes R \xrightarrow{\mu_\ell} V \otimes R).$$

If S is an algebra, Tambara [32] has discussed the conditions on σ necessary and sufficient to make μ_ℓ an S-module structure, namely $\sigma(\eta \otimes V) = V \otimes \eta$ and $\sigma(\nabla \otimes V) = (V \otimes \nabla)(\sigma \otimes S)(S \otimes \sigma)$ (Tambara treats only the case where $\mathcal{B}$ is the category of modules over a commutative ring, but the general case should be considered a trivial generalization). Assume that $S = R$. Any right module morphism

$$\nabla \colon (V \otimes R) \underset{R}{\otimes} (V \otimes R) = V \otimes V \otimes R \to V \otimes R$$

has the form

$$\nabla = (V \otimes V \otimes R \xrightarrow{\chi \otimes R} V \otimes R \otimes R \xrightarrow{V \otimes \nabla} V \otimes R)$$

with

$$\chi = (V \otimes V \xrightarrow{V \otimes V \otimes \eta} V \otimes V \otimes R \xrightarrow{\nabla} V \otimes R).$$

We shall not work out the conditions necessary on χ to ensure that $V \otimes R$ is an R-ring with multiplication ∇. When it is, we will refer to it as the generalized crossed product $V \#_{\sigma,\chi} R$. We have seen that it is the most general form of an R-ring structure on $V \otimes R$ with the given right R-module structure. Similarly, any right R-module morphism

$$\Delta \colon V \otimes R \to (V \otimes R) \underset{R}{\otimes} (V \otimes R) = V \otimes V \otimes R$$

has the form

$$\Delta = (V \otimes R \xrightarrow{\vartheta \otimes R} V \otimes V \otimes R \otimes R \xrightarrow{V \otimes V \otimes \nabla} V \otimes V \otimes R)$$

with

$$\vartheta = (V \xrightarrow{V \otimes \eta} V \otimes R \xrightarrow{\Delta} V \otimes V \otimes R).$$

If $V \otimes R$ is a coring in this way, we will refer to it as the crossed coproduct $V \#^{\vartheta}_{\sigma} R$; it is the most general form of an R-coring structure on $V \otimes R$ with the given right R-module structure.

2.5. **Duality.** Let us first fix conventions about dual objects in monoidal categories. For simplicity we restrict ourselves to the strict case. So let X be an object in the strict monoidal category $\mathcal{C}$. A left dual for X is a triple $(X^*, \mathrm{ev}, \mathrm{db})$ in which X^* is another object of $\mathcal{C}$, and $\mathrm{ev} \colon X^* \otimes X \to I$ and $\mathrm{db} \colon I \to X \otimes X^*$ are morphisms in $\mathcal{C}$ satisfying

$$\left(X \xrightarrow{\mathrm{db} \otimes X} X \otimes X^* \otimes X \xrightarrow{X \otimes \mathrm{ev}} X\right) = \mathrm{id},$$

$$\left(X^* \xrightarrow{X^* \otimes \mathrm{db}} X^* \otimes X \otimes X^* \xrightarrow{\mathrm{ev} \otimes X^*} X^*\right) = \mathrm{id}\,.$$

We will call an object (left) rigid if it has a left dual, denote by $\mathcal{C}_0$ the full subcategory of $\mathcal{C}$ of all left rigid objects, and call $\mathcal{C}$ rigid if $\mathcal{C} = \mathcal{C}_0$.

The dual of a rigid object is unique up to isomorphism, and the choice of a dual object for each $X \in \mathcal{C}_0$ can be extended canonically to a functor $(-)^*$ that is a duality between $\mathcal{C}_0$ and $(\mathcal{C}^{\mathrm{op}})_0$. More generally, let $X, Y \in \mathcal{C}$ have left duals X^* and Y^*. Then for any objects $V, W \in \mathcal{C}$ we have a bijection

$$\mathcal{C}(V \otimes X, Y \otimes W) \ni f \mapsto f^\flat \in \mathcal{C}(Y^* \otimes V, W \otimes X^*)$$

characterized by

$$(\mathrm{ev} \otimes W)(Y^* \otimes f) = (W \otimes \mathrm{ev})(f^\flat \otimes X) \colon Y^* \otimes V \otimes X \to W.$$

Following Joyal and Street [12] we call $f^\flat$ the mate of f. The dual $f^* \colon Y^* \to X^*$ of a morphism $f \colon X \to Y$ occurs as the special case $V = W = I$.

If $X, Y \in \mathcal{C}$ have duals X^* and Y^*, respectively, then $Y^* \otimes X^*$ is a dual of $X \otimes Y$. This shows that $\mathcal{C}_0 \subset \mathcal{C}$ is a monoidal subcategory. If we have chosen a duality functor $(-)^*$, then uniqueness of left duals provides a natural isomorphism $(X \otimes Y)^* \cong Y^* \otimes X^*$ for $X, Y \in \mathcal{C}_0$. In this way duality is a monoidal equivalence $\mathcal{C}_0 \cong (\mathcal{C}^{\mathrm{op}})_0^{\mathrm{sym}}$.

In particular, if C is a coalgebra in $\mathcal{C}_0$, then C^* has a natural algebra structure, corresponding to the coalgebra structure of C; we will at times apply this for noncoassociative coalgebras as well. We note that if $\mathcal{C}$ is the category of k-modules, and C is a finitely generated projective coalgebra, then the algebra structure on C^* thus described is opposite to the "usual" dual algebra of a coalgebra.

Now let $\mathcal{D}$ be a left $\mathcal{C}$-category, for simplicity strict, and C a coalgebra in $\mathcal{C}_0$. Then the bijection

$$\mathcal{D}(V, C\Diamond V) \ni \lambda \mapsto (\mathrm{ev}\Diamond V)(C^*\Diamond\lambda) \in \mathcal{D}(C^*\Diamond V, V)$$

induces a bijection between left C-comodule structures and left C^*-module structures on $V \in \mathcal{D}$.

We will now carry these facts on duality for coalgebras in monoidal categories over to corings.

If R is an ordinary ring, and Γ is an R-coring, then $\Gamma^\vee := \mathrm{Hom}_{-R}(\Gamma, R)$ is an R-ring, and in particular a ring, with the multiplication defined by $(\lambda\lambda')(\gamma) = \lambda(\lambda'(\gamma_{(1)})\gamma_{(2)})$. (Note that $\Gamma^\vee = (\Gamma^*)^{\mathrm{op}}$ if R is commutative and Γ is an R-coalgebra.) There is a natural functor ${}^\Gamma\mathcal{M} \to {}_{\Gamma^\vee}\mathcal{M}$, since every Γ-comodule V is a $\Gamma^\vee$-module by $\lambda \cdot v = \lambda(v_{(-1)})v_{(0)}$ for $\lambda \in \Gamma^\vee$ and $v \in V$. If Γ is finitely generated projective as right R-module, the functor is an equivalence.

These, to be sure, are the first facts one is bound to think about when defining corings and their comodules. In the case where Γ is finitely generated projective as right R-module, thus has a left dual $\Gamma^\vee$ in the category of R-R-bimodules, the results are also special cases of the more general duality results in monoidal categories above. The results in the rest of this subsection do little more than establish these same facts with the ring R replaced by an algebra R in some monoidal category (but the general categorical version may not be available because bimodules may not form a monoidal category).

Definition 2.9. Let R be a right coflat algebra in $\mathcal{B}$. A module $M \in \mathcal{B}^!_R$ is dualizable if there is a module $M^\vee \in {}_R\mathcal{B}$, right coflat in $\mathcal{B}$, together with morphisms $\mathrm{ev}\colon M^\vee \otimes M \to R$ in ${}_R\mathcal{B}_R$ and $\mathrm{db}\colon \mathsf{b} \to M \otimes_R M^\vee$ in $\mathcal{B}$ such that the morphisms

$$M \xrightarrow{\mathrm{db}\otimes M} M \underset{R}{\otimes} M^\vee \otimes M \xrightarrow{M\otimes_R \mathrm{ev}} M$$

(in which the middle term needs no parentheses since M is robust), and

$$M^\vee \xrightarrow{M^\vee\otimes \mathrm{db}} M^\vee \otimes M \underset{R}{\otimes} M^\vee \xrightarrow{\mathrm{ev}\otimes_R M^\vee} M^\vee$$

(in which the middle term needs no parentheses since $M^\vee$ is right coflat) are identity morphisms. The triple $(M^\vee, \mathrm{ev}, \mathrm{db})$ is called a dual of M.

Lemma 2.10. *Let $M \in \mathcal{B}^!_R$ be dualizable with dual $M^\vee$.*

(1) *The functor $M^\vee \otimes (-)\colon \mathcal{B} \to {}_R\mathcal{B}$ is left adjoint to $M \otimes_R (-)\colon {}_R\mathcal{B} \to \mathcal{B}$ with adjunction morphisms*

$$\mathrm{db}\otimes B\colon B \to M \underset{R}{\otimes} M^\vee \otimes B$$

for $B \in \mathcal{B}$ (note M is robust), and

$$\mathrm{ev} \underset{R}{\otimes} V\colon M^\vee \otimes M \underset{R}{\otimes} V \to V$$

for $V \in {}_R\mathcal{B}$ (note that $M^\vee$ is right coflat).

(2) *The adjoint functors in (1) are strict right $\mathcal{B}$-functors, and the adjunction morphisms are $\mathcal{B}$-transformations, from which it is easy to deduce that for any algebra S in $\mathcal{B}$ we obtain a pair of adjoint functors $M^\vee\otimes(-)\colon \mathcal{B}_S \to {}_R\mathcal{B}_S$ and $M \otimes_R (-)\colon {}_R\mathcal{B}_S \to \mathcal{B}_S$.*

(3) *The functor* $-\otimes M$ *from the category of right coflat objects in* $\mathcal{B}$ *to* $\mathcal{B}^!_R$ *is left adjoint to the functor* $-\otimes_R M^\vee$ *from* $\mathcal{B}^!_R$ *to the category of right coflat objects in* $\mathcal{B}$*, with adjunction morphisms*

$$B\otimes \mathrm{db}\colon B\to B\otimes M\underset{R}{\otimes} M^\vee$$

for $B\in\mathcal{B}$ *right coflat, and*

$$W\underset{R}{\otimes}\mathrm{ev}\colon W\underset{R}{\otimes} M^\vee\otimes M\to W$$

for $W\in\mathcal{B}^!_R$.

(4) *In particular, if* $N\in\mathcal{B}^!_R$ *is also dualizable with dual* $N^\vee$*, then we obtain bijections*

$${}_R\mathcal{B}(M^\vee\otimes B, W\underset{R}{\otimes} N^\vee)\cong\mathcal{B}(B, M\underset{R}{\otimes} W\underset{R}{\otimes} N^\vee)\cong\mathcal{B}_R(B\otimes N, M\underset{R}{\otimes} W)$$

for all right coflat $B\in\mathcal{B}$ *and* $W\in{}_R\mathcal{B}_R$ *which is right robust.*

As a consequence, we can dualize the structure of a coring M in ${}_R\mathcal{B}^!_R$ when M is dualizable as right R-module. We shall fall short of showing that $M^\vee$ is an R-ring in this situation, since $M^\vee$ does not seem to have to be right robust.

To begin with let $M\in{}_S\mathcal{B}^!_R$ for right coflat algebras R and S, and assume that M is dualizable as right module. Through the bijection

$$\mathcal{B}(M^\vee\otimes S, M^\vee)\cong\mathcal{B}(S\otimes M, M)$$

the left S-module structure of M induces a right S-module structure on $M^\vee$.

Now let $\Gamma\in{}_R\mathcal{B}^!_R$ be a coring, and assume Γ is dualizable as right R-module. Through the chain of inclusions and bijections

$$\begin{aligned}\mathcal{B}_R(\Gamma,\Gamma\underset{R}{\otimes}\Gamma)&\cong{}_R\mathcal{B}_R(\Gamma^\vee\otimes\Gamma,\Gamma)\\ &\subset\mathcal{B}_R(\Gamma^\vee\otimes\Gamma,\Gamma)\cong\mathcal{B}(\Gamma^\vee,\Gamma\underset{R}{\otimes}\Gamma^\vee)\cong{}_R\mathcal{B}(\Gamma^\vee\otimes\Gamma^\vee,\Gamma^\vee)\end{aligned}$$

the comultiplication $\Gamma\to\Gamma\otimes_R\Gamma$ induces a multiplication $\Gamma^\vee\otimes\Gamma^\vee\to\Gamma^\vee$ on its dual. We omit checking that $\Gamma^\vee$ is an algebra in $\mathcal{B}$ in this way. The counit of Γ composes with multiplication to give a right R-module morphism $R\otimes\Gamma\to R$, which is sent by the bijection

$$\mathcal{B}_R(R\otimes\Gamma, R)\cong\mathcal{B}(R, R\underset{R}{\otimes}\Gamma^\vee)=\mathcal{B}(R,\Gamma^\vee)$$

to an algebra morphism $R\to\Gamma^\vee$. If $V\in{}_R\mathcal{B}$, then the bijection

$$\mathcal{B}(V,\Gamma\underset{R}{\otimes} V)\cong{}_R\mathcal{B}(\Gamma^\vee\otimes V, V)$$

induces a bijection between left Γ-comodule structures on V and left $\Gamma^\vee$-module structures inducing the given R-module structure.

Note once again that dualizing a dualizable R-coring does not yield an R-ring in the sense of our definition above, since the dual might not end up being in ${}_R\mathcal{B}^!_R$.

We will now turn to the special case of (bi)modules or (co)rings that are free as right R-modules. So let R be a right coflat algebra in $\mathcal{B}$.

Let $V\in\mathcal{B}$ be a rigid object with dual V^*. Then $V\otimes R\in\mathcal{B}^!_R$ is dualizable with $(V\otimes R)^\vee = R\otimes V^*$,

$$\mathrm{db}=(b\xrightarrow{\mathrm{db}} V\otimes V^*\xrightarrow{V\otimes\eta\otimes V^*} V\otimes R\otimes V^* = (V\otimes R)\underset{R}{\otimes}(R\otimes V^*)$$

$$\mathrm{ev} = ((R \otimes V^*) \otimes (V \otimes R) \xrightarrow{R \otimes \mathrm{ev} \otimes R} R \otimes R \xrightarrow{\nabla} R)$$

Assume further $V \otimes R$ is an S-R-bimodule with left module structure μ_ℓ, and let $\sigma \colon S \otimes V \to V \otimes R$ be the corresponding morphism. Let $\sigma^\flat \colon V^* \otimes S \to R \otimes V^*$ denote the "mate" of σ under the bijection

$$\mathcal{B}(S \otimes V, V \otimes \mathcal{B}) \cong \mathcal{B}(V^* \otimes S, R \otimes V^*).$$

Then the right module structure μ_r of $(V \otimes R)^\vee$ is given by

$$R \otimes V^* \otimes S \xrightarrow{R \otimes \sigma^\flat} R \otimes R \otimes V^* \xrightarrow{\nabla \otimes V^*} R \otimes V^*.$$

(For $\mathcal{B} = \mathcal{M}_k$, this description of the dual bimodule is due to Tambara [32]).

Now assume further that ($S = R$ and) $V \otimes R$ is an R-coring. Write

$$\vartheta := (V \xrightarrow{V \otimes \eta} V \otimes R \xrightarrow{\Delta} (V \otimes R) \underset{R}{\otimes} (V \otimes R) = V \otimes V \otimes R)$$

and write $\vartheta^\flat \colon V^* \otimes V^* \to R \otimes V^*$ for the mate of ϑ under the bijection

$$\mathcal{B}(V, V \otimes V \otimes R) \cong \mathcal{B}(V^* \otimes V^*, R \otimes V^*).$$

Then multiplication in the dual algebra $(V \otimes R)^\vee = R \otimes V^*$ is given by

$$R \otimes V^* \otimes R \otimes V^* \xrightarrow{\mu_r \otimes V^*} R \otimes V^* \otimes V^* \xrightarrow{R \otimes \vartheta^\flat} R \otimes R \otimes V^* \xrightarrow{\nabla \otimes V^*} R \otimes V^*.$$

3. The general constructions

We begin by recalling a (simplified version of a) result of Pareigis [20, Thm. 4.2] generalizing a theorem of Watts [34]. To reformulate it according to our needs we define, for right $\mathcal{B}$-categories $\mathcal{C}, \mathcal{D}$ having coequalizers, $\mathfrak{Fun}^c_{\mathcal{B}}(\mathcal{C}, \mathcal{D})$ to be the category whose objects are coequalizer-preserving strict $\mathcal{B}$-functors from $\mathcal{C}$ to $\mathcal{D}$, and whose morphisms are $\mathcal{B}$-natural transformations.

Theorem 3.1. *Let R, S be coflat algebras in $\mathcal{B}$. The functor*

$$\mathcal{T} \colon {}_S\mathcal{B}^!_R \ni M \mapsto M \underset{R}{\otimes} (-) \in \mathfrak{Fun}^c_{\mathcal{B}}({}_R\mathcal{B}, {}_S\mathcal{B})$$

is a category equivalence. Its quasi-inverse equivalence maps a functor $F \colon {}_R\mathcal{B} \to {}_S\mathcal{B}$ to $\mathcal{T}^{-1}(F) := F(R)$, with the right R-module structure

$$F(R) \otimes R = F(R \otimes R) \xrightarrow{F(\nabla)} F(R).$$

As a morsel of the proof we note that for $F \in \mathfrak{Fun}^c_{\mathcal{B}}({}_R\mathcal{B}, {}_S\mathcal{B})$ the morphism

$$F(R) \otimes V = F(R \otimes V) \xrightarrow{F(\mu)} F(V)$$

for a left R-module V with module structure μ induces, by the assumption that F preserves the coequalizer defining the tensor product $R \otimes_R V = V$, the relevant isomorphism $v \colon F(R) \otimes_R V \cong F(V)$ which can serve as the counit of an adjoint equivalence in which $\mathcal{T}$ is left adjoint to its quasi-inverse $\mathcal{T}$. The unit of this adjoint equivalence is the identity $M \otimes_R R = M$ for $M \in {}_S\mathcal{B}_R$.

Now if $\mathcal{D}$ is a $\mathcal{B}$-category with coequalizers, then the category $\mathfrak{Fun}^c_{\mathcal{B}}(\mathcal{D}, \mathcal{D})$ is a strict monoidal category with composition as tensor product. For any coflat algebra R in $\mathcal{B}$ the functor

$$\mathcal{T} \colon {}_R\mathcal{B}^!_R \ni M \mapsto M \underset{R}{\otimes} (-) \in \mathfrak{Fun}^c_{\mathcal{B}}({}_R\mathcal{B}, {}_R\mathcal{B})$$

is a strict monoidal functor, and an equivalence by the above. It follows that the inverse equivalence is also a monoidal functor. Explicitly, the monoidal functor structure of

$$\mathcal{T}^{-1}\colon \mathfrak{Fun}^c_{\mathcal{B}}({}_R\mathcal{B}, {}_R\mathcal{B}) \ni F \mapsto F(R) \in {}_R\mathcal{B}^!_R$$

is the instance $v\colon F(R) \otimes_R G(R) \to F(G(R))$ of the counit of adjunction.

Definition 3.2. Let $\mathcal{D}$ be a strict right $\mathcal{B}$-category. and let $\mathcal{C}$ be a monoidal category. We say that a left $\mathcal{C}$-category structure $(\diamondsuit, \Psi)$ on $\mathcal{D}$ is compatible with its right $\mathcal{B}$-category structure, if for each $X \in \mathcal{C}$ the functor $X \diamondsuit (-)\colon \mathcal{D} \to \mathcal{D}$ is a strict right $\mathcal{B}$-functor, and Ψ is a $\mathcal{B}$-natural transformation.

For example, the ${}_R\mathcal{B}^!_R$-category structure of ${}_R\mathcal{B}$ is compatible with its right $\mathcal{B}$-category structure.

Let R be a right coflat algebra in $\mathcal{B}$, and $\mathcal{C}$ a monoidal category. Let $(\diamondsuit, \Psi)$ give ${}_R\mathcal{B}$ the structure of a left $\mathcal{C}$-category, compatible with its canonical right $\mathcal{B}$-category structure. For such a situation we can collect our key results below without much further work. We start by noting that, for any algebra S in $\mathcal{B}$, the category ${}_R\mathcal{B}_S$ is also a left $\mathcal{C}$-category, such that the underlying functor ${}_R\mathcal{B}_S : {}_R\mathcal{B}$ is a strict $\mathcal{C}$-functor.

Theorem 3.3. *Let $\mathcal{B}$ be a strict monoidal category and R a right and left coflat algebra in $\mathcal{B}$. Let $\mathcal{C}$ be a monoidal category, and let $(\diamondsuit, \Psi)$ be the structure of a left $\mathcal{C}$-category on ${}_R\mathcal{B}$, compatible with its right $\mathcal{B}$-category structure. Assume that the functors $X \diamondsuit -\colon {}_R\mathcal{B} \to {}_R\mathcal{B}$ preserve coequalizers for all $X \in \mathcal{C}$.*

(1) *For all $X \in \mathcal{C}$, $M \in {}_R\mathcal{B}_R$, and $V \in {}_R\mathcal{B}$, the identity on $X \diamondsuit M \otimes V$ induces an isomorphism $\Theta\colon (X \diamondsuit M) \otimes_R V \to X \diamondsuit (M \otimes_R V)$; we note the special case $\Theta\colon (X \diamondsuit R) \otimes_R V \to X \diamondsuit V$.*

(2) *The isomorphisms*

$$\Xi = \Xi_{XY} := \left((X \diamondsuit R) \underset{R}{\otimes} (Y \diamondsuit R) \xrightarrow{\Theta} X \diamondsuit (Y \diamondsuit R) \xrightarrow{\Psi^{-1}} (X \otimes Y) \diamondsuit R\right)$$

endow the functor $-\diamondsuit R\colon \mathcal{C} \to {}_R\mathcal{B}^!_R$ with the structure of a monoidal functor.

(3) *The isomorphisms $\Theta\colon (X \diamondsuit R) \otimes_R V \to X \diamondsuit V$ endow the identity endofunctor of ${}_R\mathcal{B}$ with the structure of a $\mathcal{C}$-functor, where the codomain has the $\mathcal{C}$-category structure induced along $(-\diamondsuit R, \Xi)$ from the ${}_R\mathcal{B}^!_R$-category structure, and the domain has the $\mathcal{C}$-category structure given by hypothesis.*

Proof. (1) is proved by the same argument used in the proof of Theorem 3.1: By assumption, the functor $X \diamondsuit -$ preserves the coequalizer defining the tensor product $M \otimes_R V$. We note that the particular instance $\Theta\colon (X \diamondsuit R) \otimes_R V \to X \diamondsuit V$ coincides with the isomorphism

$$v\colon \mathcal{T}(\mathcal{T}^{-1}(X \diamondsuit -))(V) \cong X \diamondsuit V$$

from the proof of Theorem 3.1.

We view the left $\mathcal{C}$-category structure of ${}_R\mathcal{B}$ as a monoidal functor

$$(\mathcal{L}, \tilde{\Psi})\colon \mathcal{C} \ni X \mapsto X \diamondsuit (-) \in \mathfrak{Fun}^c_{\mathcal{B}}({}_R\mathcal{B}, {}_R\mathcal{B})$$

as in Section 2.2. Composing with the monoidal functor $(\mathcal{T}^{-1}, v)$ yields a monoidal functor $\mathcal{T}^{-1}\mathcal{L}\colon \mathcal{C} \to {}_R\mathcal{B}^!_R$, whose monoidal functor structure is by definition the composition

$$\mathcal{T}^{-1}(\mathcal{L}(X)) \underset{R}{\otimes} \mathcal{T}^{-1}(\mathcal{L}(Y)) \xrightarrow{v} \mathcal{T}^{-1}(\mathcal{L}(X) \circ \mathcal{L}(Y)) \xrightarrow{\mathcal{T}^{-1}(\tilde{\Psi})} \mathcal{T}^{-1}(\mathcal{L}(X \otimes Y))$$

which coincides with

$$(X\diamondsuit R)\underset{R}{\otimes}(Y\diamondsuit R)\xrightarrow{\Theta}X\diamondsuit(Y\diamondsuit R)\xrightarrow{\Psi^{-1}}(X\otimes Y)\diamondsuit R$$

This proves (2). By definition, the $\mathcal{C}$-category structure induced on ${}_R\mathcal{B}$ along the functor $-\diamondsuit R$ by the canonical ${}_R\mathcal{B}^!_R$-category structure corresponds to the monoidal functor $\mathcal{T}\mathcal{T}^{-1}\mathcal{L}\colon\mathcal{C}\to\mathfrak{Fun}^c_{\mathcal{B}}({}_R\mathcal{B},{}_R\mathcal{B})$, while the original $\mathcal{C}$-category structure corresponds to $\mathcal{L}$. Since $\mathcal{T}\mathcal{T}^{-1}\cong Id$ as monoidal functors, the isomorphism being the counit v of adjunction, we conclude that the identity is a $\mathcal{C}$-category equivalence between the two structures with $\mathcal{C}$-functor structure Θ, as claimed in (3). □

Example 3.4. Assume that $X\in\mathcal{C}$ has a left dual $X^\vee$. Since monoidal functors preserve duals, it follows that $X\diamondsuit R\in{}_R\mathcal{B}^!_R$ has a left dual as well, so in particular $X\diamondsuit R$ is dualizable as right R-module, with dual isomorphic to $X^\vee\diamondsuit R$. If R is a quasibialgebra over a field k, then we may take the special case $\mathcal{C}={}_R\mathcal{M}$, with the canonical $\mathcal{C}$-category structure on ${}_R\mathcal{M}$ coming from the fact that ${}_R\mathcal{M}$ is monoidal. We conclude that $X^\vee\diamondsuit R\cong\operatorname{Hom}_{-R}(X\diamondsuit R,R)\cong R\otimes X^*$, where X^* denotes the vector space dual of X. If R is finite dimensional, it follows that X and $X^\vee$ have the same dimension. See [29] for a more complicated proof, but also see [28] for an example that $X\in\mathcal{M}^R$ and its dual may have different dimensions if R is a coquasibialgebra, even if the whole category of finite dimensional R-comodules is left and right rigid.

Example 3.5. Take $\mathcal{B}$ to be the category of abelian groups, so that R is a ring. If we are given the structure of a monoidal category on ${}_R\mathcal{M}=:\mathcal{C}$, then in particular ${}_R\mathcal{M}$ is a left $\mathcal{C}$-category. If we assume in addition that the abstract tensor product $\diamondsuit$ on $\mathcal{C}$ is right exact in each argument, then it follows from Theorem 3.3 that we have a right exact monoidal functor $\mathcal{C}\to{}_R\mathcal{M}_R$. This functor is faithful, since there is a neutral object in $\mathcal{C}={}_R\mathcal{M}$. Thus an abstract abelian monoidal category with right exact tensor products, whose underlying abelian category happens to be a category of left modules, admits a faithful right exact monoidal embedding into the category of bimodules over the same ring, with the usual tensor product. Such an embedding was constructed in a recent paper by Phung Ho Hai [23] as a key stage in his construction of a faithful exact monoidal embedding for arbitrary rigid abelian monoidal categories into bimodule categories.

While the preceding two examples show that the monoidal functor constructed in Theorem 3.3 can have interesting consequences by itself, the main applications in the present paper occur when we are given in addition an algebra (or coalgebra) in $\mathcal{C}$ and consider its modules (or comodules) in ${}_R\mathcal{B}$.

Corollary 3.6. *Assume given a coflat algebra R in $\mathcal{B}$, a monoidal category $\mathcal{C}$, the structure $(\diamondsuit,\Psi)$ of a $\mathcal{C}$-category on ${}_R\mathcal{B}$, compatible with its $\mathcal{B}$-category structure. Let A be an algebra in $\mathcal{C}$ such that the functor $A\diamondsuit -\colon{}_R\mathcal{B}\to{}_R\mathcal{B}$ preserves coequalizers.*

Then $A\diamondsuit R$ is an R-ring in $\mathcal{B}$ with multiplication given by

$$(A\diamondsuit R)\underset{R}{\otimes}(A\diamondsuit R)\xrightarrow{\Theta}A\diamondsuit(A\diamondsuit R)\xrightarrow{\Psi^{-1}}(A\otimes A)\diamondsuit R\xrightarrow{\nabla\diamondsuit R}A\diamondsuit R,$$

and unit $R\xrightarrow{\eta\diamondsuit R}A\diamondsuit R$.

For $V \in {}_R\mathcal{B}$ a bijection between A-comodule structures μ and $A\diamondsuit R$-comodule structures $\hat\mu$ on V is given by

$$\hat\mu = \left((A\diamondsuit R)\underset{R}{\otimes} V \xrightarrow{\Theta} A\diamondsuit V \xrightarrow{\mu} V\right).$$

In particular, we have a category equivalence ${}_A\left({}_R\mathcal{B}\right) \cong {}_{A\diamondsuit R}\mathcal{B}$.

Proof. The objects $X \in \mathcal{C}$ for which $X\diamondsuit -$ preserves coequalizers and is a strict $\mathcal{B}$-functor clearly form a monoidal subcategory of $\mathcal{C}$. Thus we may as well assume that $\mathcal{C}$ meets the hypotheses of Theorem 3.3. Then $A\diamondsuit R$ is just the usual image of the algebra A under the monoidal functor $\mathcal{F} := (-)\diamondsuit R$ from Theorem 3.3. For the category equivalence recall the equality ${}_{\mathcal{F}(A)}\left({}_R\mathcal{B}\right) = {}_A\left({}_R\mathcal{B}\right)$ that holds when ${}_R\mathcal{B}$ on the right hand side is given the $\mathcal{C}$-category structure induced along $\mathcal{F}$. But since this is equal to the $\mathcal{C}$-category structure given by hypothesis, there is no difference between the corresponding two versions ${}_A\left({}_R\mathcal{B}\right)$. □

Example 3.7. Let k be a commutative ring, H a k-flat Hopf algebra over k, and A a left faithfully flat H-Galois extension of its coinvariant subalgebra B. By Schneider's structure theorem on Hopf modules [31] we have a category equivalence

$$\mathcal{M}_B \ni N \mapsto N \underset{B}{\otimes} A \in \mathcal{M}_A^H$$

with inverse $M \mapsto M^{\operatorname{co} H}$. We put $R = B^{\mathrm{op}}$. Since $\mathcal{M}_A^H$ is naturally a left $\mathcal{M}^H$-category, so is $\mathcal{M}_B = {}_R\mathcal{M}$. We consider the algebra $A \in \mathcal{M}^H$ and find

$${}_A\mathcal{M}_A^H \cong {}_A\left(\mathcal{M}_A^H\right) \cong {}_A(\mathcal{M}_B) \cong {}_A({}_R\mathcal{M}) \cong {}_{A\diamondsuit R}\mathcal{M},$$

where $\diamondsuit$ denotes the $\mathcal{M}^H$-category structure on ${}_R\mathcal{M}$ induced by that on $\mathcal{M}_A^H$. To see how the algebra $A\diamondsuit R$ looks in detail, we note first $X\diamondsuit N = (X \otimes N \otimes_B A)^{\operatorname{co} H}$ for any $X \in \mathcal{M}^H$ and $N \in {}_R\mathcal{M}$. In particular $X\diamondsuit R = (X \otimes A)^{\operatorname{co} H}$, and $A\diamondsuit R = (A \otimes A)^{\operatorname{co} H}$. The right R-module structure of $X\diamondsuit R$ is induced by the left B-module structure of A. The isomorphism $\Theta\colon (X\diamondsuit R) \otimes_R N \to X\diamondsuit N$ for $N \in {}_R\mathcal{M}$ is

$$(X \otimes A)^{\operatorname{co} H} \underset{R}{\otimes} N \ni x \otimes a \otimes n \mapsto x \otimes n \otimes a \in (X \otimes N \underset{B}{\otimes} A)^{\operatorname{co} H}$$

and $\Psi\colon (X \otimes Y)\diamondsuit N \to X\diamondsuit(Y\diamondsuit N)$ turns out to be the map

$$(X \otimes Y \otimes N \underset{B}{\otimes} A)^{\operatorname{co} H} \to (X \otimes (Y \otimes N \underset{B}{\otimes} A)^{\operatorname{co} H} \underset{B}{\otimes} A)^{\operatorname{co} H}$$

induced by the inverse of the counit of the adjoint equivalence $\mathcal{M}_B \cong \mathcal{M}_A^H$, that is, $\Psi^{-1}(x \otimes y \otimes n \otimes a \otimes a') = x \otimes y \otimes n \otimes aa'$. Putting all this information together, we obtain multiplication on $A\diamondsuit R = (A \otimes A)^{\operatorname{co} H}$ in the form

$$\begin{aligned}(x \otimes a)(y \otimes a') = (\nabla\diamondsuit R)\Psi^{-1}\Theta(x \otimes a \otimes y \otimes a') &= (\nabla\diamondsuit R)\Psi^{-1}(x \otimes y \otimes a' \otimes a)\\ &= (\nabla\diamondsuit R)(x \otimes y \otimes a'a) = xy \otimes a'a,\end{aligned}$$

that is, $A\diamondsuit R$ is a subalgebra of $A \otimes A^{\mathrm{op}}$.

This means that $A\diamondsuit R = L(A,H)$ as studied in [27], following the case $B = k$ treated in [26] and previously in the commutative case in [33]. The category equivalence ${}_{A\diamondsuit R}\mathcal{M} \cong {}_A\mathcal{M}_A^H$ was the main motivation to study $A\diamondsuit R$ in [27].

Closing this section, we record the coalgebra version of Corollary 3.6, whose proof is dual (except for the fact, of course, that ${}_R\mathcal{B}$ stays as it is and is not turned into a comodule category, whereas R-rings are turned into R-corings).

Corollary 3.8. *Assume given a coflat algebra R in $\mathcal{B}$, a monoidal category $\mathcal{C}$, the structure $(\diamondsuit, \Psi)$ of a $\mathcal{C}$-category on ${}_R\mathcal{B}$, compatible with its $\mathcal{B}$-category structure.*

Let C be a coalgebra in $\mathcal{C}$ such that the functor $C\diamondsuit -\colon {}_R\mathcal{B} \to {}_R\mathcal{B}$ preserves coequalizers.

Then $C\diamondsuit R$ is an R-coring in $\mathcal{B}$ with comultiplication given by

$$C\diamondsuit R \xrightarrow{\Delta\diamondsuit R} (C \otimes C)\diamondsuit R \xrightarrow{\Psi} C\diamondsuit(C\diamondsuit R) \xrightarrow{\Theta^{-1}} (C\diamondsuit R) \underset{R}{\otimes} (C\diamondsuit R)$$

and counit $C\diamondsuit R \xrightarrow{\varepsilon\diamondsuit R} R$.

For $V \in {}_R\mathcal{B}$ a bijection between C-comodule structures δ and $C\diamondsuit R$-comodule structures $\hat{\delta}$ on V is given by

$$\delta = \left(V \xrightarrow{\hat{\delta}} (C\diamondsuit R) \underset{R}{\otimes} V \xrightarrow{\Theta} C\diamondsuit V \right).$$

In particular, we have a category equivalence ${}^C({}_R\mathcal{B}) \cong {}^{C\diamondsuit R}\mathcal{B}$.

4. Examples

In all the examples we will discuss, with the exception of Example 3.7 above, the $\mathcal{C}$-category structure on ${}_R\mathcal{B}$ in Theorem 3.3 and its corollaries will be modelled on the tensor product in $\mathcal{B}$, in particular we will always have objects in $\mathcal{B}$ assigned to objects of $\mathcal{C}$. More formally, we will be dealing with

Situation 4.1. R is an algebra in $\mathcal{B}$, $\mathcal{C}$ is a monoidal category, $(\nu, \xi)\colon \mathcal{C} \to \mathcal{B}$ is an incoherent tensor functor. We are given a $\mathcal{C}$-category structure $(\diamondsuit, \Psi)$ on ${}_R\mathcal{B}$, compatible with its right $\mathcal{B}$-category structure, and such that the functors $X\diamondsuit -$ for fixed $X \in \mathcal{C}$ preserve coequalizers. Moreover, we have a $\mathcal{B}$-natural isomorphism $\zeta\colon \nu(X) \otimes V \to X\diamondsuit V$ such that the underlying functor is an incoherent $\mathcal{C}$-functor $(\mathcal{U}, \zeta)\colon {}_R\mathcal{B} \to \mathcal{B}$, where the codomain has the incoherent $\mathcal{C}$-category structure induced via (ν, ξ).

In less technical language, this means that we have fixed isomorphisms $\nu(X) \otimes V \cong X\diamondsuit V$ for all $X \in \mathcal{C}$ and $V \in {}_R\mathcal{B}$. Note that $X\diamondsuit -$ will automatically preserve coequalizers if $\nu(X)$ is right coflat in $\mathcal{B}$. The assumption that ζ is a $\mathcal{B}$-natural transformation makes sure that ζ is a right R-module morphism for the obvious right R-module structure on $\nu(X) \otimes M$ whenever $M \in {}_R\mathcal{B}_R$. Also, we find that

$$\begin{tikzcd}
(X\diamondsuit R) \otimes_R V \arrow[r, "\Theta"] & X\diamondsuit V \\
(\nu(X) \otimes R) \otimes_R V \arrow[u, "\zeta"] \arrow[r, equal] & \nu(X) \otimes V \arrow[u, "\zeta"']
\end{tikzcd}$$

commutes for all $X \in \mathcal{C}$ and $V \in {}_R\mathcal{B}$.

Given Situation 4.1, we can assume without loss of generality that ζ is the identity: Otherwise define a new $\mathcal{C}$-category structure on ${}_R\mathcal{B}$ by $X\diamondsuit' V := \nu(X) \otimes V$ for $X \in \mathcal{C}$ and $V \in {}_R\mathcal{B}$, with the R-module structure making ζ_{XV} an R-module morphism, and with Ψ' defined by commutativity of

$$\begin{array}{ccc}
(X\otimes Y)\diamondsuit' V & \xrightarrow{\Psi'} & X\diamondsuit'(Y\diamondsuit' V) \\
 & & \downarrow \zeta \\
\downarrow \zeta & & X\diamondsuit(Y\diamondsuit' V) \\
 & & \downarrow X\diamondsuit\zeta \\
(X\otimes Y)\diamondsuit V & \xrightarrow{\Psi} & X\diamondsuit(Y\diamondsuit V).
\end{array}$$

This defnition ensures that $(Id,\zeta)\colon ({}_R\mathcal{B},\diamondsuit,\Psi)\to({}_R\mathcal{B},\diamondsuit',\Psi')$ is a $\mathcal{C}$-category equivalence, and of course it is a strict $\mathcal{B}$-functor. If we assume $\zeta=\mathrm{id}$, then it follows that $\Theta=\mathrm{id}$.

If we are in Situation 4.1, and A is an algebra in $\mathcal{C}$, then we know that $A\diamondsuit R\cong\nu(A)\otimes R$, so the right hand side is an R-ring, with the obvious underlying right R-module structure. In particular, $\nu(A)\otimes R=\nu(A)\#_{\sigma,\chi}R$ is a generalized crossed product for some $\sigma\colon R\otimes\nu(A)\to\nu(A)\otimes R$, and some $\chi\colon\nu(A)\to\nu(A)\otimes\nu(A)\otimes R$. Similarly, if C is a coalgebra in $\mathcal{C}$, then $C\diamondsuit R\cong\nu(C)\#^{\vartheta}_{\sigma}R$ is a generalized crossed coproduct for some $\sigma\colon R\otimes\nu(C)\to\nu(C)\otimes R$ and $\vartheta\colon\nu(C)\to\nu(C)\otimes\nu(C)\otimes R$. If $\nu(C)$ happens to have a left dual in $\mathcal{B}$, we also know how to compute the dual algebra to the coring $\nu(C)\otimes R$. All that remains to be done is to distill the morphisms σ,χ and ϑ from the given situation.

Note first that the morphisms $\sigma\colon R\otimes\nu(X)\to\nu(X)\otimes R$ responsible for the left R-module structure of $\nu(X)\otimes R\cong X\diamondsuit R$ form a natural transformation. Since the left R-module structure on $\nu(X)\otimes V\cong X\diamondsuit V$ for $V\in{}_R\mathcal{B}$ is $\mathcal{B}$-natural in V, it is given by

$$R\otimes\nu(X)\otimes V\xrightarrow{\sigma\otimes V}\nu(X)\otimes R\otimes V\xrightarrow{\nu(X)\otimes\mu}\nu(X)\otimes V$$

Since the top morphism in the diagram

$$\begin{array}{ccc}
\nu(X)\otimes\nu(Y)\otimes V & \xrightarrow{\Psi^{(\xi,\zeta)}_{XYV}} & \nu(X)\otimes\nu(Y)\otimes V \\
\downarrow \xi\otimes V & & \downarrow \nu(X)\otimes\zeta \\
\nu(X\otimes Y)\otimes V & & \nu(X)\otimes(Y\diamondsuit V) \\
\downarrow \zeta & & \downarrow \zeta \\
(X\otimes Y)\diamondsuit V & \xrightarrow{\Psi_{XYV}} & X\diamondsuit(Y\diamondsuit V)
\end{array}$$

is $\mathcal{B}$-natural in V, it is given by

$$\Psi^{(\xi,\zeta)}_{XYV}=(\nu(X)\otimes\nu(Y)\otimes V\xrightarrow{\psi_{XY}\otimes V}\nu(X)\otimes\nu(Y)\otimes R\otimes V\xrightarrow{\mathrm{id}\otimes\mu}\nu(X)\otimes\nu(Y)\otimes V)$$

with

$$\psi_{XY}=(\nu(X)\otimes\nu(Y)\xrightarrow{\nu(X)\otimes\nu(Y)\otimes\eta}\nu(X)\otimes\nu(Y)\otimes R\xrightarrow{\Psi^{(\xi,\zeta)}_{XYR}}\nu(X)\otimes\nu(Y)\otimes R).$$

Abusing notation, we write $\psi^{-1}_{XY}:\nu(X)\otimes\nu(Y)\to\nu(X)\otimes\nu(Y)\otimes R$ for the morphism corresponding in the same way to $(\Psi^{(\xi,\zeta)})^{-1}$, and sum up our findings

Corollary 4.2. *Assume Situation 4.1. Then there are natural transformations*

$$\psi_{XY}\colon \nu(X)\otimes\nu(Y)\to\nu(X)\otimes\nu(Y)\otimes R$$
$$\psi^{-1}_{XY}\colon \nu(X)\otimes\nu(Y)\to\nu(X)\otimes\nu(Y)\otimes R$$

such that

$$\begin{tikzcd}
\nu(X)\otimes\nu(Y)\otimes V \arrow[r, "\psi_{XY}"] \arrow[d, "\xi\otimes V"] & \nu(X)\otimes\nu(Y)\otimes R\otimes V \arrow[r, "\mathrm{id}\otimes\mu_V"] & \nu(X)\otimes\nu(Y)\otimes V \arrow[d, "\nu(X)\otimes\zeta"] \\
\nu(X\otimes Y)\otimes V \arrow[d, "\zeta"] & & \nu(X)\otimes(Y\diamondsuit V) \arrow[d, "\zeta"] \\
(X\otimes Y)\diamondsuit V \arrow[rr, "\Psi_{XYV}"] & & X\diamondsuit(Y\diamondsuit V)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\nu(X)\otimes\nu(Y)\otimes V \arrow[d, "\xi\otimes V"] & \nu(X)\otimes\nu(Y)\otimes R\otimes V \arrow[l, "\mathrm{id}\otimes\mu_V"'] & \nu(X)\otimes\nu(Y)\otimes V \arrow[l, "\psi^{-1}_{XY}"'] \arrow[d, "\nu(X)\otimes\zeta"] \\
\nu(X\otimes Y)\otimes V \arrow[d, "\zeta"] & & \nu(X)\otimes(Y\diamondsuit V) \arrow[d, "\zeta"] \\
(X\otimes Y)\diamondsuit V \arrow[rr, "\Psi_{XYV}"] & & X\diamondsuit(Y\diamondsuit V)
\end{tikzcd}$$

commute.

If A is an algebra in $\mathcal{C}$, then $A\diamondsuit R\cong\nu(A)\#_{\sigma,\chi}R$ with $\chi=(\nabla_{\nu(A)}\otimes R)\psi^{-1}_{AA}$, where $\nabla_{\nu(A)}=(\nu(A)\otimes\nu(A)\xrightarrow{\xi}\nu(A\otimes A)\xrightarrow{\nu(\nabla_A)}\nu(A))$.

If C is a coalgebra in $\mathcal{C}$, then $C\diamondsuit R\cong\nu(C)\#^{\vartheta}_{\sigma}R$, with $\vartheta=\psi_{CC}\Delta_{\nu(C)}$, where $\Delta_{\nu(C)}=(\nu(C)\xrightarrow{\nu(\Delta)}\nu(C\otimes C)\xrightarrow{\xi^{-1}}\nu(C)\otimes\nu(C))$.

If in addition $\nu(C)$ is left rigid with dual $\nu(C)^*$, and we denote by $\nabla_{\nu(C)^*}$ the multiplication on $\nu(C)^*$ dual to comultiplication in $\nu(C)$, then $(C\diamondsuit R)^\vee\cong R\#_{\sigma^\flat,\vartheta^\flat}\nu(C)^*$, and $\vartheta^\flat=(\nabla_{\nu(C)^*}\otimes R)\psi^\flat_{CC}$.

Proof. It remains to check the assertions involving (co)algebras in $\mathcal{C}$. We will do this only in the algebra case, and only when $\zeta=\mathrm{id}$. Here multiplication on $A\diamondsuit R=\nu(A)\otimes R$ is defined to be

$$(\nu(A)\otimes R)\underset{R}{\otimes}(\nu(A)\otimes R)=\nu(A)\otimes\nu(A)\otimes R\xrightarrow{\Psi^{-1}}\nu(A\otimes A)\otimes R\xrightarrow{\nu(\nabla)\otimes R}\nu(A)\otimes R,$$

and χ is by definition the composition of

$$\nu(A)\otimes\nu(A)\xrightarrow{\nu(A)\otimes\nu(A)\otimes\eta}\nu(A)\otimes\nu(A)\otimes R=(\nu(A)\otimes R)\underset{R}{\otimes}(\nu(A)\otimes R)$$

with multiplication. Hence the outer rectangle of

$$\begin{tikzcd}
\nu(A)\otimes\nu(A) \arrow[rr, "\chi"] \arrow[dd, "\mathrm{id}\otimes\eta"'] \arrow[dr, "\psi^{-1}"] & & \nu(A)\otimes R \\
& \nu(A)\otimes\nu(A)\otimes R \arrow[ur, "\nabla"] \arrow[dr, "\xi\otimes R"] & \\
\nu(A)\otimes\nu(A)\otimes R \arrow[rr, "\Psi^{-1}"] & & \nu(A\otimes A)\otimes R \arrow[uu, "\nu(\nabla)\otimes R"']
\end{tikzcd}$$

commutes. The upper triangle is our claim, the right hand triangle is the definition of $\nabla_{\nu(A)}$, and the remaining quadrangle is the definition of ψ^{-1}. □

Remark 4.3. Assume $\mathcal{B}=\mathcal{M}_k$ is the category of modules over the commutative base ring k. If $\nu(X)$ and $\nu(Y)$ are finitely generated projective, we can identify $\psi_{XY}\in\mathrm{Hom}(\nu(X)\otimes\nu(Y),\nu(X)\otimes\nu(Y)\otimes R)$ with an element $\psi_{XY}\in\mathrm{End}(\nu(X))\otimes\mathrm{End}(\nu(Y))\otimes R$, which we formally write as $\psi_{XY}=\psi_X^{(1)}\otimes\psi_Y^{(2)}\otimes\psi^{(3)}$, so that

$$\psi_{XY}(x\otimes y)=\psi_X^{(1)}(x)\otimes\psi_Y^{(2)}(y)\otimes\psi^{(3)}. \tag{4.1}$$

Similarly, we write $\psi_{XY}^{-1}=\psi_X^{(-1)}\otimes\psi_Y^{(-2)}\otimes\psi^{(-3)}$. (With these identifications, ψ_{XY}^{-1} is really the inverse of ψ_{XY} in $\mathrm{End}(\nu(X))\otimes\mathrm{End}(\nu(Y))\otimes R$, so our former abuse of notation is now justified.)

Also, if $\nu(X)$ is finitely generated projective, then we can write

$$\sigma(r\otimes x)=:r_X(x)\otimes r_R \tag{4.2}$$

for some $r_X\otimes r_R\in\mathrm{End}(\nu(X))\otimes R$. These notations allow us to write multiplication in $\nu(A)\#R$ in the form

$$(x\#r)(y\#s)=\psi_A^{(-1)}(x)\psi_A^{(-2)}(r_A(y))\#\psi^{(-3)}r_Rs. \tag{4.3}$$

The mate

$$\psi^\flat_{XY}\in\mathrm{Hom}(\nu(Y)^*\otimes\nu(X)^*,R\otimes\nu(Y)^*\otimes\nu(X)^*\otimes R)\cong R\otimes\mathrm{End}(\nu(Y)^*)\otimes\mathrm{End}(\nu(X)^*)$$

of ψ_{XY} is given by

$$\psi^\flat_{XY}=\psi^{(3)}\otimes(\psi_Y^{(2)})^*\otimes(\psi_X^{(1)})^*.$$

Thus if C is a coalgebra in $\mathcal{C}$, and $\nu(C)$ is finitely generated projective, then multiplication in the dual of $C\diamondsuit R$ has the form

$$(r\#\beta)(s\#\gamma)=rs_R\psi^{(3)}\#\psi_C^{(1)*}(\gamma)\psi^{(2)^*}(s_C^*(\beta))$$

where, for better handling in explicit calculations, we have written the product of two elements in $\nu(C)^*$ in the usual order, so $(\gamma\beta)(c)=\gamma(c_{(1)})\beta(c_{(2)})$.

Remark 4.4. If we still assume $\mathcal{B}=\mathcal{M}_k$, but drop the assumption that all $\nu(X)$ for $X\in\mathcal{C}$ are finitely generated projective, we may still keep (4.1) and (4.2) in the usual sense as "formal notations", giving us "formally" the same expression (4.3) for multiplication in $\nu(A)\#R$.

Example 4.5. Let R be a k-algebra. Define the category $\mathcal{C}$ of transfer maps through R to have objects (X, σ_{RX}) where X is a k-module and $\sigma_{RX}\colon R \otimes X \to X \otimes R$ is an isomorphism making

$$\begin{tikzcd} R \otimes R \otimes X \arrow[r, "R\otimes\sigma"] \arrow[d, "\nabla\otimes X"] & R \otimes X \otimes R \arrow[r, "\sigma\otimes R"] & X \otimes R \otimes R \arrow[d, "X\otimes\nabla"] \\ R \otimes X \arrow[rr, "\sigma"] & & X \otimes R \end{tikzcd} \tag{4.4}$$

commute and satisfying $\sigma_{RX}(\eta \otimes X) = X \otimes \eta$. It is straightforward to check that $\mathcal{C}$ is a monoidal category with a strict monoidal underlying functor $\mathcal{C} \to \mathcal{M}_k$; one just has to set $(X, \sigma_{RX}) \otimes (Y, \sigma_{RY}) = (X \otimes Y, \sigma_{R,X\otimes Y})$ after defining $\sigma_{R,X\otimes Y} = (X \otimes \sigma_{RY})(\sigma_{RX} \otimes Y)$. Moreover, the category ${}_R\mathcal{M}$ of left R-modules is a strict $\mathcal{C}$-category such that the underlying functor to $\mathcal{M}_k$ is a strict $\mathcal{C}$-functor. Here, we have to set $X \diamondsuit V := X \otimes V$ for $V \in {}_R\mathcal{M}$, and to endow $X \diamondsuit V$ with the left R-module structure

$$\mu_{X\diamondsuit V} := (R \otimes X \otimes V \xrightarrow{\sigma\otimes V} X \otimes R \otimes V \xrightarrow{X\otimes\mu_V} X \otimes V).$$

We observe that $\mathcal{C}$ has a couniversal property: If $\mathcal{C}'$ is another monoidal category with a strict monoidal functor $\nu'\colon \mathcal{C}' \to \mathcal{M}_k$, and if ${}_R\mathcal{M}$ is a strict left $\mathcal{C}'$-category in such a way that the underlying functor ${}_R\mathcal{M} \to \mathcal{M}_k$ is a strict $\mathcal{C}$-functor, then a monoidal functor $\hat{\nu}'\colon \mathcal{C}' \to \mathcal{C}$ can be defined by sending $X \in \mathcal{C}'$ to $(\nu(X), \sigma)$, where $\sigma\colon R \otimes \nu(X) \to \nu(X) \otimes R$ is the morphism responsible for the left R-module structure of $\nu(X) \otimes R$.

Algebras in $\mathcal{C}$ were studied by Tambara [32]. An algebra in $\mathcal{C}$ is a k-algebra A equipped with an isomorphism $\sigma_{RA}\colon R \otimes A \to A \otimes R$ making (4.4) and a left-right switched version of that diagram commute (and fulfilling the obvious identities involving the units of A and R). Tambara shows that this enables us to endow $A \otimes R$ with an associative algebra structure given by

$$A \otimes R \otimes A \otimes R \xrightarrow{A\otimes\sigma\otimes R} A \otimes A \otimes R \otimes R \xrightarrow{\nabla\otimes\nabla} A \otimes R.$$

This is precisely the algebra structure on $A \diamondsuit R$ provided by Corollary 3.6.

A coalgebra in $\mathcal{C}$ is called an entwining structure (R, C, σ) ([2], we have built in the extra condition that the entwining map be a bijection). The objects in ${}^C({}_R\mathcal{M})$ are usually called entwined modules. We see that we can describe them equivalently by comodules over a certain R-coring $C \diamondsuit R$. According to the review article [18] of Masuoka this observation is due to Takeuchi. If C is finitely generated projective, we obtain ${}^C({}_R\mathcal{M}) \cong {}_{R\#C^*}\mathcal{M}$ for a certain generalized smash product algebra $R\#C^*$; this was described by Hobst and Pareigis in [11].

Remark 4.6. Assume the situation of Remark 4.3. If in addition $\mathcal{C} = {}_H\mathcal{M}$ for some quasibialgebra H, with ν the underlying functor, and ζ the identity, then the natural transformations σ, ψ, and ψ^{-1} have the forms

$$\sigma(r \otimes x) = r_{(-1)}x \otimes r_{(0)}$$
$$\psi_{XY}(x \otimes y) = \psi^{(1)}x \otimes \psi^{(2)}y \otimes \psi^{(3)}$$
$$\psi^{-1}_{XY}(x \otimes y) = \psi^{(-1)}x \otimes \psi^{(-2)}y \otimes \psi^{(-3)}$$

for the map

$$R \ni r \mapsto r_{(-1)} \otimes r_{(0)} := \sigma(r \otimes 1_H) \in H \otimes R$$

and the elements $\psi^{(1)} \otimes \psi^{(2)} \otimes \psi^{(3)} := \psi := \psi_{HH}(1 \otimes 1) \in H \otimes H \otimes R$, and $\psi^{(-1)} \otimes \psi^{(-2)} \otimes \psi^{(-3)} := \psi^{-1} := \psi^{-1}_{HH}(1 \otimes 1)$. In particular, we have

$$(x\#r)(y\#s) = (\psi^{(-1)}x) \cdot (\psi^{(-2)}r_{(-1)}y)\#\psi^{(-3)}r_{(0)}s$$

in $A\#R$ when A is an algebra in ${}_H\mathcal{M}$, and

$$(r\#\beta)(s\#\gamma) = rs_{(0)}\psi^{(3)}\#(\gamma \leftharpoonup \psi^{(1)})(\beta \leftharpoonup s_{(1)}\psi^{(2)})$$

in $(C\#R)^\vee$ when C is a coalgebra in ${}_H\mathcal{M}$, where $\beta \leftharpoonup h$ for $h \in H$ and $\beta \in X^*$ denotes the transpose of the action of H on $X \in {}_H\mathcal{M}$.

Example 4.7. A special case of the preceding remark occurs when $R = H$, and the ${}_H\mathcal{M}$-category structure of ${}_R\mathcal{M}$ is that given by the monoidal category structure of ${}_H\mathcal{M}$. The resulting smash products of a quasi-Hopf algebra H and an algebra A in the monoidal category of H-modules occur in [3].

Remark 4.8. Assume again the situation of Remark 4.3, and assume in addition that $\mathcal{C} = \mathcal{M}^H$ is the category of right H-comodules for some coquasibialgebra H. Then

$$\sigma(r \otimes x) = x_{(0)} \otimes r \leftharpoonup x_{(1)}$$
$$\psi_{XY}(x \otimes y) = x_{(0)} \otimes y_{(0)} \otimes \tau(x_{(1)} \otimes y_{(1)})$$
$$\psi^{-1}_{XY}(x \otimes y) = x_{(0)} \otimes y_{(0)} \otimes \tau^{-1}(x_{(1)} \otimes y_{(1)})$$

for $r \leftharpoonup h := (\varepsilon \otimes R)\sigma_H(r \otimes h)$, $\tau(g \otimes h) = (\varepsilon \otimes \varepsilon \otimes R)\psi_{HH}(g \otimes h)$, and τ^{-1} the convolution inverse of $\tau\colon H \otimes H \to R$. In particular

$$(x\#r)(y\#s) = x_{(0)}y_{(0)}\#\tau^{-1}(x_{(1)} \otimes y_{(1)})(r \leftharpoonup y_{(2)})s$$

in $A\Diamond R$ for A an algebra in ${}_H\mathcal{M}$.

Example 4.9. The formula for multiplication in the preceding remark looks quite like that for an ordinary Hopf crossed product (with sides switched compared to the usual conventions as used in [19]). Assume that H is an ordinary bialgebra, and we are given a map $\leftharpoonup\colon R \otimes H \to R$, and a convolution invertible $\tau\colon H \otimes H \to R$. It is easy to check that

$$R \otimes X \otimes V \ni r \otimes x \otimes v \mapsto x_{(0)} \otimes (r \leftharpoonup x_{(1)})v \in X \otimes V$$

defines an R-module structure on $X \otimes V$ for all $X \in \mathcal{M}^H$ and $V \in {}_H\mathcal{M}$ if and only if

$$(rs) \leftharpoonup h = (r \leftharpoonup h_{(1)})(s \leftharpoonup h_{(2)}), \quad 1 \leftharpoonup h = \varepsilon(h)1$$

hold for all $h \in H$ and $r, s \in R$, that is, $\leftharpoonup$ is a measuring action. Assuming this to be the case, we have defined a functor

$$\mathcal{M}^H \times {}_R\mathcal{M} \ni (X, V) \mapsto X \otimes V \in {}_R\mathcal{M}.$$

Next, one checks that the natural transformation

$$\begin{aligned}\Psi_{XYV}: (X \otimes Y) \otimes V &\to X \otimes (Y \otimes V)\\ x \otimes y \otimes v &\mapsto x_{(0)} \otimes y_{(0)} \otimes \tau(x_{(1)} \otimes y_{(1)})v\end{aligned}$$

consists of R-linear maps if and only if

$$\tau(g_{(1)} \otimes h_{(1)})(r \leftharpoonup g_{(2)}h_{(2)}) = (r \leftharpoonup g_{(1)} \leftharpoonup h_{(1)})\tau(g_{(2)} \otimes h_{(2)})$$

holds for all $g, h \in H$ and $r \in R$, that is, if and only if $(\leftharpoonup, \tau^{-1})$ is a twisted module structure. Assuming this to be the case, we have defined an incoherent $\mathcal{M}^H$-category structure on ${}_R\mathcal{M}$, which turns out to be coherent if and only if

$$(\tau(f_{(1)} \otimes g_{(1)}) \leftharpoonup h_{(1)})\tau(f_{(2)}g_{(2)} \otimes h_{(2)}) = \tau(g_{(1)} \otimes h_{(1)})\tau(f \otimes g_{(2)}h_{(2)})$$

holds for all $f, g, h \in H$, that is, if and only if $(\leftharpoonup, \tau^{-1})$ is a crossed product action.

As we pointed out at the beginning of this example, a special case of the algebras $A \diamondsuit R$ coming from this $\mathcal{M}^H$-category structure and an algebra $A \in \mathcal{M}^H$ is an ordinary Hopf crossed product (where $A = H$).

Example 4.10 (The double of a quasi-Hopf algebra). The construction of the double of a quasi-Hopf algebra was proposed by Majid in [16] to explain and generalize the example by Dijkgraaf, Pasquier and Roche [5] of the double of a finite group modified by a three-cocycle.

Let $(H, \phi, S, \alpha, \beta)$ be a quasi-Hopf algebra, finitely generated projective over some base ring k. The double $D(H)$ of H is again a quasi-Hopf algebra, which (as pointed out by Majid) should be constructed to have the same conceptual interpretations that the Drinfeld double of an ordinary Hopf algebra has: It should be constructed to satisfy ${}_{D(H)}\mathcal{M} \cong \mathcal{Z}({}_H\mathcal{M})$, where $\mathcal{Z}$ denotes the center construction for monoidal categories. As an intermediate step, it is convenient to introduce a suitable notion of Yetter-Drinfeld modules so that $\mathcal{Z}({}_H\mathcal{M}) \cong {}_H^H\mathcal{YD}$. In the end, it turns out that the quasi-Hopf algebra $D(H)$ can be modelled on the k-module $H \otimes H^*$. Now it is certainly not obvious how to endow $H \otimes H^*$ with an associative multiplication. A formula for the relevant multiplication was given by Hausser and Nill [9] using [8]. The paper [29] is an attempt to redo their constructions with more conceptual arguments and less computations, using another interpretation of modules over the double of an ordinary Hopf algebra: The category of Yetter-Drinfeld modules is equivalent to the category ${}_H^H\mathcal{M}_H^H$ of Hopf modules [25]. In [29] we have explained that the same holds for quasi-Hopf algebras, and that the equivalence can be used to construct the double.

We shall now discuss how the Drinfeld double of a quasi-Hopf algebra fits into the scheme developed in the present paper.

Let (H, ϕ) as above denote a finitely generated projective quasi-Hopf algebra from now on. The category ${}_H\mathcal{M}_H$ of H-H bimodules is a monoidal category, in which tensor product is defined as tensor product over k, and the associativity constraint is conjugation by ϕ. The coassociativity axiom for H says precisely that H with its natural H-bimodule structure is a coalgebra in ${}_H\mathcal{M}_H$. Although there is at first no natural notion of H-comodule, since H is not an ordinary coalgebra, this does give us a natural notion of Hopf module over H, first considered by Hausser and Nill: A Hopf module in ${}_H\mathcal{M}_H^H$ is a right H-comodule in the monoidal category $\mathcal{C} = {}_H\mathcal{M}_H$. It is clear from this categorical definition that one has a natural functor ${}_H\mathcal{M}_H \to {}_H\mathcal{M}_H^H$ assigning to $P \in {}_H\mathcal{M}_H$ the cofree right H-comodule $P \otimes H$ over P in the monoidal category ${}_H\mathcal{M}_H$. Explicitly, $P \otimes H$ has the diagonal H-module structures, and the right H-comodule structure given by $\rho(p \otimes h) = \phi^{(-1)}p\phi^{(1)} \otimes \phi^{(-2)}h_{(1)}\phi^{(2)} \otimes \phi^{(-3)}h_{(2)}\phi^{(3)}$.

Hausser and Nill [10] have proved the following structure theorem for Hopf modules: The functor

$$\mathcal{R}\colon {}_H\mathcal{M} \ni V \mapsto V \otimes H \in {}_H\mathcal{M}_H^H$$

given by restricting the above functor to left modules is a category equivalence. We note that is $\mathcal{R}(V) = V \otimes H$ has the obvious right H-module structure, the diagonal left H-module structure, and the right H-comodule structure given by

$$\rho(v \otimes h) = \phi^{(-1)}v \otimes \phi^{(-2)}h_{(1)} \otimes \phi^{(-3)}h_{(2)}$$

for $v \in V$ and $h \in H$.

It is obvious that there is also a natural Hopf module category ${}_H^H\mathcal{M}_H^H$, namely the category of bicomodules over the coalgebra H within the monoidal category ${}_H\mathcal{M}_H$. We can view this category as follows: The category ${}_H\mathcal{M}_H^H$ of right H-comodules in ${}_H\mathcal{M}_H$ is naturally a left ${}_H\mathcal{M}_H$-category, and we have ${}_H^H\mathcal{M}_H^H = {}^C\left({}_H\mathcal{M}_H^H\right)$, where we have denoted by C the coalgebra $H \in {}_H\mathcal{M}_H$. As shown in [29], the category equivalences ${}_H^H\mathcal{YD} \cong \mathcal{Z}({}_H\mathcal{M}) \cong {}_H^H\mathcal{M}_H^H$ between the (suitably defined!) category of Yetter-Drinfeld modules over H, the center of the monoidal category of H-modules, and the just mentioned category of Hopf modules works just as well for the quasi-Hopf algebra case as for ordinary Hopf algebras. Thus an algebra $D(H)$ such that there is a category equivalence ${}_{D(H)}\mathcal{M} \cong {}_H^H\mathcal{M}_H^H$ induced by $\mathcal{R}$ would be a fit replacement for the double of an ordinary Hopf algebra.

Since ${}_H\mathcal{M}$ is equivalent to ${}_H\mathcal{M}_H^H$, it is a left ${}_H\mathcal{M}_H$-category as well, in such a way that the functor $\mathcal{R}$ is a ${}_H\mathcal{M}_H$-functor. We denote the relevant ${}_H\mathcal{M}_H$-category structure by $(\diamondsuit, \Psi)$. We conclude that

$${}_H^H\mathcal{M}_H^H \cong {}^C\left({}_H\mathcal{M}_H^H\right) \cong {}^C\left({}_H\mathcal{M}\right) \cong {}^{C\diamondsuit H}\mathcal{M}$$

for some H-coring $C\diamondsuit H$. It remains to compute that coring, or rather, in the case that H is finite dimensional, its dual algebra $(C\diamondsuit H)^\vee$.

As shown in [29], we have an isomorphism

$$\tau\colon {}_{.}P_{.} \otimes {}_{.}H_{.}^{.} \to \mathcal{R}({}_{\mathrm{ad}}P).$$

for $P \in {}_H\mathcal{M}_H$, where ${}_{\mathrm{ad}}P$ is P with the adjoint module structure given by $h \rightharpoonup p = h_{(1)}pS(h_{(2)})$. It follows that we have an isomorphism

$$\zeta = \left(P \otimes \mathcal{R}(V)) = P \otimes (V \otimes H) \xrightarrow{\Phi^{-1}} (P \otimes V) \otimes H \xrightarrow{\tau} \mathcal{R}({}_{\mathrm{ad}}(P \otimes V))\right).$$

Hence we can define $\diamondsuit$ by $P\diamondsuit V = {}_{\mathrm{ad}}(P \otimes V)$, in particular we are in Situation 4.1, and if we identify ${}_H\mathcal{M}_H$ with ${}_{H\otimes H^{\mathrm{op}}}\mathcal{M}$, then we are more precisely in the situation of Remark 4.6. If H is finitely generated projective over the base ring k, we conclude that $(C\diamondsuit H)^\vee = H\#H^*$ with multiplication given by

$$(g\#\varphi)(h\#\psi) = gh_{(1)(2)}\omega^{(3)}\#(\omega^{(5)} \rightharpoonup \psi \leftharpoonup \omega^{(1)})(\omega^{(4)}S(h_{(2)}) \rightharpoonup \varphi \leftharpoonup h_{(1)(1)}\omega^{(2)})$$

for $g, h \in H$ and $\varphi, \psi \in H^*$, where we have denoted by $\omega = \omega^{(1)} \otimes \ldots \otimes \omega^{(5)} \in H^{\otimes 5}$ the element with $\Psi(p \otimes q \otimes v) = \omega^{(1)}p\omega^{(5)} \otimes \omega^{(2)}q\omega^{(4)} \otimes \omega^{(3)}v$, and where $\rightharpoonup$ and $\leftharpoonup$ denote the transposes of the natural actions of H on itself.

There is one last thing to do, namely to determine the element ω explicitly. This is done in [29, Sec. 4], and we merely cite the *ansatz* and the result: The isomorphism Ψ is defined by commutativity of

$$\begin{tikzcd}
\mathcal{R}((P\otimes Q)\diamondsuit V) \arrow[r, "\mathcal{R}(\Psi)"] \arrow[dd, "\zeta"] & \mathcal{R}(P\diamondsuit(Q\diamondsuit V)) \arrow[d, "\zeta"] \\
 & P\otimes\mathcal{R}(Q\diamondsuit V) \arrow[d, "P\otimes\zeta"] \\
(P\otimes Q)\otimes\mathcal{R}(V) \arrow[r, "\Phi"] & P\otimes(Q\otimes\mathcal{R}(V))
\end{tikzcd},$$

and the element ω is given by

$$\omega = (1\otimes 1\otimes 1\otimes\tau(\mathsf{f}^{-1}))\cdot(H\otimes\Delta\otimes S\otimes S)(\chi)\cdot(\phi\otimes 1\otimes 1).$$

where $\chi = (\phi\otimes 1)\cdot(\Delta\otimes H\otimes H)(\phi)$ and

$$\mathsf{f} = S(\phi^{(-1)}{}_{(2)})\gamma^{(1)}(\phi^{(-2)}\beta\phi^{(-3)})_{(1)}\otimes S(\phi^{(-1)}{}_{(1)})\gamma^{(2)}(\phi^{(-2)}\beta\phi^{(3)})_{(2)}$$

with

$$\delta := \phi^{(-1)}\beta S(\phi^{(-3)}{}_{(2)}\phi^{(3)})\otimes\phi^{(-2)}\phi^{(1)}\beta S(\phi^{(-3)}{}_{(1)}\phi^{(2)})$$
$$\gamma := S(\phi^{(-1)}\phi^{(2)})\alpha\phi^{(-2)}\phi^{(3)}{}_{(1)}\otimes S(\phi^{(1)})\alpha\phi^{(-3)}\phi^{(3)}{}_{(2)}.$$

Example 4.11. Hausser and Nill have constructed the Drinfeld double of a quasi-Hopf algebra as an example of their diagonal crossed product construction [8]. A diagonal crossed product $H^*\bowtie_\delta A$ of an algebra A with the dual of a quasi-Hopf algebra H is formed using a so-called two-sided H-coaction (δ,ψ) of H on A, consisting by definition of a map $\delta\colon A\to H\otimes A\otimes H$, and an invertible element $\psi\in H\otimes H\otimes A\otimes H\otimes H$, subject to a set of axioms. In the case of a Drinfeld double, $A=H$, the map δ is built from comultiplication in H, and ψ is built from the coassociator ϕ of H. We will not give complete details on the general diagonal crossed product construction. However, in Section 9 of [8] give a "representation theoretic interpretation" of their two-sided coaction, which is very close to the ideas exploited in the present paper. In [8] this interpretation is used to gain a better understanding of the complicated axioms for a two-sided coaction. We will now discuss (without computational details) how to use it to construct diagonal crossed products through our general scheme.

The representation-theoretic interpretation of the axioms of a two-sided coaction is a functor

$${}_H\mathcal{M}\times{}_A\mathcal{M}\times{}_H\mathcal{M}\to{}_A\mathcal{M}$$

mapping (X,V,Y) to $X\otimes V\otimes Y$ with the A-module structure induced by δ. The axioms on δ and ψ are such that the triple tensor product is associative in a suitable sense, coherently with the associativity constraints within the two copies of ${}_H\mathcal{M}$. In our language, ${}_A\mathcal{M}$ is thus shown in [8, Sec. 9] to be a ${}_H\mathcal{M}\times({}_H\mathcal{M})^{\mathrm{sym}}$-category. We note that $({}_H\mathcal{M})^{\mathrm{sym}}\cong\mathcal{M}_{H^{\mathrm{opcop}}}$. Further, we note that by results already proved in Drinfeld's paper [6] introducing quasi-Hopf algebras, the antipode of H is a quasi-Hopf algebra homomorphism $H^{\mathrm{opcop}}\to H$, up to a twist. This means that $\mathcal{M}_{H^{\mathrm{opcop}}}$ and $\mathcal{M}_H$ are equivalent monoidal categories. Hence we find that ${}_A\mathcal{M}$ is a left ${}_H\mathcal{M}\times\mathcal{M}_H$-category. From this in turn it is easy to deduce that ${}_A\mathcal{M}$ is a ${}_H\mathcal{M}_H$-category. We are then in a situation where our results apply directly to the coalgebra H in the monoidal category ${}_H\mathcal{M}_H$ to yield an A-coring $H\diamondsuit A$, and a dual algebra $(H\diamondsuit A)^\vee\cong A\otimes H^*$.

Example 4.12. In the Example 4.10 we used a generalization of the category equivalence ${}_{D(H)}\mathcal{M} \cong {}^H_H\mathcal{M}^H_H$ for a finitely generated projective Hopf algebra H. Another, more direct way of viewing Hopf modules in ${}^H_H\mathcal{M}^H_H$ as modules was studied by Cibils and Rosso [4], who constructed an algebra structure on $H \otimes H \otimes H^* \otimes H^*$ such that ${}^H_H\mathcal{M}^H_H \cong {}_{H\otimes H\otimes H^*\otimes H^*}\mathcal{M}$.

We will point out now how such constructions (even more generally for quasibialgebras) fit into our picture. For the rest of this example we let $H = (H, \phi)$ be a quasibialgebra, finitely generated projective over the base ring k.

As above, we can view H as a coalgebra in the monoidal category ${}_H\mathcal{M}_H$ of H-H-bimodules. We can now arrive directly in a situation where we can apply Theorem 3.3 by setting $\mathcal{C} := {}_H\mathcal{M}_H \times ({}_H\mathcal{M}_H)^{\mathrm{sym}}$, where $(-)^{\mathrm{sym}}$ denotes reversing the order of tensor products in an abstract monoidal category. Then $C := (H, H)$ is a coalgebra in $\mathcal{C}$. Now since ${}_H\mathcal{M}_H$ is a monoidal category (with tensor product over k!), it is easy to check that $({}_H\mathcal{M}_H, \diamondsuit, \Psi)$ is a left $\mathcal{C}$-category with $(P, Q)\diamondsuit M = P \otimes (M \otimes Q)$, and

$$\Psi\colon ((P, Q) \otimes (P' \otimes Q'))\diamondsuit M \to (P, Q)\diamondsuit((P', Q')\diamondsuit M)$$

given by the coherence isomorphism

$$(P \otimes P') \otimes (M \otimes (Q' \otimes Q)) \to P \otimes ((P' \otimes (M \otimes Q')) \otimes Q)$$

in the monoidal category ${}_H\mathcal{M}_H$. We only have to identify ${}_H\mathcal{M}_H = {}_{H\otimes H^{\mathrm{op}}}\mathcal{M}$ to conclude

$${}^H_H\mathcal{M}^H_H \cong {}^{(H,H)}\left({}_{H\otimes H^{\mathrm{op}}}\mathcal{M}\right) \cong {}^{(H,H)\diamondsuit(H\otimes H^{\mathrm{op}})}\mathcal{M} \cong {}_{((H,H)\diamondsuit(H\otimes H^{\mathrm{op}}))^\vee}\mathcal{M}.$$

The algebra $((H, H)\diamondsuit(H \otimes H^{\mathrm{op}}))^\vee$ is modelled on the k-module $(H \otimes H)^* \otimes H \otimes H^{\mathrm{op}}$.

Here is a slightly modified way of viewing the construction: We can first analyze the category

$${}_H\mathcal{M}^H_H = ({}_H\mathcal{M}_H)^H \cong {}^C\left({}_{H\otimes H^{\mathrm{op}}}\mathcal{M}\right) \cong {}^{C\diamondsuit(H\otimes H^{\mathrm{op}})}\mathcal{M} \cong {}_{(C\diamondsuit(H\otimes H^{\mathrm{op}}))^\vee}\mathcal{M}$$

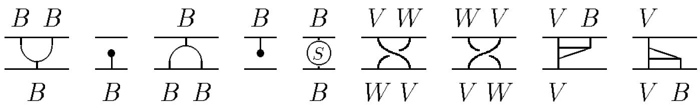
where we have used that ${}_H\mathcal{M}_H$ is a right ${}_H\mathcal{M}_H$-category, hence a left $({}_H\mathcal{M}_H)^{\mathrm{sym}}$-category, and denoted by C the coalgebra H in ${}_H\mathcal{M}_H$ considered as a coalgebra in $({}_H\mathcal{M}_H)^{\mathrm{sym}}$. The algebra $R := (C\diamondsuit(H \otimes H^{\mathrm{op}}))^\vee$ is modelled on the k-module $H^* \otimes H \otimes H$; such an algebra whose modules classify Hopf modules in ${}_H\mathcal{M}^H_H$ was announced by Hausser and Nill in [10]. We can continue from here: ${}_R\mathcal{M}$ is a left ${}_H\mathcal{M}_H$-category since ${}_H\mathcal{M}^H_H$ is, and using again that H is a coalgebra in ${}_H\mathcal{M}_H$, we find

$${}^H_H\mathcal{M}^H_H = {}^H\left({}_H\mathcal{M}^H_H\right) = {}^H\left({}_R\mathcal{M}\right) \cong {}^{H\diamondsuit R}\mathcal{M} \cong {}_{(H\diamondsuit R)^\vee}\mathcal{M},$$

with $(H\diamondsuit R)^\vee \cong R \otimes H^* \cong H \otimes H \otimes H^* \otimes H^*$ as k-modules.

By [27] both the algebra R we have used as an intermediate step and the final algebra $(H\diamondsuit R)^\vee$ have $\times_H$-bialgebra structures due to the fact that the categories ${}_H\mathcal{M}^H_H$ and ${}^H_H\mathcal{M}^H_H$ are monoidal categories with respect to tensor products over H. As observed by Hausser and Nill [10] this shows that the category of representations of the quasi-Hopf algebra H, which has a non-coassociative comultiplication, can also be viewed, through the equivalence ${}_H\mathcal{M} \cong {}_H\mathcal{M}^H_H$, as the category of representations of a $\times_H$-bialgebra, which has a coassociative comultiplication.

Example 4.13. Assume that our base category $\mathcal{B}$ is braided monoidal with braiding σ. Let B be a Hopf algebra in $\mathcal{B}$. The usual structure theorems for Hopf modules have been generalized to this case by Bespalov and Drabant [1]. We use graphical notations to recall them, writing

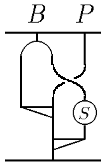


for multiplication, unit, comultiplication, counit, antipode, braiding, inverse braiding, a right module structure, and a right comodule structure, respectively. Bespalov and Drabant have shown that there are category equivalences $\mathcal{B} \cong \mathcal{B}^B_B$, sending $V \in \mathcal{B}$ to $V \otimes B \in \mathcal{B}^B_B$, with the module and comodule structure induced by the right tensor factor. Moreover, this category equivalence induces an equivalence ${}^B_B\mathcal{YDB} \cong {}^B_B\mathcal{B}^B_B$, where ${}^B_B\mathcal{YDB}$ is the category of all left-left Yetter-Drinfeld B-modules within $\mathcal{B}$. The equivalence sends V to $V \otimes B$ with the diagonal, resp. codiagonal, left module and comodule structures. In the case where $\mathcal{B}$ is the category of vector spaces and B is finite dimensional, the category of Yetter-Drinfeld modules is isomorphic to the category of modules over the Drinfeld double, so it would be desirable to find a Hopf algebra in $\mathcal{B}$ whose module category is ${}^B_B\mathcal{YDB}$.

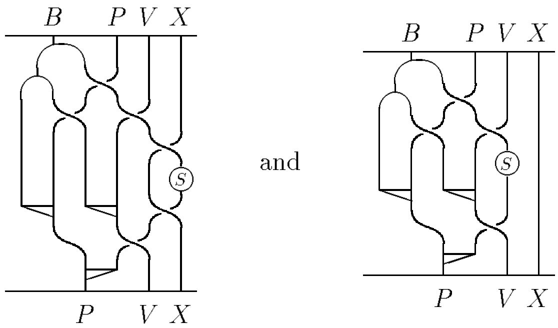
We attempt to apply our general scheme. To do so, we consider the category ${}_B\mathcal{B}_B$ of B-B-bimodules in $\mathcal{B}$, which becomes monoidal when we endow the tensor product in $\mathcal{B}$ of two B-B-bimodules with the diagonal left and right module structures. B is an algebra in ${}_B\mathcal{B}_B$, and we can view ${}^B_B\mathcal{B}^B_B$ as the category of left B-comodules in the category ${}_B\mathcal{B}^B_B$ of right B-comodules in $\mathcal{B}$. Now ${}_B\mathcal{B}^B_B \cong {}_B\mathcal{B}$, hence the right hand category is naturally a ${}_B\mathcal{B}_B$-category, and

$$ {}^B_B\mathcal{B}^B_B \cong {}^B\left({}_B\mathcal{B}^B_B\right) \cong {}^B\left({}_B\mathcal{B}\right). $$

Next, we will have to find explicitly the ${}_B\mathcal{B}_B$-category structure $(\diamondsuit, \Psi)$ of ${}_B\mathcal{B}$. Using the results in [1] it is not hard to verify that for each $P \in {}_B\mathcal{B}_B$ the category equivalence ${}_B\mathcal{B}^B_B \cong {}_B\mathcal{B}$ takes the Hopf module ${}_{.}P_{.} \otimes {}_{.}B^{.}_{.}$ to ${}_{\mathrm{ad}}P \otimes {}_{.}B^{.}_{.}$, where ${}_{\mathrm{ad}}P \in {}_B\mathcal{B}$ denotes P with the adjoint module structure



and thus we can take $P \diamondsuit V := {}_{\mathrm{ad}}(P \otimes V)$ for all $P \in {}_B\mathcal{B}_B$ and $V \in {}_B\mathcal{B}$. Nevertheless, we have reached a dead end: In order to apply Theorem 3.3 and its corollaries, we would have to verify that the ${}_B\mathcal{B}_B$-category structure of ${}_B\mathcal{B}$ is compatible with its right $\mathcal{B}$-category structure. In other words, we would have to have ${}_{\mathrm{ad}}(P \otimes V \otimes X) = {}_{\mathrm{ad}}(P \otimes V) \otimes X$ as left B-modules. The module structures of both sides have the respective graphical representations



which shows that they cannot be expected to agree if the braiding of $\mathcal{B}$ is not a symmetry.

On the other hand, if $\mathcal{B}$ *is* symmetric, then the left ${}_B\mathcal{B}_B$-category structure of ${}_B\mathcal{B}$ is compatible with its right $\mathcal{B}$-category structure. We find that ${}^B_B\mathcal{B}^B_B \cong {}^{B\Diamond B}\mathcal{B}$ for some B-coring $B\Diamond B$ in $\mathcal{B}$, and if B has a dual object in $\mathcal{B}$, then we can proceed to conclude that ${}^B_B\mathcal{B}^B_B \cong {}_{(B\Diamond B)^\vee}\mathcal{B}$, where $(B\Diamond B)^\vee \cong B\otimes B^*$ is a legitimate counterpart of Drinfeld's double within $\mathcal{B}$.

The serious obstacles to the construction of a Drinfeld double in the nonsymmetric braided case were observed by Majid [17], who also provided a fruitful "solution" which works in the case where $\mathcal{B}$ happens to be the category of left H-modules over some (necessarily quasitriangular) Hopf algebra.

In this case we have ${}_B\mathcal{B} \cong {}_{B\#H}\mathcal{M}$ for the smash product algebra $B\#H$, and of course the left ${}_B\mathcal{B}_B$-category structure of ${}_{B\#H}\mathcal{M}$ is compatible with its right $\mathcal{M}_k$-category structure. Hence

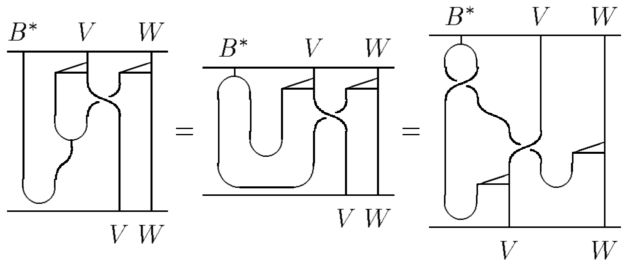
$$ {}^B_B\mathcal{B}^B_B \cong {}^B({}_B\mathcal{B}) \cong {}^B({}_{B\#H}\mathcal{M}) \cong {}^{B\Diamond(B\#H)}\mathcal{M} \cong {}_{(B\Diamond(B\#H))^\vee}\mathcal{M} $$

where the last step uses the assumption that B be finitely generated projective over k; the dual algebra of the $B\#H$-coring $B\Diamond(B\#H)$ is modelled on the k-module $B\otimes H\otimes B^*$, so it has a "triangular decomposition". We will not give any more details on its structure, but rather sketch a modified approach, which we will generalize in the next example.

We consider the category ${}^B\mathcal{B}^B$ of B-B-bicomodules in $\mathcal{B}$. This is a monoidal category with the tensor product of two bicomodules defined to be their tensor product in $\mathcal{B}$, with the two codiagonal comodule structures.

We assume that B has a left dual B^* in $\mathcal{B}$ and consider the dual Hopf algebra structure on B^*. Then $\mathcal{C} := {}^B\mathcal{B}^B$ is the category of B^*-B-dimodules in $\mathcal{B}$, that is, left B^*-modules and right B-comodules such that the structures "commute". As to the description of the tensor product in ${}^B\mathcal{B}^B$ in terms of left B^*-module structures, we find, for $V, W \in {}^B\mathcal{B}$, the B^*-module structure on the tensor product

B-comodule $V \otimes W$ to be



If we define C to be B^* with the opposite comultiplication defined by the inverse of the braiding, then C is a bialgebra in the category $\mathcal{B}$ with the inverse braiding, and the above calculation shows that the tensor product of left B-comodules corresponds to the tensor product of left C-modules (defined with the inverse braiding, with respect to which C is a bialgebra).

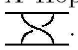
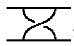
Now ${}^B\mathcal{B}^B_B \cong {}^B\mathcal{B} \cong {}_C\mathcal{B}$. Since the left hand side is naturally a left $\mathcal{C}$-category, so is ${}_C\mathcal{B}$, with structure $\diamondsuit$, and ${}^B_B\mathcal{B}^B_B \cong {}_B\left({}^B\mathcal{B}^B_B\right) \cong {}_B\left({}_C\mathcal{B}\right) \cong {}_B\left({}_{C\#H}\mathcal{M}\right) \cong {}_{B\diamondsuit(C\#H)}\mathcal{M}$.

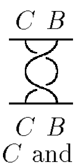
The core of the generalization below (which is also essentially covered by Majid's paper [17]) is that we do not need the full strenght of the assumptions made above to carry it through. First, we do not need $\mathcal{B}$ to be braided. All we need to consider a Hopf algebra B is the instance σ_{BB} of the braiding. All we need to consider Hopf modules in ${}^B\mathcal{B}^B_B$ are the instances σ_{BM} of the braiding for objects $M \in \mathcal{B}$. All we need to say that C is a Hopf algebra in $\mathcal{B}$ with respect to the inverse braiding is the instance σ^{-1}_{CC} of that inverse. All we need to consider the tensor product of C-modules are the instances σ^{-1}_{CV} of the inverse braiding for $V \in \mathcal{B}$. Finally, we also do not need C to be the dual of the rigid object B. All we need is a pairing between C and B, compatible in a suitable way with multiplications and comultiplications.

Example 4.14. We let H be a k-bialgebra, and $\mathcal{B} := {}_H\mathcal{M}$.

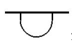
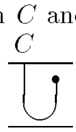
Recall that the *weak center* of $\mathcal{B}$, is a monoidal category whose objects are by definition pairs $(V, \sigma_{V,-})$ in which V is an object of $\mathcal{B}$, and $\sigma_{VX}\colon V \otimes X \to X \otimes V$ is natural in $X \in \mathcal{B}$ and satisfies $\sigma_{V,X\otimes Y} = (X \otimes \sigma_{VY})(\sigma_{VX} \otimes Y)$ for all $X, Y \in \mathcal{B}$ as well as $\sigma_{V\mathsf{b}} = \mathrm{id}_V$. The weak center is a prebraided (i. e. braided without the assumption that the braiding is an isomorphism) monoidal category. We refer the reader to Kassel's book [13] for more information on the center construction.

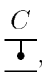
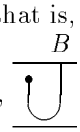
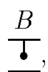
After the discussion at the end of the preceding example, it will hopefully appear natural to assume given the following data and axioms

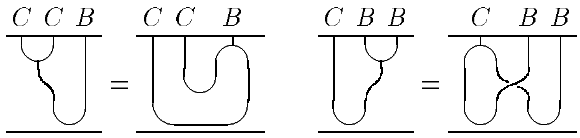
- A Hopf algebra $(B, \sigma_{B,-})$ in the left weak center of $\mathcal{B}$. We write $\sigma_{BV} =$ .
- A bialgebra $(C, \kappa_{C,-})$ in the left weak center of $\mathcal{B}$. We write $\kappa_{CV} =$ , and assume



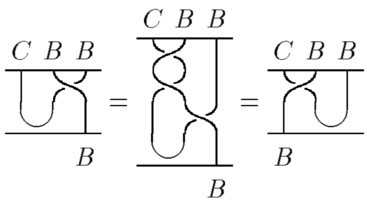
$= \mathrm{id}\,.$

- A "skew pairing" between C and B, that is, a morphism $C \otimes B \to \mathsf{b}$ in $\mathcal{B}$, depicted , such that 

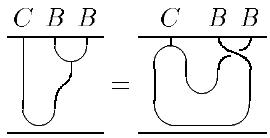
$=$ ,  $=$ ,



We note that the axioms imply

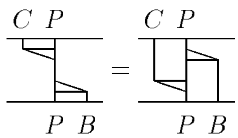


and

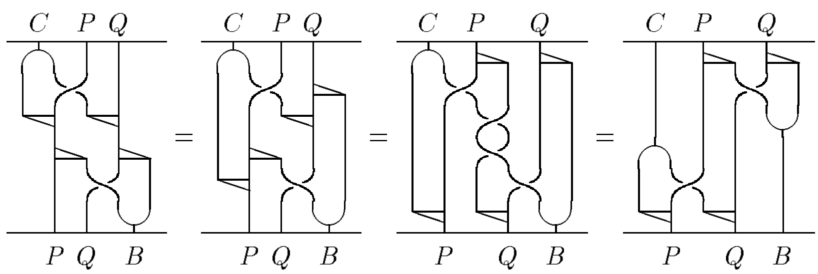


We will continue to be led by the idea that C is a replacement for the dual of B. In ${}^B_B\mathcal{B}^B_B$ we want to replace the left comodule structure by a C-module structure. We denote the result (to be defined below) by ${}^{C\downarrow}_{B}\mathcal{B}^B_B$. The idea behind this admittedly awkward notation is that ${}^{C\downarrow}\mathcal{B} = {}_C\mathcal{B}$, that is, the arrow indicates that C "should go down", whereas the position of C still indicates the type of compatibility condition the module structure should satisfy with respect to the B-(co)module structures, for example ${}^{C\downarrow}\mathcal{B}^B$ should be defined by the compatibility condition dual to that of a bicomodule, whereas ${}^{C\downarrow}_{B}\mathcal{B}$ should be defined by the compatibility condition dual to that of a Hopf module.

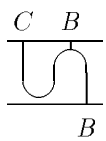
We shall get more formal now. We define ${}^{C\downarrow}\mathcal{B}^B$ to be the category of C-B-dimodules, that is, left C-modules and right B-comoudules P satisfying



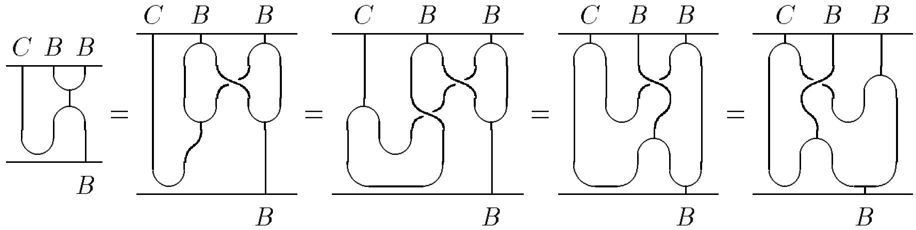
${}^{C\downarrow}\mathcal{B}^B$ is a monoidal category, with the tensor product of $P, Q \in {}^{C\downarrow}\mathcal{B}^B$ defined as their tensor product in $\mathcal{B}$, with the usual diagonal left C-module structure (defined using $\kappa_{C,-}$), and the usual codiagonal right B-comodule structure (defined using $\sigma_{B,-}$. In fact one checks



B is an algebra in ${}^{C\downarrow}\mathcal{B}^B$ with the coregular right comodule structure, and the C-module structure

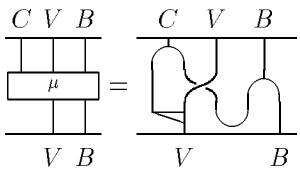


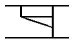
In fact B-colinearity of multiplication in B holds by definition of a Hopf algebra, and

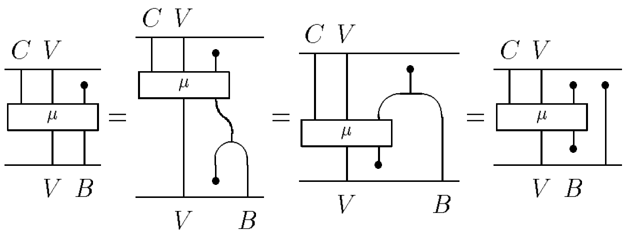


shows that multiplication in B is also C-linear.

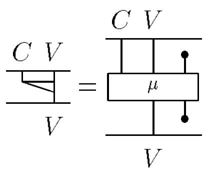
We define ${}^{C\downarrow}\mathcal{B}_B^B$ and ${}^{C\downarrow}_{B}\mathcal{B}_B^B$ to be the categories of right B-modules, respectively B-B-bimodules in the monoidal category ${}^{C\downarrow}\mathcal{B}^B$. We have a functor ${}^{C\downarrow}\mathcal{B}^B \to {}^{C\downarrow}\mathcal{B}_B^B$ assigning to P the free right B-module $P \otimes B$ over P, and in particular a functor ${}_C\mathcal{B} = {}^{C\downarrow}\mathcal{B} \to {}^{C\downarrow}\mathcal{B}_B^B$. We claim that the latter is an equivalence. By the structure theorem on Hopf modules in $\mathcal{B}_B^B$, this means that for an object $V \in \mathcal{B}$, all C-module structures μ on $V \otimes B \in \mathcal{B}_B^B$ making it an object in ${}^{C\downarrow}\mathcal{B}_B^B$ are of the form



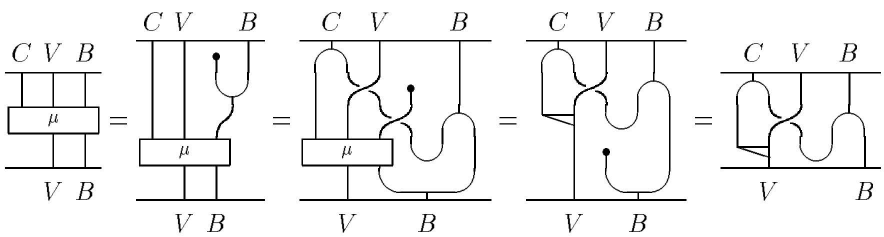
for some left C-module structure  on V. Now the calculation



shows that the split monic $V \otimes \eta \colon V \to V \otimes B$ makes V a submodule of $V \otimes B$, with module structure given by



and then

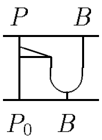


shows that the C-module structure of $V \otimes B$ is of the claimed form.

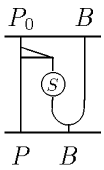
Now ${}^{C\downarrow}\mathcal{B}^B_B$ is a left ${}^{C\downarrow}\mathcal{B}^B$-category, and ${}^{C\downarrow}_{B}\mathcal{B}^B_B \cong {}_B\left({}^{C\downarrow}\mathcal{B}^B_B\right)$. Also, the category ${}_C\mathcal{B}$, being equivalent to ${}^{C\downarrow}\mathcal{B}^B_B$, is also a left ${}^{C\downarrow}\mathcal{B}^B$-category, whose structure we denote by $(\diamondsuit, \Psi)$, and

$$ {}^{C\downarrow}_{B}\mathcal{B}^B_B \cong {}_B\left({}^{C\downarrow}\mathcal{B}^B_B\right) \cong {}_B\left({}_C\mathcal{B}\right) \cong {}_B\left({}_{C\#H}\mathcal{M}\right) \cong {}_{B\diamondsuit(C\#H)}\mathcal{M}. $$

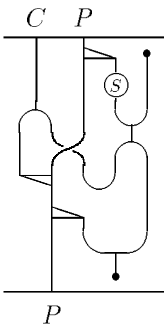
Our next task is to determine the relevant ${}^{C\downarrow}\mathcal{B}^B$-category structure. If $P \in \mathcal{B}^B$, and P_0 is P with the trivial B-comodule structure, then



and



are mutually inverse isomorphisms in the category $\mathcal{B}^B_B$. Hence for $P \in {}^{C\downarrow}\mathcal{B}^B$ we have $P \otimes B \cong {}_{\mathrm{ad}}P \otimes B$ where ${}_{\mathrm{ad}}P$ is P endowed with the trivial B-module structure and the "adjoint" C-module structure



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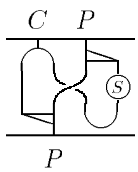
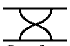
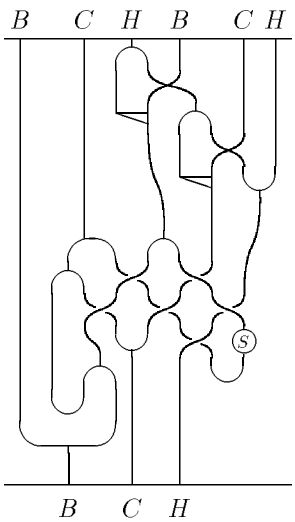


We can conclude that the ${}^{C\downarrow}\mathcal{B}^B$-category structure of ${}_C\mathcal{B}$ is given by $P\diamondsuit V = {}_{\mathrm{ad}}(P \otimes V)$; we will skip showing that the associator morphism Ψ is trivial.

We are ready to reap the algebra structure of $B\diamondsuit(C\#H) \cong B \otimes C \otimes H$. Writing  for the ordinary flip of tensor factors in a tensor product of k-modules, we find

$\nabla_{B\diamondsuit(C\#H)} =$



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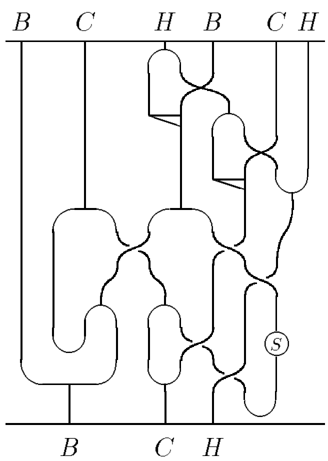
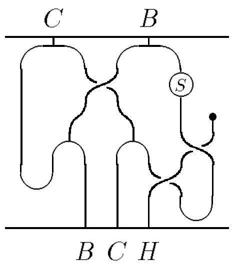


The left hand form of multiplication is a direct application of the general results: First, the left H acts on the right $B \otimes C \otimes H$, then the left hand C acts (by the adjoint action), which completes the action of $C\#H$ on $B \otimes C \otimes H$, and is followed by multiplication in B. The right hand side is merely rearranged a little to have less crossings, using coassociativity of C. As we mentioned above, this multiplication in $B \otimes C \otimes H$ is a version of Majid's "double-bosonization". We have obtained here the form treated in [7], with switched sides. We should note that the order of tensor factors is not essential if H is a Hopf algebra, since then the "right smash product" $C\#H$ can be written as a "left smash product" $H\#C$.

We can view the formula for multiplication in $D := D(B, C) := B\diamondsuit(C\#H)$ as follows: We have algebra inclusions $\iota_X : X \to D$ for $X \in \{B, C, H\}$, given by inserting units in the remaining two places. Denoting by $\nabla_D^{(2)} : D \otimes D \otimes D \to D$ twofold multiplication, we have $\nabla_D^{(2)}(\iota_B \otimes \iota_C \otimes \iota_H) = \mathrm{id}_D$. The multiplication within D is then described by giving "commutation relations" between the three factors. In particular the commutation relations between B and H and between C and H are such that we have algebra inclusions $B\#H \to D$ and $C\#H \to D$. The commutation relation between C and B produces a factor in H:

(4.5) $\nabla_D(\iota_C \otimes \iota_B) =$



Example 4.15. Although this was already explained in [17], we would like to point out once more, in our specific setup, how quantized enveloping algebras with their triangular decomposition occur as a special case of "double-bosonization".

We use the notations of Lusztig's book [14]. In particular, we let $(I, \cdot)$ be a Cartan datum, that is, I a finite set and $\cdot : \mathbb{Z}I \times \mathbb{Z}I \to \mathbb{Z}$ a symmetric bilinear form such that $i \cdot i \in 2\mathbb{N}$ and $2\frac{i \cdot j}{i \cdot i} \in -\mathbb{N}_0$ for all $i, j \in I$.

We fix the base field $k = \mathbb{Q}(v)$, the function field in one variable over the rationals, and put $v_i = v^{i \cdot i/2}$ for all $i \in I$. The category $\mathcal{G}$ of $\mathbb{Z}I$-graded vector spaces is braided with the braiding $a \otimes b \mapsto v^{|a|\,|b|} b \otimes a$. Lusztig's algebra $\mathbf{f}$ with its comultiplication r is a Hopf algebra in $\mathcal{G}$. Let $\overline{\mathbf{f}}$ be $\mathbf{f}$ with the braided opposite comultiplication formed by the inverse of the braiding, which is denoted $\overline{r}$ in [14], is a Hopf algebra in the category of $\mathbb{Z}I$-graded vector spaces with the inverse braiding; we shall denote this category by $\overline{\mathcal{G}}$. Lusztig defines a bilinear form $(,) : \mathbf{f} \otimes \mathbf{f} \to k$ which is a coalgebra and an algebra pairing of $\mathbf{f}$ with itself, satisfying $(\theta_i, \theta_j) = \delta_{ij}(1 - v_i^{-2})^{-1}$ for the standard generators θ_i with $i \in I$. In addition he defines an anti-multiplicative and anti-comultiplicative automorphism σ of $\mathbf{f}$ fixing the generators θ_i. If we define another pairing $\overline{\mathbf{f}} \otimes \mathbf{f} \to k$ as sending $a \in \overline{\mathbf{f}}$ and $b \in \mathbf{f}$ to $(a, \sigma(b))$, then this is compatible with the grading (so a morphism in $\mathcal{G}$), and satisfies the conditions on a skew pairing in the preceding example.

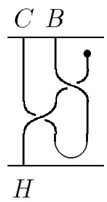
Now we also fix a root datum of type $(I, \cdot)$. That is, we let Y, X be free abelian groups, and $\langle , \rangle \colon Y \times X \to \mathbb{Z}$ a perfect bilinear pairing. We assume to be given inclusions $I \ni i \mapsto i' \in X$ and $I \ni i \mapsto i \in Y$, which we extend to group homomorphisms from $\mathbb{Z}I$ with the same notations, such that $\langle i, j' \rangle = 2\frac{i \cdot j}{i \cdot i}$ for all $i, j \in I$.

In addition, we define a group homomorphism $\mathbb{Z}I \ni \nu \mapsto \tilde{\nu} \in Y$ by $\tilde{i} = \frac{i \cdot i}{2} i$ for $i \in I$. We note that this entails $\langle \tilde{\mu}, \nu' \rangle = \mu \cdot \nu$ for all $\mu, \nu \in \mathbb{Z}I$.

We let H be the group algebra of Y, denoting the canonical grouplike corresponding to $\mu \in Y$ by K_μ.

Then we can define a braided monoidal functor $\mathcal{G} \to {}^H_H\mathcal{YD}$ and $\overline{\mathcal{G}} \to {}^H_H\mathcal{YD}$ as follows: To $A \in \mathcal{G}$ we assign the underlying vector space of A with the left H-module structure $K_\mu \cdot a := v^{\langle \mu, |a|' \rangle} a$ and the comodule structure $a \mapsto \tilde{K}_{|a|} \otimes a$, where $\tilde{K}_\mu := K_{\tilde{\mu}}$. To $A \in \overline{\mathcal{G}}$ we assign A with the same H-comodule structure as before, but the module structure $K_\mu \cdot a := v^{-\langle \mu, |a|' \rangle} a$. We take $B \in {}^H_H\mathcal{YD}$ to be the image of $\mathbf{f} \in \mathcal{G}$, and $C \in {}^H_H\mathcal{YD}$ the image of $\overline{\mathbf{f}} \in \overline{\mathcal{G}}$, respectively.

If we put $E_i := \iota_B(\theta_i)$ for $i \in I$ and $X_i := \iota_C(\theta_i)$, then the relations in $D(B, C)$ are, first of all, those of smash products $B \# H$ and $C \# H$, that is, $K_\mu E_i = (K_\mu \cdot E_i) K_\mu = v^{\langle \mu, i' \rangle} E_i K_\mu$, and $K_\mu X_i = v^{-\langle \mu, i' \rangle} X_i K_\mu$. In addition, there is the commutation relation (4.5) between C and B. Since the elements θ_i are primitive, the image of $Y_i \otimes E_j$ has, at first sight, nine summands (there are double comultiplications on each factor). However, since the pairing between C and B preserves the grading, only three summands remain. We note that

$\mathrm{ev}' :=$ 

is graded, maps $1 \otimes 1$ to 1, and $Y_i \otimes E_j$ to $\delta_{ij} \tilde{K}_i^2 (1 - v_i^{-2})^{-1}$, since the braidings between B and H as well as between C and H map $a \otimes h$ to $\tilde{K}_{|a|} h \otimes a$. Finally the antipode of $\mathbf{f}$ maps θ_i to $-\theta_i$, so we finally obtain the formula

$$\begin{aligned} X_i E_j &= \delta_{ij}(1 - v_i^{-2})^{-1} + v^{\langle \tilde{i}, j' \rangle} E_j X_i - \delta_{ij}(1 - v_i^{-2})^{-1} \tilde{K}_i^2 \\ &= v^{i \cdot j} E_j X_i - \delta_{ij} \frac{\tilde{K}_i^2 - 1}{1 - v_i^{-2}} \end{aligned}$$

Now putting $F_i := v_i X_i \tilde{K}_i^{-1} = v_i^{-1} \tilde{K}_i^{-1} X_i$, we conclude

$$\begin{aligned} F_i E_j &= v_i X_i \tilde{K}_i^{-1} E_j = v^{-i \cdot j} v_i X_i E_j \tilde{K}_i^{-1} = v_i E_j X_i \tilde{K}_i^{-1} - \delta_{ij} v^{-i \cdot i} v_i \frac{\tilde{K}_i^2 - 1}{1 - v_i^{-2}} \tilde{K}_i^{-1} \\ &= E_j F_i - \delta_{ij} \frac{\tilde{K}_i - \tilde{K}_i^{-1}}{v_i - v_i^{-1}} \end{aligned}$$

and have regained the familiar form of the defining relation of Lusztig's version $\mathbf{U}$ of a quantum enveloping algebra.

Example 4.16. Let (H, ϕ) be a coquasibialgebra, K a Hopf algebra, and $\iota \colon K \to H$ an injective bialgebra map (treated as inclusion below) such that $\phi(\iota \otimes H \otimes H) = \varepsilon$. Then H is an ordinary left K-module coalgebra. We assume that H, K, and the quotient coalgebra and left module $Q := H/K^+H$ are flat k-modules, and that H

is a faithfully coflat K-Galois coextension of Q, that is, see [31], the canonical map $H \otimes K \to H \Box_Q H$ is a bijection and H is faithfully coflat as a Q-comodule. As a consequence [31]

$$^{Q}\mathcal{M} \ni V \mapsto H \underset{Q}{\Box} V \in {}^{H}_{K}\mathcal{M}$$

is a category equivalence. The quasi-inverse equivalence is given by

$${}^{H}_{K}\mathcal{M} \ni M \mapsto \overline{M} := K\backslash M := M/K^{+}M \in {}^{Q}\mathcal{M}.$$

By the assumption on ϕ, we can view K as an algebra in the category ${}^{H}\mathcal{M}$, and ${}^{H}_{K}\mathcal{M}$ as the category of left K-modules in ${}^{H}\mathcal{M}$. Thus ${}^{H}_{K}\mathcal{M}$ is naturally a right ${}^{H}\mathcal{M}$-category, and by restriction a right ${}^{K}\mathcal{M}$-category, hence so is ${}^{Q}\mathcal{M}$. We view the right ${}^{K}\mathcal{M}$-category structures as left $\left({}^{K}\mathcal{M}\right)^{\mathrm{sym}}$-category structures, and observe $\left({}^{K}\mathcal{M}\right)^{\mathrm{sym}} \cong {}^{K^{\mathrm{op}}}\mathcal{M}$. Denote the ${}^{K^{\mathrm{op}}}\mathcal{M}$-category structure of ${}^{Q}\mathcal{M}$ by $(\diamondsuit, \Psi)$, then we find

$$X \diamondsuit V = K\backslash((H \underset{Q}{\Box} V) \otimes X) \cong K\backslash(H \underset{Q}{\Box} V) \otimes X \cong V \otimes X \cong X \otimes V,$$

with the Q-comodule structure given by $x \otimes v \mapsto v_{(-1)} \cdot x_{(-1)} \otimes x_{(0)} \otimes v_{(0)}$ for $v \in V \in {}^{Q}\mathcal{M}$ and $x \in X \in {}^{K}\mathcal{M}$, where the dot indicates the right action of K on Q induced by the right action of K on H. Thus we can write the left Q-comodule structure of $X \otimes V$ in the form

$$X \otimes V \xrightarrow{X \otimes \lambda} X \otimes Q \otimes V \xrightarrow{\sigma \otimes V} Q \otimes X \otimes V$$

with

$$\sigma\colon X \otimes Q \ni x \otimes q \mapsto q \cdot x_{(-1)} \otimes x_{(0)} \in Q \otimes X$$

The associator Ψ is given by

$$\begin{aligned}\Psi_{XYV}\colon (X \otimes Y) \otimes V &\to X \otimes (Y \otimes V)\\ x \otimes y \otimes v &\mapsto x_{(0)} \otimes y_{(0)} \otimes v_{(0)}\phi(v_{(-1)} \otimes y_{(-1)} \otimes x_{(-1)})\end{aligned}$$

(where by abuse of notation ϕ also denotes the map $\phi\colon Q \otimes H \otimes H \to k$ induced by $\phi \in (H \otimes H \otimes H)^{*}$.) Hence

$$\Psi_{XYV} = \left(X \otimes Y \otimes V \xrightarrow{X \otimes Y \otimes \lambda} X \otimes Y \otimes Q \otimes V \xrightarrow{\psi_{XY} \otimes V} X \otimes Y \otimes V\right)$$

with

$$\begin{aligned}\psi_{XY}\colon X \otimes Y \otimes Q &\to X \otimes Y\\ x \otimes y \otimes q &\mapsto \phi(q \otimes y_{(-1)} \otimes x_{(-1)})x_{(0)} \otimes y_{(0)}\end{aligned}$$

Now we assume in addition that K is finitely generated projective over k, so that ${}^{K^{\mathrm{op}}}\mathcal{M} \cong \mathcal{M}_{K^{*\mathrm{cop}}}$. We let $Q \ni q \mapsto q_{[-1]} \otimes q_{[-0]} \in K^{*} \otimes Q$ denote the transpose of the right action of K on Q. Then we can rewrite

$$\sigma\colon X \otimes Q \ni x \otimes q \mapsto xq_{[-1]} \otimes q_{[-0]} \in X \otimes Q.$$

We define

$$\tau\colon Q \ni q \mapsto \tau^{(1)}(q) \otimes \tau^{(2)}(q) \in K^{*} \otimes K^{*}$$

by $\tau^{(1)}(q)(x)\tau^{(2)}(q)(y) = \phi(q \otimes x \otimes y)$. Then we can rewrite

$$\psi_{XY}(x \otimes y \otimes q) = x\tau^{(2)}(q) \otimes y\tau^{(1)}(q).$$

We proceed to compute, for finite dimensional $X \in \mathcal{M}_{K^{*\mathrm{cop}}}$, the mates

$$\sigma^{\flat}\colon Q \otimes X^{*} \ni q \otimes \varphi \mapsto q_{[-1]} \cdot \varphi \otimes q_{[0]} \in X^{*} \otimes Q$$

$$\psi^\flat_{XY}: Q\otimes Y^*\otimes X^*\ni q\otimes\psi\otimes\varphi\mapsto\tau^{(1)}(q)\cdot\psi\otimes\tau^{(2)}(q)\cdot\varphi\in Y^*\otimes X^*.$$

Finally, we consider K^{op} as an algebra in the category ${}^{K^{\mathrm{op}}}\mathcal{M}\cong\mathcal{M}_{K^{*\mathrm{cop}}}$, to find

$${}^H_K\mathcal{M}_K\cong{}_{K^{\mathrm{op}}}\left({}^H_K\mathcal{M}\right)\cong{}_{K^{\mathrm{op}}}\left({}^Q\mathcal{M}\right).$$

To this setup we can apply the results from the beginning of the section, using $\mathcal{B}:=\mathcal{M}_k^{\mathrm{op}}$ as our base category.

We conclude that

$${}^H_K\mathcal{M}_K\cong{}_{K^{\mathrm{op}}\diamondsuit Q}\mathcal{M}\cong{}^{(K^{\mathrm{op}}\diamondsuit Q)^\vee}\mathcal{M}$$

where $K^{\mathrm{op}}\diamondsuit Q$ is a Q-coring in the category $\mathcal{B}$, that is, it has a multiplication $(K^{\mathrm{op}}\diamondsuit Q)\,\square_Q\,(K^{\mathrm{op}}\diamondsuit Q)\to K^{\mathrm{op}}\diamondsuit Q$, and $(K^{\mathrm{op}}\diamondsuit Q)^\vee$ is its dual coalgebra (that is, algebra in $\mathcal{B}$) according to Section 2.5. Thus $(K^{\mathrm{op}}\diamondsuit Q)^\vee\cong Q\otimes K^*$ with comultiplication given by

$$\Delta(q\otimes\varphi)=q_{(1)}\otimes q_{(2)[-1]}\tau^{(1)}(q_{(2)})\varphi_{(1)}\otimes q_{(2)[0]}\otimes\tau^{(2)}(q_{(2)})\varphi_{(2)}.$$

This coalgebra was studied extensively in [30]. If H is cocleft as K-module coalgebra, then $Q\otimes K^*$ is a coquasibialgebra due to the natural monoidal category structure of ${}^H_K\mathcal{M}_K$.

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