MATHEMATICAL STATISTICAL PHYSICS I

Course given at LMU in the SoSe 14

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C^*-algebras, states & representations (R. Helling)

MATHEMATICAL STATISTICAL PHYSICS I – SOSE14

MATHEMATICAL STRUCTURE OF PHYSICS: ALGEBRAIC APPROACH

Postulate 1. *The set of physical observables of a system is described by the set of self-adjoint elements of a C^*-algebra $\mathcal{A}$.*

Note: It is customary and mathematically easier to call the whole of $\mathcal{A}$ the algebra of observables and work with it.

Postulate 2. *The system is classical if $\mathcal{A}$ is commutative.*

Theorem 1. *Let $\mathcal{A}$ be a commutative C^*-algebra. Then there exists a unique locally compact Hausdorff space X such that $\mathcal{A}$ is isomorphic to the C^*-algebra $C_0(X)$ of continuous functions on X which vanish at infinity.*

Postulate 3. *The physical 'state' of the system is a mathematical state over $\mathcal{A}$, namely a normalized, positive linear map $\omega : \mathcal{A} \to \mathbb{C}$.*

Theorem 2. *Let X be a locally compact space. Every state ω on $C_0(X)$ is of the form*

$$\omega(f) = \int f d\mu$$

where μ is a (Baire) probability measure.

In complete generality:

Theorem 3. *Let $\mathcal{A}$ be a C^*-algebra with a unit and let ω be a state on $\mathcal{A}$. Then there exists a Hilbert space $\mathcal{H}_\omega$, a representation π_ω of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H}_\omega)$ and a unit vector $\Omega_\omega \in \mathcal{H}_\omega$ such that*

$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle_{\mathcal{H}_\omega}$$

for all $A \in \mathcal{A}$, and such that $\{\pi_\omega(A)\Omega_\omega : A \in \mathcal{A}\}$ is dense in $\mathcal{H}_\omega$. Such a representation is unique up to unitary isomorphism.

Remark 1. *The proofs of these theorems are constructive.*

Remark 2. *In the commutative case, the space X depends on $\mathcal{A}$ but not on the state. In general, this is not true: $\mathcal{H}_\omega$ depends on $\mathcal{A}$ and the state, and there are truly inequivalent representations.*

Remark 3. $\mathrm{Ran}(\pi_\omega)$ *does not necessarily cover all of $\mathcal{B}(\mathcal{H}_\omega)$: it is only a C^*-subalgebra of it.*

The morale of the story:

i. For any classical system, the set of observables is the set of functions over a phase space and all states are probability measures on that phase space.
ii. For any system and given a state, the set of observables is a subset of the bounded operators on a Hilbert space and the state is the vector state associated to a unit vector in that Hilbert space.

iic) UHF algebras, quantum spin systems

- Quasi-local algebras: I_0: directed set.

* $\perp$ is an orthogonality relation if

a) $\alpha \in I_0 \Rightarrow \exists \beta \in I_0 : \alpha \perp \beta$

b) $\alpha \leq \beta$ and $\beta \perp \gamma \Rightarrow \alpha \perp \gamma$

c) $\alpha \perp \beta$ and $\alpha \perp \gamma \Rightarrow \exists \delta : \alpha \perp \delta$ and $\beta, \gamma \leq \delta$.

also: assume that I_0 has a least upper bound for each pair $\alpha, \beta \in I_0$: $\alpha \vee \beta$ i.e.

i) $\alpha, \beta \leq \alpha \vee \beta$

ii) $\cancel{\alpha} \leq \gamma$ and $\beta \leq \gamma \Rightarrow \alpha \vee \beta \leq \gamma$.

Example: I_0 as the set of finite subsets of $\mathbb{R}^d$, with "$\perp$" given by disjointness and "$\vee$" given by the union of sets.

* Assume: there is an automorphism of $\mathcal{A}$ s.t. $\sigma^2 = 1$
Then, for each $A \in \mathcal{A}$:

$$A = A^+ + A^- \qquad \text{with} \qquad A^{\pm} = \frac{1}{2}(A \pm \sigma(A))$$

i.e. $\sigma(A^{\pm}) = \pm A^{\pm}$ (+ : "even", − : "odd")

* A quasi-local algebra is a C^*-algebra with a net $\{\mathcal{A}_\alpha\}_{\alpha \in I_0}$ of C^*-subalgebras, s.t. I_0 has an "$\perp$" and:

1) $\alpha \leq \beta \Rightarrow \mathcal{A}_\alpha \subseteq \mathcal{A}_\beta$.

2) $\mathcal{A} = \overline{\bigcup_{\alpha \in I_0} \mathcal{A}_\alpha}^{\|\cdot\|}$ (denote $\mathcal{A}_{loc} = \bigcup_{\alpha \in I_0} \mathcal{A}_\alpha$)

3) $\mathcal{A}_\alpha$ have a common identity

4) $$\left.\begin{aligned} [\mathcal{A}_\alpha^e, \mathcal{A}_\beta^e] &= \{0\} \\ [\mathcal{A}_\alpha^e, \mathcal{A}_\beta^o] &= \{0\} \\ \{\mathcal{A}_\alpha^o, \mathcal{A}_\beta^o\} &= \{0\} \end{aligned}\right\} \text{ whenever } \alpha \perp \beta, \quad \begin{aligned} e &: \text{even} \\ o &: \text{odd.} \end{aligned}$$

Note: often: $\sigma = 1$ so that $\mathcal{A}_\alpha^c = \mathcal{A}_\alpha$; $\mathcal{A}_\alpha^0 = \{0\}$.

- UHF algebras: I: <u>countable</u> index set, I_0: set of finite subsets.
  (think of $I = \mathbb{Z}^d$) (or I = graph).

For each $x \in I$, $\mathcal{H}_x$ is a finite <u>dimensional</u> Hilbert space

$$\Lambda \in I_0 \quad : \quad \mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x .$$

and

$$\mathcal{A}_\Lambda := \mathcal{L}(\mathcal{H}_\Lambda)$$

a finite-dimensional matrix algebra.

Check: $\Lambda_1 \cap \Lambda_2 = \emptyset$: $\mathcal{H}_{\Lambda_1 \cup \Lambda_2} = \mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda_2}$

natural embedding $\mathcal{A}_{\Lambda_1} \simeq \mathcal{A}_{\Lambda_1} \otimes 1\!\!1_{\Lambda_2} \subset \mathcal{A}_{\Lambda_1 \cup \Lambda_2}$ (3),(1) ✓

define $\mathcal{A} := \overline{\bigcup_{\alpha \in I_0} \mathcal{A}_\alpha}^{\|\cdot\|}$ (2) ✓

$$\Lambda_1 \cap \Lambda_2 = \emptyset \Rightarrow \underset{\mathcal{A}_{\Lambda_1}}{\underset{\cap}{A_1}}\ \underset{\mathcal{A}_{\Lambda_2}}{\underset{\cap}{A_2}} = A_1 \otimes A_2 = A_2 A_1 \qquad (4) ✓$$

- Why "quantum spin system"? Take $\mathcal{H}_x = \mathcal{H}$ for all $x \in I$. and $\mathcal{H} = \mathbb{C}^{2S+1}$, $S \in \frac{1}{2}\mathbb{N}$ carries the $(2S+1)$-irreducible representation of $SU(2)$: i.e. a spin-S is attached to each site.

- Let $\mathcal{A}$ be a UHF algebra, $\omega \in \mathcal{E}(\mathcal{A})$. Then ω is <u>locally</u> normal, i.e. $\omega\restriction_{\mathcal{A}_\alpha} = \mathrm{Tr}(\rho_\alpha^\omega \cdot)$ $\alpha \in I_0$:

$$A \in \mathcal{A}_\alpha \quad : \quad \omega(A) = \mathrm{Tr}(\rho_\alpha^\omega A)$$

in other words ω is equivalently defined by the set $\{\rho_\alpha^\omega\}_{\alpha \in I}$ of local density matrices satisfying consistency conditions (cf Kolmogorov's representation thm for stochastic processes).

- Recall: ω_1 and ω_2 are quasi-equivalent if any ω_1-normal state is also an ω_2-normal state

Proposition: $\omega_1, \omega_2 \in \mathcal{E}(\mathcal{A})$, $\mathcal{A}$: UHF algebra. Then ω_1 and ω_2 are quasi-equivalent iff $\forall \varepsilon > 0$, ~~s.t.~~ $\exists \Lambda \in I_0$:

$$|\omega_1(A) - \omega_2(A)| < \varepsilon \|A\|$$

for all $A \in \mathcal{A}_{\Lambda'}$, with $\Lambda \cap \Lambda' = \emptyset$

in other words: equivalent states are "equal at infinity".

- In practice: we consider a set of vectors $\{\Psi_\Lambda\}_{\Lambda \in I_0}$ and the associated set of states $\rho_\Lambda = P_{\Psi_\Lambda}$ i.e. $\omega_\Lambda(A) = \langle \Psi_\Lambda, A \Psi_\Lambda \rangle$

Note: the ρ_Λ do not need to satisfy any consistency condition.

Let $\{\tilde{\omega}_\Lambda\}_{\Lambda \in I_0}$ be extensions of ω_Λ to the whole algebra (exists by Hahn-Banach)

Since $\mathcal{E}(\mathcal{A})$ is the unit sphere in $\mathcal{A}^*$, it is compact in the weak-* topology (Banach-Alaoglu), hence for any $A \in \mathcal{A}_\Lambda$, $\Lambda \in I_0$, there exists limits along subsequences.

$$\omega(A) := \lim_{k \to \infty} \omega_{\Lambda_k}(A)$$

"thermodynamic limit"

- Remarks: i) If ω_1 & ω_2 are quasi-equivalent and pure, then they are unitarily equivalent.

ii) If ω is pure, then $\mathrm{span}\{\pi_\omega(A)\xi : A \in \mathcal{A}\} = \mathcal{H}_\omega$ for any $\xi \in \mathcal{H}_\omega$: the representation is said to be irreducible. In that case the weak closure of $\pi_\omega(\mathcal{A})$ is all of $\mathcal{L}(\mathcal{H}_\omega)$.

- Translation symmetry. Take $I = \mathbb{Z}^d$ on which there is a natural translation (action of $\mathbb{Z}^d$ on itself by addition).
For $A \in \mathcal{A}_\Lambda$: $\tau_x(A) \in \mathcal{A}_{\Lambda+x}$ for any $x \in \mathbb{Z}^d$.
we have: $\tau_x : \mathcal{A}_{loc} \to \mathcal{A}_{loc}$
and $\|\tau_x(A)\| = \|A\|$, $A \in \mathcal{A}_{loc}$
hence τ_x can be extended to all of $\mathcal{A}$.
and $\tau_x(\tau_y(A)) = \tau_{x+y}(A)$; $\tau_0(A) = A$
i.e. $\tau : \mathbb{Z}^d \to \mathrm{Aut}(\mathcal{A})$ is a group of automorphisms.

<u>Theorem</u>: Let $\omega \in \mathcal{E}(\mathcal{A})$ be s.t.

$$(*) \quad (\omega \circ \tau_x)(A) = \omega(A) \quad \forall x \in \mathbb{Z}^d, A \in \mathcal{A}.$$

(translation-invariant state).

Then there exist $U(x)$, unitary on $\mathcal{H}_\omega$ s.t.

$$\pi_\omega(\tau_x(A)) = U(x)\pi_\omega(A)U(x)^* \quad \& \quad U(x)\Omega_\omega = \Omega_\omega$$

Moreover, $U(0) = \mathbb{1}$ and

$$U(x)U(y) = U(y)U(x) = U(x+y) \quad \forall x, y \in \mathbb{Z}^d.$$

<u>Proof</u>: By $(*)$, for any $x \in \mathbb{Z}^d$, $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ and $(\mathcal{H}_\omega, \pi_\omega \circ \tau_x, \Omega_\omega)$ are two cyclic reps of ω. By uniqueness of the GNS rep, $\exists$ a unitary $U(x)$ on $\mathcal{H}_\omega$ s.t.

$$\pi_\omega \circ \tau_x(A) = U(x)\pi_\omega(A)U(x)^*$$

and $U_x\Omega_\omega = \Omega_\omega$.

It immediately follows that $U(0) = \mathbb{1}$ is a possible choice.
Now: $U(x)U(y)\pi_\omega(A)\Omega_\omega = U(x)\pi_\omega(\tau_y(A))\Omega_\omega$

$$= \pi_\omega(\tau_x \circ \tau_y(A))\Omega_\omega = \pi_\omega(\tau_{x+y}(A))\Omega_\omega$$
$$= U(x+y)\pi(A)\Omega_\omega$$

The group property follows by density of $\{\pi_\omega(A)\Omega_\omega : A \in \mathcal{A}_{loc}\}$ $\square$

- Note: In this case, since the group of translations is discrete, the "generators" are simply given by
$$U_i = U(e_i)$$
where $\{e_i\}_{i=1}^d$ are the unit vectors of the lattice $\mathbb{Z}^d$. We have
$$U(x) = \prod_{i=1}^d U_i^{x_i}$$
since the group is abelian.

- Theorem: If $\mathcal{A}$ is a quantum spin system,
$$\lim_{|x|\to\infty} \| [\tau_x(A), B] \| = 0$$
for all $A, B \in \mathcal{A}$.

Proof: If $A, B \in \mathcal{A}_{loc}$, $\exists x_0$ s.t. $\tau_x(A)$ and B have disjoint support, hence $[\tau_x(A), B] = 0$, for all $|x| > |x_0|$. The general case follows by norm approximation of A, B by strictly local elements $A_L, B_L \in \mathcal{A}_{loc}$. $\square$

- Corollary: Any translation invariant state can be decomposed uniquely as a convex combination of pure, translation invariant states.

(non-trivial proof).

- Final remark: in complete generality, $\mathcal{A}$: C^*-algebra and α is an automorphism of $\mathcal{A}$. If $\omega \in \mathcal{E}(\mathcal{A})$ s.t.
$$\omega \circ \alpha = \omega ,$$
then α is unitarily implementable in π_ω: $\exists\, U_\alpha$:
$$\pi_\omega(\alpha(A)) = U_\alpha \, \pi_\omega(A) \, U_\alpha^* .$$

Proof: As above, by uniqueness of the GNS representation.

iii) The CAR & CCR algebras

- Algebras of observables for fermions & bosons, respectively.
  - many particle systems (one, N, infinite)
  - the Hilbert space for one particle is $\mathcal{H}$
- CAR algebra $\mathcal{A}_-(\mathcal{H})$: C^*-algebra generated by
$$\{\mathbb{1}, a(f), a^*(f) : f \in \mathcal{H}\}.$$

s.t.

i) $f \mapsto a(f)$ is antilinear

ii) $a(f)^* = a^*(f)$
$\quad \uparrow$ C^*-conjugation

iii) $\{a(f), a^*(g)\} = \langle f, g\rangle_{\mathcal{H}} \cdot \mathbb{1}.$
$$\{a(f), a(g)\} = \{a^*(f), a^*(g)\} = 0$$

Note: In fact: $\mathcal{H}$ needs only be a prehilbert space, and $\mathcal{A}_-(\mathcal{H}) = \mathcal{A}_-(\overline{\mathcal{H}})$.

Proposition:

(1) $\mathcal{A}_-(\mathcal{H})$ is unique: if $a_1(f)$ and $a_2(f)$ generate $\mathcal{A}_-^{(1)}(\mathcal{H})$ and $\mathcal{A}_-^{(2)}(\mathcal{H})$, then there exists an unique $*$-isomorphism s.t.
$$\alpha(a_1(f)) = a_2(f)$$

(2) $\|a(f)\| = \|f\|$ (bounded!)

(3) if $\mathcal{H}$ is n-dimensional, $n<\infty$, then $\mathcal{A}(\mathcal{H}) \simeq \mathcal{M}_{2^n}(\mathbb{C})$.

(4) if U is a bounded linear operator and V a bounded antilinear operator on $\mathcal{H}$ s.t.
$$V^*U + U^*V = 0 = UV^* + VU^*$$
$$U^*U + V^*V = \mathbb{1} = UU^* + VV^*$$
there exists a unique $*$-isomorphism γ of $\mathcal{A}_-(\mathcal{H})$ s.t.

$$\gamma(a(f)) = a(Uf) + a^*(Vf) \qquad (B)$$

$$\left(\text{and} \quad \gamma^{-1}(a(f)) = a(U^*f) + a^*(V^*f) \qquad \right)$$

(1) $\mathcal{A}(\mathcal{H})$ is an UHF algebra.

<u>Remarks</u>: • The transformation (B) is a Bogoliubov transformation.

• The Prop. does not say anything about its implementability in a given rep.

• Simple case: $V=0$, U unitary

$$\gamma(a(f)) = a(Uf)$$

"each particle evolves independently"

<u>Sketch of proof</u>: (2) follows from the CAR relations:

$$(a^*(f)a(f))^2 = a^*(f)\{a(f), a^*(f)\}a(f) = \|f\|^2 a^*(f)a(f)$$

take $\|\cdot\|$ and use the C^*-property $\|A^*A\| = \|A\|^2$.

(3): $\{f_1, \dots, f_n\}$: ON basis of $\mathcal{H}$. Define

$$e_{11}^{(k)} := a(f_k)a^*(f_k) \qquad e_{12}^{(k)} := V_{k-1}a(f_k)$$

$$e_{21}^{(k)} := V_{k-1}a^*(f_k) \qquad e_{22}^{(k)} := a^*(f_k)a(f_k)$$

with

$$V_k = \prod_{i=1}^{k}(1 - 2a^*(f_i)a(f_i))$$

From the CAR:

$$\begin{cases} e_{ij}^{(k)}e_{rs}^{(k)} = \delta_{jr}e_{is}^{(k)} \\ e_{ij}^{(k)}e_{rs}^{(\ell)} = e_{rs}^{(\ell)}e_{ij}^{(k)} \qquad (k \neq \ell) \end{cases}$$

i.e. n families of commuting 2×2 matrix units.

<u>and</u>: the $e_{ij}^{(k)}$'s generate the same algebra, as

$$a(f_k) = \left(\prod_{i=1}^{k-1}(e_{11}^{i} - e_{22}^{(i)})\right)e_{12}^{(k)}$$

Hence $\mathcal{A}(\mathcal{H}) \simeq \mathcal{M}_2(\mathbb{C})^{\otimes n} \simeq \mathcal{M}_{2^n}(\mathbb{C})$.

$\leadsto$ Prop. (1)–(4) established for $\mathcal{H}$ finite dim.

For infinite dimensional $\mathcal{H}$. Choose ON-basis $\{f_\alpha\}_{\alpha \in \mathbb{N}}$ and consider matrix units as above for finite subsets of α's. But since $f \mapsto \tilde{a}(f)$ is bounded by (2), it suffices to construct the isomorphism on a dense subset and extend by continuity, uniquely.
This in fact also proves (5): $\mathcal{A}(\mathcal{H})$ is generated by finite-dimensional matrix algebras.

(4): $$\{\gamma(a(f))^*, \gamma(a(g))\} = \{a(Uf)^*, a(Ug)\} + \{a(Vf), a(Vg)^*\}$$
$$\overset{CAR}{=} \langle Ug, Uf\rangle + \langle Vf, Vg\rangle$$
$$= \langle g, (U^*U + V^*V) f\rangle$$

(for an antilinear V: $\langle f, Vg\rangle = \overline{\langle V^* f, g\rangle}$) $a(Uf) + a^*(Vf)$

Claim follows from (1) with $a_1(f) = a(f)$; $a_2(f) = \cancel{\gamma(a_1(f))}$ $\square$

- Natural quasi-local structure:
  - index set: closed subsets of $\mathcal{H}$, directed by inclusion, orthogonality given by orthogonality in $\mathcal{H}$.
  - $\mathbb{Z}_2$-automorphism σ on $\mathcal{A}_-(\mathcal{H})$:
  $$\sigma(a(f)) = -a(f)$$
  $\leadsto$ even and odd parts of $\mathcal{A}_-(\mathcal{H})$ are generated by even and odd polynomials in the a, a^*'s, because A^2 commutes with B if A anticommutes with B.
- Note: $\{a^*(f), a^*(f)\} = 0$ "there cannot be two particles in the same state". cf Pauli principle.
- CCR algebra, $\mathcal{A}_+(\mathcal{H})$. We would like to talk about X, P operators, but they are unbounded $\leadsto$ take their complex exponential and consider the unitaries.
  C^*-algebra generated by $\{W(f);\ f \in \mathcal{H}\}$ s.t.

  (i) $W(f)^* = W(-f)$
  (ii) $W(f)W(g) = \exp\left(-\frac{i}{2}\operatorname{Im}\langle f, g\rangle\right) W(f+g)$

Note: in fact $\mathcal{H}$ needs only be a real-linear space equipped with a non-degenerate symplectic form.
Careful here: $\mathcal{A}_+(\mathcal{H}) \neq \mathcal{A}_+(\overline{\mathcal{H}})$, not UHF.

Proposition:

(1) $\mathcal{A}_+(\mathcal{H})$ is unique (as above)

(2) $W(0) = \mathbb{1}$

$W(f)^* W(f) = W(f) W(f)^* = \mathbb{1}$ (unitarity)

$\| W(f) - \mathbb{1} \| = 2$ for all $f \in \mathcal{H}$, $f \neq 0$.

(3) if $\mathcal{H}$ is finite dimensional $\leadsto$ Stone-von Neumann uniqueness theorem (see later).

(4) If S is a real linear invertible operator on $\mathcal{H}$ s.t.

$$\operatorname{Im} \langle Sf, Sg \rangle = \operatorname{Im} \langle f, g \rangle$$

there exists a unique *-isomorphism γ of $\mathcal{A}_+(\mathcal{H})$ s.t.

$$\gamma(W(f)) = W(Sf)$$

Some remarks about (4)

The proof is harder here: (1) is non-trivial (really)
(3) also (related to above).

(2) is easy from the CCRelations:

$W(f) W(0) = W(0) W(f) = W(f)$ for all $f \Rightarrow W(0) = \mathbb{1}$.

$W(-f) W(f) = W(0) = W(f) W(-f) \Rightarrow$ unitarity.

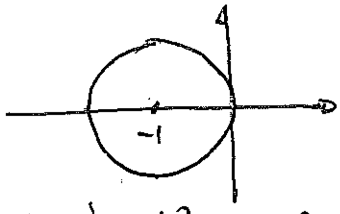
Also:

$$W(g)^* W(f) W(g) = e^{i \operatorname{Im} \langle g, f \rangle} W(f)$$

Since $W(g)$ is unitary, the spectrum of the L.H.S. is equal to the spectrum of $W(f)$, hence (since $W(f)$ is also unitary)

$$\operatorname{Spec}(W(f)) = S^1, \text{ the unit circle.}$$

Moreover, $W(f) - \mathbb{1}$ is a normal operator, and its spectrum is



⇝ its spectral radius is $+2$, and therefore $\|W(f)-1\| = +2$.

Important note: the map $f \mapsto W(f)$ is not norm continuous!

- The quasi-local structure here is given by $\sigma = \mathrm{Id}$.
- On reps of $\mathcal{A}_+(\mathcal{H})$.

A regular representation of $\mathcal{A}_+(\mathcal{H})$ is a representation π s.t.

$$t \longmapsto \pi(W(tf))$$

is continuous in the strong operator topology on $\mathcal{H}_\pi$.
(i.e. $\forall \psi \in \mathcal{H}_\pi : \|\pi(W(tf))\psi - \psi\| \to 0 \quad (t \to 0)$).

A regular state over $\mathcal{A}_+(\mathcal{H})$ is s.t. π_ω is a regular rep.

$\pi(W(tf))$ is a strongly continuous group $\to$ by Stone's theorem, it is generated by a self-adjoint generator $\Phi_\pi(f)$:

$$\pi(W(tf)) = e^{it\Phi_\pi(f)}$$

In the representation: creation/annihilation operators

$$a_\pi^\dagger(f) = \frac{1}{\sqrt{2}}\left(\Phi_\pi(f) - i\,\Phi_\pi(if)\right)$$
$$a_\pi(f) = \frac{1}{\sqrt{2}}\left(\Phi_\pi(f) + i\,\Phi_\pi(if)\right)$$

It can be shown that $\Phi_\pi(f)$ & $\Phi_\pi(if)$ are densely defined on $\mathcal{H}_\pi$, and $\Phi_\pi(f)$, $\Phi_\pi(if)$ are essentially self-adjoint on D.

For any $\psi \in D$, take $\frac{d}{ds}\frac{d}{dt}\left[W_\pi(sg)W_\pi(tf)\psi = e^{ist\,\mathrm{Im}\langle f,g\rangle}W_\pi(tf)W_\pi(sg)\psi\right]$ s.t. to get

$$\left(\Phi_\pi(f)\Phi_\pi(g) - \Phi_\pi(g)\Phi_\pi(f)\right)\psi = i\,\mathrm{Im}\langle f,g\rangle\psi$$

and further

$$[a_\pi(f), a_\pi^\dagger(g)]\psi = \langle f,g\rangle\psi \quad \text{usual form of CCR.}$$

Note: by considering $a_\pi^+(f)a_\pi(f)$, it is easy to see that $a_\pi(f)$ is always unbounded, so they cannot form a C^*-algebra.

Finally, representation with $\dim\mathcal{H} = n < \infty$. Check for $n=1$:

$$W(s) = e^{is\Phi} \quad ; \quad W(is) = e^{is\Pi} \qquad (s \in \mathbb{R})$$

(i.e. $\Phi = \Phi(1)$, $\Pi = \Phi(i)$ as above)

then: $[\Phi, \Pi] = i \operatorname{Im}(1 \cdot i) = i \cdot \mathbb{1}$.

Define a regular representation of $\mathcal{A}_+(\mathbb{C})$ on $L^2(\mathbb{R})$:

$$\pi(W(s+it)) = e^{\frac{i}{2}st}\, U(s)\, V(t) \qquad \text{with}$$

$$(U(s)\psi)(x) = e^{isx}\psi(x)$$
$$(V(t)\psi)(x) = \psi(x+t)$$

with s.a. generators $\Phi_\pi = X$; $\Pi_\pi = -i\frac{\partial}{\partial x}$

$\Rightarrow$ "Schrödinger representation of QM." !

On $\mathbb{C}^n$: representation on $L^2(\mathbb{R}^n)$

$$\prod_{k=1}^{n} e^{\frac{i}{2}s_k t_k}\, U_k(s_k)\, V_k(t_k)$$

Stone-von Neumann uniqueness thm: Any regular irreducible representation of $\mathcal{A}_+(\mathcal{H})$ with $\dim\mathcal{H} = n < \infty$ is unitarily equivalent to the Schrödinger representation.

In statistical mechanics: we are interested in the case $\dim\mathcal{H} = \infty$, typically $\mathcal{H} = L^2(\mathbb{R}^d)$ itself.

- Fock representations.
  - Fock space : $\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}$

    i.e. $\Psi \in \mathcal{F}(\mathcal{H})$: $\Psi = (\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \dots)$

    A dense subspace. $\mathcal{F}^{fin}(\mathcal{H}) := \{\Psi \in \mathcal{F}(\mathcal{H}) : \exists n_0 : \psi^{(n)} = 0, n > n_0\}$.
  - Creation / annihilation operators (concrete ones!).

    $a(f) : \mathcal{H}^{\otimes p} \to \mathcal{H}^{\otimes p-1}$; $a^*(f) : \mathcal{H}^{\otimes p} \to \mathcal{H}^{\otimes p+1}$;

    $$a(f)(\psi_1 \otimes \dots \otimes \psi_p) = \sqrt{p}\, \langle f, \psi_1 \rangle\, \psi_2 \otimes \dots \otimes \psi_p \qquad (*)$$
    $$a^*(f)(\psi_1 \otimes \dots \otimes \psi_p) = \sqrt{p+1}\, f \otimes \psi_1 \otimes \psi_2 \otimes \dots \otimes \psi_p$$

    and $a(f)\Omega_0 = 0$ for $\Omega_0 \in \mathcal{H}^{\otimes 0} = \mathbb{C}$.

    Extend by linearity to $\mathcal{F}^{fin}(\mathcal{H})$, where they are closable ops.

    Check: $a^*(f) = a(f)^*$, but no definite commutation relations.
  - $\mathcal{H}^{\otimes p}$ naturally carries a representation of the symmetric group S_p.

    For $\sigma \in S_p$, a permutation, define

    $$P_\sigma(\psi_0 \otimes \dots \otimes \psi_p) = \psi_{\sigma^{-1}(0)} \otimes \psi_{\sigma^{-1}(1)} \otimes \dots \otimes \psi_{\sigma^{-1}(p)}$$

    and define:

    $$\mathcal{H}_{\pm}^{\otimes p} = \{\Psi \in \mathcal{H}^{\otimes p} : P_\sigma \Psi = (\pm 1)^{\varepsilon(\sigma)} \Psi\}$$

    where $\varepsilon(\sigma)$ is the signature of $\sigma \in S_p$.

    Let $P_\pm$ be the $\perp$ projections onto $\mathcal{H}_\pm^{\otimes p}$.

    $\leadsto$ bosonic (+) and fermionic (−) Fock spaces:

    $$\mathcal{F}_\pm(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \underbrace{\mathcal{H}_\pm^{\otimes n}}_{\text{space of } n \text{ bosons/fermions.}}$$

    and $a_\pm^\#(f) := P_\pm\, a^\#(f)\, P_\pm$

    so that $a_\pm^\#(f) : \mathcal{F}_\pm(\mathcal{H}) \to \mathcal{F}_\pm(\mathcal{H})$.
  - Note: Fock space contains states with an arbitrary but finite number of particles.

- Number operator N : $N\Psi = n\Psi$ if $\Psi \in \mathcal{H}_\pm^{(n)}$

Proposition :

i) $a(f)$ defined in (*) maps $\mathcal{F}_\pm(\mathcal{H}) \to \mathcal{F}_\pm(\mathcal{H})$

ii) Let $a^*(f) := a(f)^*$. Then

$$a^*(f)\Psi = \frac{1}{\sqrt{p}} \sum_{k=1}^{p} (\pm 1)^{k-1} P_{\pi_k}(f \otimes \Psi)$$

where $\pi_k^{-1} = (k, 1, 2, \dots, k-1)$

iii) $f \mapsto a(f)$ is antilinear, $f \mapsto a^*(f)$ is linear

iv) $N a_\pm(f) = a_\pm(f)(N-1)$, $N a_\pm^*(f) = a_\pm^*(f)(N+1)$

v) Let

$$[A, B]_\pm = AB \mp BA$$

Then

$$[a_\pm(f), a_\pm(g)]_\pm = [a_\pm^*(f), a_\pm^*(g)]_\pm = 0$$
$$[a_\pm(f), a_\pm^*(g)]_\pm = \langle f, g \rangle$$

$\rightsquigarrow$ We have a representation of the CAR/Weyl algebras. Note:

$$\langle \Omega_0, a_\pm^*(f) a_\pm(g) \Omega_0 \rangle = 0$$

This is a simple example of the quasi-free representation

- Quasi-free states of the CCR: Finite linear combinations of $W(f)$'s are dense in $\mathcal{A}_-(\mathcal{H}) \Rightarrow \omega \in \mathcal{E}(\mathcal{A}_+(\mathcal{H}))$ is completely determined by its characteristic function

$$\mathcal{H} \ni f \mapsto S_\omega(f) := \omega(W(f))$$

and ω is regular if $\lambda \mapsto S_\omega(\lambda f)$ is continuous

Definition : A state $\omega \in \mathcal{E}(\mathcal{A}_+(\mathcal{H}))$ is quasi-free if

$$S_\omega(f) = \exp\left(-\frac{1}{4}\|f\|^2 - \frac{1}{2}\langle f, \rho f \rangle\right)$$

where $\rho = \rho^* \geq 0$ on $\mathcal{H}$.

Remark: The map $\lambda \mapsto S_\omega(\lambda f)$ is in fact analytic in a neighbourhood of $\lambda = 0$. Hence, $S_\omega(f)$ and the state ω are completely determined by $\partial_\lambda^n S_\omega(\lambda f)|_{\lambda=0}$ or equivalently by the correlation functions

$$\langle \Omega_\omega, \Phi_\omega(f_1) \cdots \Phi_\omega(f_n) \Omega_\omega \rangle$$

or even the <u>truncated correlation functions</u>

$$(-i\partial_\lambda)^n (\log S_\omega(\lambda f))|_{\lambda=0} \quad : \text{ note: } = 0 \ \forall n > 2$$

"Gaussian state"

We get:

$$\langle \Omega_\omega, \Phi_\omega(f) \Omega_\omega \rangle = 0$$

$$\langle \Omega_\omega, \Phi_\omega(f) \Phi_\omega(g) \Omega_\omega \rangle = \langle f, (\rho + \tfrac{1}{2}) g \rangle$$

and $\langle \Omega_\omega, a_\omega^*(f) a_\omega(g) \Omega_\omega \rangle = \langle g, \rho f \rangle$

- Quasi-free states of the CAR: As above, it suffices to define the $(n+m)$-point functions:

<u>Def</u>: A state $\omega \in \mathcal{E}(\mathcal{A}_-(\mathcal{H}))$ is a quasi-free state if it satisfies

$$\omega(a^*(g_m) \cdots a^*(g_1) a(f_1) \cdots a(f_n)) = \delta_{nm} \det(\langle f_i, \rho g_j \rangle)$$

for some $0 \le \rho = \rho^* \le 1$ on $\mathcal{H}$.

<u>Remarks</u>: * $\omega(a^*(g) a(f)) = \langle f, \rho g \rangle$

but also $= \langle f, g \rangle - \omega(a(f) a^*(g))$

taking $f = g$: $\langle f, \rho f \rangle \le \|f\|^2$, hence $\rho \le 1$.

* In fact, one needs to prove that the formulas for $\mathcal{A}_\pm(\mathcal{H})$ indeed define states on the respective algebras. We skip this.
* Quasi-free states are simple: any multiparticle expectation value can be expressed in terms of <u>one-particle</u> quantities.

This is known as <u>Wick's lemma</u>.

- These states are <u>gauge-invariant</u>: they are invariant under the algebra automorphisms $a(f) \mapsto a(e^{i\theta} f)$, resp. $W(f) \mapsto W(e^{i\theta} f)$, for all $\theta \in \mathbb{R}$.
- In both cases: $\rho = 0$ defines the Fock state
- As in the CCR case, the <u>truncated</u> correlation functions for the CAR quasi-free states satisfy:

$$\omega_T\left(a^*(g_n) \cdots a^*(g_1) a(f_1) \cdots a(f_n)\right) = 0 \quad \forall n > 2$$

(i.e. generalisation of the variance

$$\omega_T\left(a^*(f) a(g)\right) = \omega\left(a^*(f) a(g)\right) - \omega\left(a^*(f)\right) \omega\left(a(g)\right) \Big)$$

→ Araki-Wyss representation: Let $0 \le \rho \le 1$.

Let $\mathcal{H}_\rho := \mathfrak{F}_-(\mathcal{H}) \otimes \mathfrak{F}_-(\mathcal{H})$

$\Omega_\rho := \Omega_0 \otimes \Omega_0$

and

$$a_\rho(f) = \pi_\rho(a(f)) = a_-(\sqrt{1-\rho}\, f) \otimes 1 + (-1)^N \otimes a_-^*(\overline{\sqrt{\rho} f})$$

Check: CAR &

$$\langle \Omega_\rho, a_\rho^*(f) a_\rho(g) \Omega_\rho \rangle_{\mathcal{H}_\rho} = \langle \Omega_\rho, \left((-1)^N \otimes a(\overline{\sqrt{\rho} f})\right)\left((-1)^N \otimes a_-^*(\overline{\sqrt{\rho} g})\right) \Omega_\rho \rangle_{\mathcal{H}_\rho}$$

$$= \langle a_-^*(\overline{\sqrt{\rho} f}) \Omega_0, a_-^*(\overline{\sqrt{\rho} g}) \Omega_0 \rangle_{\mathfrak{F}_-(\mathcal{H})}$$

$$= \langle \overline{\sqrt{\rho} f}, \overline{\sqrt{\rho} g} \rangle_{\mathcal{H}} = \langle g, \rho f \rangle_{\mathcal{H}}$$

and of course $\langle \Omega_\rho, a_\rho^\#(f) \Omega_\rho \rangle_{\mathcal{H}_\rho} = 0$

Hence $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ is a GNS representation for ω_ρ on $\mathcal{A}_-(\mathcal{H})$.

Example: Let $H = -\Delta$ on $L^2(\mathbb{R}^d)$ and $\rho = \Theta(\mu - H)$ be the spectral projection associated to the interval $[0,\mu]$. (i.e. the Fermi energy is μ). If $f \in \operatorname{Ran} \rho$, i.e. its Fourier transform is supported in $(0,\mu)$, then

$$a_\rho^*(f) = (-1)^N \otimes a_-(\overline{f})$$

while for g s.t. $\hat{g}$ is supported in (μ, ∞):

$$a_\rho^*(g) = a_-^*(g) \otimes 1.$$

Interpretation: Filled Fermi sea. Creating a particle below μ is creating a hole (i.e. removing a particle), creating a particle above μ just creates a particle.

In fact: $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ is quasi-equivalent to the Fock representation iff ρ is trace-class. (non-trivial result).

* Araki-Woods representation. Altough it can be on a general one-particle $\mathcal{H}$, we shall write it here for $\mathcal{H} = L^2(\mathbb{R}^3, d^3k)$ i.e. "in Fourier space".

Let $\mu = \mu(k) \geq 0$ and $\mu_0 \geq 0$.

Let
$$\mathcal{H}_{\mu,\mu_0} := \mathcal{F}_+(\mathcal{H}) \otimes \mathcal{F}_+(\mathcal{H}) \otimes L^2(S^1)$$
$$\Omega_{\mu,\mu_0} := \Omega_0 \otimes \Omega_0 \otimes 1$$

and
$$a_{\mu,\mu_0}(f) := a_+(\sqrt{1+\mu}\, f) \otimes 1 \otimes 1 + 1 \otimes a_+^*(\overline{\sqrt{\mu}\, f}) \otimes 1$$
$$+ 1 \otimes 1 \otimes e^{i\theta} \cdot (-\sqrt{\mu_0})\, \overline{f(0)}$$

Check again: CCR &

$$\langle \Omega_{\mu,\mu_0}, a^*_{\mu,\mu_0}(f)\, a_{\mu,\mu_0}(g) \Omega_{\mu,\mu_0} \rangle = \langle g, \mu f \rangle + \mu_0 f(0) \overline{g(0)}$$
$$= \int_{\mathbb{R}^3} \overline{g(k)} f(k) \mu(k)\, d^3k + \mu_0 \overline{g(0)} f(0).$$

smooth momentum density

condensate density

e.g. $\mu(k) = \left(e^{\beta\omega(k)} - 1\right)^{-1} \ldots$ (see later!).

black-body radiation, for $\omega(k) = |k|$

see chapter on Bose-Einstein condensation.

Note: From the general framework of quasi-free states, $\rho = \mu + \delta_0$ to be understood as a quadratic form on $\mathcal{H}$.

- Why the representations? Again: ∞-extended gas at non-zero density do not live in Fock space.

However: if $\rho = 0$ (in both cases), then the A-W. reps reduce to Fock representations.

- The ideal Fermi gas: a pedestrian approach

One-particle Hilbert space $\mathcal{H}$, one-particle Hamiltonian H
no interactions.

Evolution on Fock space: Think of finite density in finite volume.

$$i\frac{d\Psi}{dt} = d\Gamma(H)\Psi \qquad (*)$$

where $d\Gamma(H)\restriction_{\mathcal{H}^{\otimes n}} = H\otimes 1\otimes\cdots\otimes 1 + 1\otimes H\otimes 1\cdots\otimes 1 + \cdots$
$+ 1\otimes\cdots\otimes 1\otimes H$

i.e. total energy = sum of one-particle energies.

Solution of $(*)$: $\Psi(t) = e^{-it\,d\Gamma(H)}\Psi(0) =: \Gamma(e^{-itH})\Psi(0)$

Observables: $\tau_t(A) = \Gamma(e^{+itH})A\,\Gamma(e^{-itH})$

in particular: $\tau_t(a_-^\#(f)) = a_-^\#(e^{itH}f)$

this follows from

$$\begin{aligned}\tau_t(a_-^*(f))\Omega_0 &= \Gamma(e^{itH})\,a_-^*(f)\,(1,0,0,\dots)\\ &= \Gamma(e^{itH})\,(0,f,0,0,\dots)\\ &= (0,e^{itH}f,0,\dots) = a_-^*(e^{itH})\Omega_0.\end{aligned}$$

and generalization to vectors of the form $\prod_{i=1} a_-^*(g_i)\Omega_0$

i.e. τ_t is a simple Bogoliubov automorphism.

Strong continuity:

$$\begin{aligned}\|\tau_t(a(f)) - a(f)\| &= \|a(e^{itH}f) - a(f)\| = \|a(e^{itH}f - f)\|\\ &= \|e^{itH}f - f\| \to 0 \quad (t\to 0).\end{aligned}$$

Gibbs state:

$$\omega_{\beta,\mu}(A) := \frac{\mathrm{Tr}\left(Ae^{-\beta(d\Gamma(H)-\mu N)}\right)}{\mathrm{Tr}\left(e^{-\beta(d\Gamma(H)-\mu N)}\right)}$$

Proposition: Let $H = H^*$ on $\mathcal{H}$, and $\beta \in \mathbb{R}$. The following are equivalent:

(i) $e^{-\beta H}$ is trace-class on $\mathcal{H}$

(ii) $e^{-\beta(d\Gamma(H) - \mu N)}$ is trace-class on $\mathcal{F}_-(\mathcal{H})$ for all $\mu \in \mathbb{R}$.

Proof ($\beta > 0$) (i) $\Rightarrow$ (ii). $(\varepsilon_n)_{n \geq 0}$: eigenvalues of H in increasing order

$\mathbb{R} \ni z = e^{\beta\mu}$. We have

$$0 \leq \mathrm{Tr}_{\mathcal{F}_-}\left(e^{-\beta(d\Gamma(H) - \mu N)}\right) = \sum_{m \geq 0} \mathrm{Tr}_{\mathcal{H}^{\wedge m}}\left(e^{-\beta(d\Gamma(H)\restriction_m - \mu \cdot m \cdot 1)}\right)$$

$$= \sum_{m \geq 0} z^m \underbrace{\sum_{0 \leq n_1 < \dots < n_m}}_{\text{by antisymmetry}} \exp\left(-\beta \sum_{p=1}^{m} \varepsilon_{n_p}\right) = \sum_{m \geq 0} \sum_{\underline{n}} \prod_{p=1}^{m} z e^{-\beta \varepsilon_{n_p}}$$

$$= \prod_{n \geq 0} \left(1 + z e^{-\beta \varepsilon_n}\right) \qquad \text{(a bit of combinatorics)}$$

$$\leq \prod_{n \geq 0} e^{z e^{-\beta \varepsilon_n}} = e^{z \sum_{n \geq 0} e^{-\beta \varepsilon_n}} = e^{z \mathrm{Tr}(e^{-\beta H})}$$

(ii) $\Rightarrow$ (i), $e^{-\beta(d\Gamma(H) - \mu N)}$ leaves the one-particle space invariant and

$$e^{-\beta(d\Gamma(H) - \mu N)}\restriction_{\mathcal{H}} = e^{-\beta H} e^{\beta\mu} = z e^{-\beta H}$$ ☑

Hence: The Gibbs state is well-defined iff $e^{-\beta H}$ is trace-class.

Proposition: Assume that $e^{-\beta H}$ is trace-class, and let $\omega_{\beta\mu}$ be the corresponding Gibbs state. Then $\omega_{\beta\mu}$ is a gauge-inv. quasi-free state s.t.

$$\rho = z e^{-\beta H}\left(1 + z e^{-\beta H}\right)^{-1}$$

Proof: We compute the $(n+m)$-point functions:

First, note that $N = d\Gamma(1)$, so that, since $[1,H]=0$

$$e^{-\beta(d\Gamma(H)-\mu N)}\, a^*(f)\, e^{\beta(d\Gamma(H)-\mu N)}$$

$$= e^{-\beta d\Gamma(H)} \underbrace{e^{\beta\mu d\Gamma(1)}\, a^*(f)\, e^{-\beta\mu d\Gamma(1)}}_{= a^*(e^{\beta\mu} f) = z a^*(f)} e^{\beta d\Gamma(H)} = z\, a^*(e^{-\beta H} f)$$

Denote $K_\mu = d\Gamma(H) - \mu N$, for which $\mathcal{H}^{\otimes n}$ is a invariant space.

Hence $\mathrm{Tr}\left(e^{-\beta K_\mu} a^*(g_m) \cdots a^*(g_1)\, a(f_1) \cdots a(f_n)\right) = 0$ if $m \neq n$, by computing the trace in a basis of vectors with fixed particle number.

Now.

$$\omega_{\beta,\mu}\left(a^*(g_n) \cdots a(f_n)\right) = Z_{\beta,\mu}^{-1} \mathrm{Tr}\left(e^{-\beta K_\mu} a^*(g_n) \cdots a(f_n)\right)$$

$$= Z_{\beta,\mu}^{-1} z\, \mathrm{Tr}\left(a^*(e^{-\beta H} g_n)\, e^{-\beta K_\mu} a^*(g_{n-1}) \cdots a(f_n)\right)$$

$$= Z_{\beta,\mu}^{-1} z \Big[\mathrm{Tr}\left(e^{-\beta K_\mu} a^*(g_{n-1}) \cdots a(f_{n-1})\right) \langle f_n, e^{-\beta H} g_n \rangle$$

$$- \mathrm{Tr}\left(e^{-\beta K_\mu} a^*(g_{n-1}) \cdots a^*(e^{-\beta H} g_n)\, a(f_n)\right)\Big]$$

$$= \sum_{i=1}^{n} (-1)^{n+i} z \langle f_i, e^{-\beta H} g_n \rangle\, \omega_{\beta,\mu}(\text{no } f_i, \text{no } g_n)$$

$$+ z\, \omega_{\beta,\mu}\left(a^*(e^{-\beta H} g_n) \cdots a(f_n)\right) (-1)^{n+(n-1)}$$

so that

$$\omega_{\beta,\mu}\left(a^*\left((1+ze^{-\beta H}) g_n\right) a^*(g_{n-1}) \cdots a(f_n)\right) =$$

$$= \sum_{i=1}^{n} (-1)^{n+i} \langle f_i, z e^{-\beta H} g_n \rangle\, \omega_{\beta,\mu}(\text{no } f_i, \text{no } g_n).$$

Since $1+ze^{-\beta H}$ is invertible, let $\eta_n = (1+ze^{-\beta H}) g_n$ to get

$$\omega_{\beta,\mu}\left(a^*(\eta_n) \cdots a(f_n)\right) = \sum_{i=1}^{n} (-1)^{n+i} \left\langle f_i, \frac{ze^{-\beta H}}{1+ze^{-\beta H}} \eta_n \right\rangle \omega_{\beta,\mu}(\text{no } f_i, \text{no } g_n)$$

$$= \det\left(\langle f_i, ze^{-\beta H}(1+ze^{-\beta H})\eta_j \rangle\right)$$

where the last line follows by induction and the Laplace expansion of the determinant $\square$

* We now concentrate on $H = -\Delta$.

<u>Note</u>: Lemma: For any bounded open $\Lambda \subset \mathbb{R}^d$, $-\Delta$ with Dirichlet B.C., $e^{-\beta(-\Delta)}$ is trace-class for $\beta > 0$.

Therefore: Well-defined Gibbs state at any temperature for finite volume.

However: $-\Delta$ defined on $L^2(\mathbb{R}^d)$ has purely absolutely continuous spectrum and $e^{-\beta(-\Delta)}$ cannot be $\mathcal{T}_1$.

But: can we make sense of a limiting state
$$\omega_\Lambda(A) \quad \text{as } \Lambda \to \mathbb{R}^d \quad ?$$

Answer is yes.

* Let $(\Lambda_n)_{n\in\mathbb{N}}$ be a sequence of finite open subsets of $\mathbb{R}^d$ s.t.
  (i) $n \le m \Rightarrow \Lambda_n \subseteq \Lambda_m$
  (ii). For any finite $\Omega \in \mathbb{R}^d$, $\exists\, n_0$ s.t. $\Omega \subseteq \Lambda_n \ \forall n > n_0$.

We shall write $\Lambda_n \to \mathbb{R}^d$ or $\Lambda \to \mathbb{R}^d$

<u>Theorem</u>. Let H_Λ be the Dirichlet Laplacian and $\omega^\Lambda_{\beta,\mu}$ the associated Gibbs state. Then:
$$\lim_{\Lambda\to\mathbb{R}^d} \omega^\Lambda_{\beta,\mu}(A) = \omega_{\beta,\mu}(A)$$
for $A \in \mathcal{A}_-(L^2(\Omega))$, where $\omega_{\beta,\mu}$ is the gauge-inv. quasi-free state over $\mathcal{A}_-(L^2(\mathbb{R}^d))$ s.t.
$$\omega_{\beta,\mu}(a_-^+(f)\, a_-(g)) = \langle g, z e^{-\beta H}(1 + z e^{-\beta H})^{-1} f\rangle$$
$$= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} dp\, \overline{\hat{g}(p)}\, \hat{f}(p)\, z e^{-\beta p^2}(1 + z e^{-\beta p^2})^{-1}$$

The proof follows: i) $\omega^{\Lambda}_{\beta,\mu}(a^{+}(f)\,a(g))$ are matrix elements of a bounded function of H_Λ

ii) Since $\tau^{\Lambda}_t(A)$ converges to some $\tau_t(A)$ $(\Lambda \to \mathbb{R}^d)$ bounded functions of H_Λ converge strongly as well

- Remarks: * The thermodynamic state is unique: no phase transitions for the free Fermi-gas.

  * It is a quasi-free state with ρ given by the Fermi-Dirac distribution

$$\frac{z e^{-\beta p^2}}{1 + z e^{-\beta p^2}}$$

  * Different picture for bosons: Bose-Einstein condensation

iii) KMS states

iii a) C^*-dynamical systems: definition & properties.

- **Def**: A pair $(\mathcal{A}, \tau^t)$ is a C^*-dynamical system if $\mathcal{A}$ is a quasi-local algebra and τ^t is a strongly continuous group of $*$-automorphisms of $\mathcal{A}$.

i.e. $\forall A \in \mathcal{A}$, $\|\tau^{t+\varepsilon}(A) - \tau^t(A)\| \to 0 \quad (\varepsilon \downarrow 0)$.

- **Proposition**: Let $\delta_t := \frac{1}{t}(\tau^t - 1)$, and
$D(\delta) := \{A \in \mathcal{A} : \lim_{t \downarrow 0} \delta_t(A) \text{ exists}\}$.
For $A \in D(\delta)$, define $\delta(A) = \lim_{t \to 0} \delta_t(A)$.
Then δ is a closed, densely defined map.
Moreover:
i) $1 \in D(\delta)$ and $\delta(1) = 0$
ii) $\delta(AB) = \delta(A)B + A\delta(B)$
iii) $\delta(A^*) = \delta(A)^*$

In other words: τ^t is generated by δ:
$$\tau^t = e^{t\delta}$$
and δ is a $*$-derivation on $\mathcal{A}$.

Proof left for functional analysis, (i)-(iii) are immediate consequences of the properties of τ^t.

- Remark: i) For a finite QM system:
$\tau^t(A) = e^{itH} A e^{-itH}$ so that $\delta(A) = i[H, A]$.
ii) see page 24

- **Def**: Let $(\mathcal{A}, \tau^t)$ be a C^*-dynamical system. A state ω of $\mathcal{A}$ is a (τ^t, β)-KMS state at $\beta > 0$ if for any $A, B \in \mathcal{A}$ there exists a function $F_\beta(A, B; z)$, analytic in the strip

$$S_\beta := \{ z \in \mathbb{C} : 0 < \operatorname{Im} z < \beta \},$$
continuous on its closure and s.t. "KMS boundary condition":

$$F_\beta(A, B; t) = \omega(A\tau^t(B))$$
$$F_\beta(A, B; t+i\beta) = \omega(\tau^t(B)A)$$

- Def: $A \in \mathcal{A}$ is analytic for τ^t if the function $t \mapsto \tau^t(A)$ extends to an entire analytic function on $\mathbb{C}$.

Lemma: The set of analytic elements is dense in $\mathcal{A}$.

Proof: Let $A_n := \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} \tau^t(A) e^{-nt^2} dt$ for any $A \in \mathcal{A}$.

i) well-defined integral: $\|\tau^t(A) e^{-nt^2}\| \leq \|A\| e^{-nt^2} \in L^1(\mathbb{R})$ for all $n > 0$.

ii) A_n is analytic for τ^t:
$$\tau^s(A_n) = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} \tau^t(A) e^{-n(t-s)^2} dt$$
Now: the r.h.s. extends to an analytic function on $\mathbb{C}$, since for $z \in \mathbb{C}$: (a little bit of contour integration)
$$\|\text{r.h.s.}\| \leq \|A\| e^{n(\operatorname{Im} z)^2}$$
Hence, the r.h.s. defines $\tau^z(A_n)$ and A_n is analytic.

Finally, we prove that $\|A_n - A\| \to 0$ $(n \to \infty)$:
First: $A = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-t^2} \cdot A \, dt.$
and changing variables: $A_n = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-t^2} \tau^{t/\sqrt{n}}(A) \, dt.$
so that
$$A_n - A = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \left(\tau^{t/\sqrt{n}}(A) - A \right) e^{-t^2} dt \to 0$$
by strong continuity of τ^t and dominated convergence.

□

from p. 12: ii) τ^t often arises as the thermodynamic limit of finite volume dynamics generated by finite volume Hamiltonians.

- **Lemma** Let ω be a (τ^t, β)-KMS state and $A, B \in \mathcal{A}$ be analytic elements of τ^t. Then

$$\omega(A\tau^{i\beta}(B)) = \omega(BA)$$

or of course

$$\omega(A\tau^{i\beta+t}(B)) = \omega(\tau^t(B)A)$$

the usual form of the KMS condition.

Proof. $z \mapsto \omega(A\tau^z(B))$ is analytic. Let

$$f(z) := \omega(A\tau^z(B)) - F_\beta(A,B;z)$$

$f(z)$ is s.th.:

i) analytic in S_β

ii) continuous in $\overline{S_\beta}$

iii) $f(t) = 0$, $t \in \mathbb{R}$ by the KMS boundary condition.

Therefore, it extends to an analytic function on $\{-\beta < \operatorname{Im} z < \beta\}$ by Schwarz's reflection principle, and by continuity to the closure $\{-\beta \le \operatorname{Im} z \le \beta\}$. In part.: $f(z) = 0$ on the closure since $f(t) = 0$ on the real line. But then $f(i\beta) = 0$ reads by the KMS B.C:

$$\omega(A\tau^{i\beta}(B)) = \omega(BA) \qquad \square$$

- We will see: KMS-state are <u>equilibrium</u> states. Here two elementary properties first:

A Gibbs state is a KMS state by cyclicity of the trace.

$$Z_\beta^{-1}\operatorname{Tr}(e^{-\beta H}Ae^{-\beta H}Be^{\beta H}) = Z_\beta^{-1}\operatorname{Tr}(BAe^{-\beta H}) = Z_\beta^{-1}\operatorname{Tr}(e^{-\beta H}BA)$$

Secondly.

- **Theorem**: Every (τ^t, β)-KMS state is τ^t-invariant, i.e.
$$\omega \circ \tau^t = \omega \qquad t \in \mathbb{R}.$$

Proof: A: analytic element, $g(z) := \omega(\tau^z(A))$, i.e. $z \mapsto g(z)$ analytic.
By lemma:
$$g(z + i\beta) = \omega\left(1 \cdot \tau^{i\beta}(\tau^z(A))\right) = \omega\left(\tau^z(A) \cdot 1\right) = g(z)$$
i.e. g is a periodic function with period $i\beta$. On $\overline{S_\beta}$:
$$|g(t + i\alpha)| \leq \|\tau^{t+i\alpha}(A)\| = \|\tau^{i\alpha}(A)\|$$
$$\leq \sup_{0 \leq \gamma \leq \beta} \|\tau^{i\gamma}(A)\| < \infty$$
since $z \mapsto \tau^z(A)$ is analytic and $[0, \beta]$ is compact.
Hence: g is bounded everywhere, and analytic
$\Rightarrow$ g is constant by Liouville's thm. $\square$

- By uniqueness of the GNS representation (see ex 8(ii)), the dynamics is unitarily implementable on $\mathcal{H}_\omega$, namely
$$\pi_\omega(\tau^t(A)) = e^{itH_\omega} \pi_\omega(A) e^{-itH_\omega}$$
$$H_\omega \Omega_\omega = 0, \quad H_\omega = H_\omega^*, \text{ the GNS Hamiltonian.}$$

iii (?) <u>energy-entropy balance inequality</u> & passivity.

- We'll first prove EEB and discuss its physical significance later.

<u>Theorem</u>. Let ω be a β-KMS state of $(\mathfrak{A}, \tau^t)$. Then the EEB inequality holds:

$$-i\beta\, \omega(A^* \delta(A)) \geq \omega(A^*A) \log\left[\omega(A^*A)/\omega(AA^*)\right] \qquad \text{(EEB)}$$

for all $A \in D(\delta)$, where

$$u \log(u/v) = \begin{cases} 0 & u=0,\ v\geq 0 \\ +\infty & u>0,\ v=0 \end{cases}$$

- <u>Remark</u>: in fact, there is <u>equivalence</u>. (EEB) $\Leftrightarrow$ ω is a (τ^t, β)-KMS state, so that (EEB) is a characterization of the KMS states.

- <u>Corollary</u>: Let ω be as above. Then for any unitary element $U \in \mathcal{A}$, s.t. $U \in D(\delta)$, we have that

$$-i\, \omega(U^* \delta(U)) \geq 0 \qquad (P)$$

<u>Proof</u>: From (EEB) with $\omega(A^*A) = \omega(AA^*) = 1$ $\square$

- <u>Remark</u>: inequality (P) is referred to as the fact that ω is a <u>passive state</u>.

Here, truly: $((\tau^t, \beta)\text{-KMS}) \Rightarrow (P)$ but not $\Leftarrow$ ..
There are conditions under which $(\Leftarrow)$ holds.

- <u>Proof of Theorem</u>. Let $H_\omega = \int_{\mathbb{R}} \lambda\, dE(\lambda)$ be the spectral decomposition of the GNS Hamiltonian. Let

$$d\mu_A(\lambda) := \langle \pi_\omega(A)\Omega_\omega,\ dE(\lambda)\, \pi_\omega(A)\Omega_\omega \rangle,$$

a well-defined measure $\forall A \in \mathcal{A}$, and for $f \in C_c^\infty(\mathbb{R})$,

$$\mu_A(f) := \int_{\mathbb{R}} f(\lambda)\, d\mu_A(\lambda)$$

Similarly for $d\nu_A(\lambda) := \langle \pi_\omega(A)^* \Omega_\omega, dE(\lambda) \pi_\omega(A)^* \Omega_\omega \rangle$

Now:

$$\mu_A(f) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \hat{f}(t) e^{it\lambda} dt \right) d\mu_A(\lambda)$$

$$= \int_{\mathbb{R}} \hat{f}(t) \left\langle \pi_\omega(A)\Omega_\omega , \int_{\mathbb{R}} e^{it\lambda} dE(\lambda) \pi_\omega(A)\Omega_\omega \right\rangle \cancel{d\mu_A(\lambda)}\, dt$$

$$= \int_{\mathbb{R}} \hat{f}(t) \langle \pi_\omega(A)\Omega_\omega , e^{itH_\omega} \pi_\omega(A) e^{-itH_\omega} \Omega_\omega \rangle dt$$

$$= \int_{\mathbb{R}} \hat{f}(t)\, \omega(A^* \tau^t(A))\, dt .$$

Similarly $\nu_A(f) = \int_{\mathbb{R}} \hat{f}(t)\, \omega(\tau^t(A) A^*)\, dt$.

Furthermore: using the analyticity of $z \mapsto \omega(A^* \tau^z(A))$, we can shift the contour to $t + i\beta$ and use the KMS condition to get (see exercises)

$$\mu_A(f) = \int_{\mathbb{R}} \hat{f}(t+i\beta)\, \omega(\tau^t(A) A^*)\, dt$$

Repeating the steps above, backwards, we obtain

$$= \iint \hat{f}(t+i\beta) e^{it\lambda} dt\, d\nu_A(\lambda)$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \hat{f}(s) e^{is\lambda} \right) e^{\beta\lambda} ds\, d\nu_A(\lambda)$$

or in terms of the measures:

$$\frac{d\mu_A}{d\nu_A}(\lambda) = e^{\beta\lambda} \qquad (M)$$

The rest of the proof is by convexity:

$$\frac{\beta \langle \pi_\omega(A)\Omega_\omega , H_\omega \pi_\omega(A)\Omega_\omega \rangle}{\langle \pi_\omega(A)\Omega_\omega , \pi_\omega(A)\Omega_\omega \rangle} = \frac{\int \beta\lambda\, d\mu_A(\lambda)}{\int d\mu_A(\lambda)}$$

$$= -\ln \left(\exp\left(-(\dots) \right) \right)$$

but e^{-x} is a convex function, so by Jensen's inequality

$$\exp\left(-\frac{\int \beta\lambda \, d\mu}{\int d\mu}\right) \leq \frac{\int e^{-\beta\lambda} d\mu_A(\lambda)}{\int d\mu_A(\lambda)} = \frac{\int d\nu_A(\lambda)}{\int d\mu_A(\lambda)} = \frac{\omega(AA^*)}{\omega(A^*A)}$$

since $-\ln(\cdot)$ is a decreasing function, we have

$$\frac{\beta \langle \cdots H_\omega \cdots \rangle}{\omega(A^*A)} \geq -\ln \frac{\omega(AA^*)}{\omega(A^*A)} = \ln \frac{\omega(A^*A)}{\omega(AA^*)}$$

It remains to observe that

$$\langle \cdot \, H_\omega \cdots \rangle = \langle \pi_\omega(A)\Omega_\omega , [H_\omega, \pi_\omega(A)] \Omega_\omega \rangle = \omega(A^* \delta(A))(-i)$$

since $H_\omega \Omega_\omega = 0$ □

* Interpretation: We consider the case $\dim \mathcal{H} < \infty$, i.e.

$$\omega_\beta(A) = Z^{-1} \operatorname{Tr}(e^{-\beta H} A).$$

1) Consider $H(t)$ s.t. $H(0) = H(T)$ and let U: evolution on $[0,T]$

The change in energy between $t=0$ and $t=T$ is given by

$$W = \operatorname{Tr}(H U \rho U^*) - \operatorname{Tr}(H\rho)$$

where ρ is the initial state. Choose the Gibbs state:

$$W_\beta = \omega_\beta(U^* H U - H) = \omega_\beta(U^* [H, U])$$
$$= -i\,\omega_\beta(U^* \delta(U)) \geq 0.$$

by passivity.

⇝ Total work done by the system on the environment, $-W_\beta$, is non-positive (on average!): the system is passive!

see 2nd law of TD

2) In fact, this is an example of the Gibbs variational principle

Let, for any state $\sigma \in \mathcal{E}(\mathcal{A})$:

$$F_\beta(\sigma) = \beta^{-1} S(\sigma) - H(\sigma)$$

where

$$S(\sigma) = -\operatorname{Tr}(\sigma \ln \sigma)$$

is the entropy, and

$$H(\sigma) = \operatorname{Tr}(\sigma H)$$

is the (mean) energy.

Proposition: The functional $F_\beta(\sigma)$ has a unique maximizer, namely the Gibbs state ω_β, and its maximal value is

$$F_\beta(\rho_\beta) = \frac{1}{\beta} \ln \operatorname{Tr}(e^{-\beta H}) = \frac{1}{\beta} \ln Z_\beta$$

is the Gibbs free energy.

Proof: Exercise.

Consequences: Let $\varphi : \mathcal{E}(A) \to \mathcal{E}(A)$, then

$$F_\beta(\varphi(\omega_\beta)) \leq F_\beta(\omega_\beta) \quad \left[\begin{array}{c} \text{i.e. (EEB).} \\ \beta \Delta E \geq \Delta S \end{array} \right]$$

e.g. $\varphi(\omega_\beta) = \omega_\beta(U^* \cdot U) \rightsquigarrow$ back to the example 1).

Other example: Strongly continuous semigroup $\varphi_t : t \geq 0$, with generator $L = \frac{d\varphi_t}{dt}\Big|_{t=0+}$:

$$L(\rho) = A^* \rho A - \frac{1}{2}(AA^* \rho + \rho A^* A)$$

φ_t is linear, trace-preserving, positive (in fact: completely positive)

in particular $\varphi_t \mathcal{E}(A) \subseteq \mathcal{E}(A)$.

$$(\text{GVP}) \Rightarrow t^{-1}\left(F(\rho_\beta) - F(\varphi_t(\rho_\beta))\right) \geq 0$$

hence $\quad -F(L(\rho_\beta)) \geq 0.$

Compute the derivatives and use the KMS condition ...

$$\rightsquigarrow \omega_\beta(A^*[H, A]) \geq \omega_\beta(A^* A) \log \frac{\omega(A^* A)}{\omega(AA^*)}$$

I.e. (GVP) implies (EEB) in the finite dim. case.

- Remark: The above is without loss of generality: The generator of a trace-preserving, completely positive, strongly continuous semigroup is of the form
$$L(\rho) = -i[H,\rho] + \sum_\alpha [\Gamma_\alpha \rho \Gamma_\alpha^\dagger - \tfrac{1}{2}\{\rho, \Gamma_\alpha^\dagger \Gamma_\alpha\}]$$
with $H = H^\dagger$ and arbitrary Γ_α. The converse is also true.
(Lindblad, Gorini-Kossakowski-Sudarshan)

iii c Stability

- The EEB as a generalization of the Gibbs variational principle is a first example of stability: ~~It is a maxi~~
KMS states are characterized by the maximality of $S(\rho) - \beta H(\rho)$.
- Here, we discuss briefly (without proofs) stability w.r.t. local perturbations.

Def: Let $(\mathcal{A}, \tau_0^t)$ be a C^*-dynamical system with generator δ_0.
A local perturbation of δ_0 is
$$\delta_V = \delta_0 + i[V, \cdot]$$
where $V = V^* \in \mathcal{A}$, and s.t. $D(\delta_V) = D(\delta_0)$.
δ_V generates a strongly continuous group of $*$-automorphisms τ_V^t on $\mathcal{A}$.

- First: Structural stability: smooth dependence of the equilibrium state on the local perturbation.

By perturbation theory: Let ω be a (τ_0, β)-KMS state. Then for every local perturbation V, there is a (τ_V, β)-KMS state s.t.
$$\|\omega_V - \omega\| = O(\|V\|),$$
and ω_V is ω-normal.
Moreover: $\omega \mapsto \omega_V$ is a bijection from the set of (τ_0, β)-KMS states and the set of (τ_V, β)-KMS states.

- Then: dynamical stability: is the structural isomorphism $\omega \mapsto \omega_V$ realized dynamically, i.e. do we have (in some form)

$$\omega\left(\tau_V^t(A)\right) \xrightarrow{t\to+\infty} \omega_V(A) \qquad ?$$

There are various forms of answers to this, all of them using some form of asymptotic abelianness. Here is one.

(AI): For any $V = V^*$ in a dense $*$-subalgebra $\mathcal{A}_0$ of $\mathcal{A}$, there is $\lambda_V > 0$ s.t.

$$\int_{-\infty}^{+\infty} \| [V, \tau_{\lambda V}^t(A)] \| \, dt < \infty$$

for all $|\lambda| < \lambda_V$, $A \in \mathcal{A}_0$.

(AII): For any $V = V^* \in \mathcal{A}_0$ and $|\lambda|$ small enough, there exists a $\tau_{\lambda V}$-invariant, ω-normal state $\omega_{\lambda V}$ s.t.

$$\lim_{\lambda \downarrow 0} \| \omega_{\lambda V} - \omega \| = 0$$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \omega \circ \tau_{\lambda V}^t \, dt = \omega_{\lambda V} \quad \text{(unique limit)}$$

<u>Theorem</u>: If ω is a τ_0-invariant, factor state, and if (AI) holds, then (AII) holds <u>iff</u> ω is a (τ_0, β)-KMS state for some β.

In other words: (dynamical and structural) stability is <u>equivalent</u> to the KMS condition, under the assumption (AI).

- Partial results are easier, e.g.: (return to equilibrium)

Proposition: Let ω_V be a (τ_V, β)-KMS state and let $\tilde{\omega}$ be a weak-* limit point of $\omega_V \circ \tau_0^t$ as $t \to \infty$. If

$$\lim_{t\to\infty} \| [A, \tau_0^t(B)] \| = 0 \qquad \forall A, B \in \mathcal{A},$$

then $\tilde{\omega}$ is a (τ_0, β)-KMS state.

Proof: Note the lower semi-continuity of

$$(u,v) \mapsto u \log(u/v)$$

Then:

$$\tilde{\omega}(A^*A) \ln \frac{\tilde{\omega}(A^*A)}{\tilde{\omega}(AA^*)} \overset{\text{up to subsequence}}{\leq} \lim_{t\to\infty} \omega_V(\tau_0^t(A^*A)) \ln \frac{\omega_V(\tau_0^t(A^*A))}{\omega_V(\tau_0^t(AA^*))}$$

$$\overset{(EEB)}{\leq} \lim_{t\to\infty} -i\beta\, \omega_V\big(\tau_0^t(A^*)\, \delta_V(\tau_0^t(A))\big)$$

now $\delta_V = \delta_0 + i[V, \cdot]$ and $\delta_0 \tau_0^t = \tau_0^t \delta_0$:

$$\leq -i\beta \lim_{t\to\infty} \omega_V\big(\tau_0^t(A^* \delta_0(A))\big)$$

$$+ \beta \lim_{t\to\infty} \omega_V\big(\tau_0^t(A^*)\, [V, \tau_0^t(A)]\big)$$

the first term is $-i\beta\, \tilde{\omega}(A^* \delta_0(A))$, while the second is bounded above by

$$\lim_{t\to\infty} \beta\, \omega_V(\tau_0^t(A^*A))^{1/2} \, \| [V, \tau_0^t(A)] \| = 0 \quad \text{by assumption}$$

$\Rightarrow$ $\tilde{\omega}$ satisfies the (EEB) for the dynamics τ_0 $\Rightarrow$ $\tilde{\omega}$ is a (τ_0, β)-KMS state

$\square$

iii.d Symmetries

- Phase transitions often associated with the breaking of symmetries:
  - Rotational symmetries for magnetic transitions
  - Translational symmetry for liquid-solid transitions
  - Gauge groups in particle physics (Higgs phenomenon).

i.e.: the thermal states are not invariant under the symmetry of the Hamiltonian.

- First, Def: $(\mathcal{A}, \tau^t)$ as usual. A symmetry of $(\mathcal{A}, \tau^t)$ is an automorphism α of $\mathcal{A}$ s.t.

$$\alpha\left(\tau^t(A)\right) = \tau^t\left(\alpha(A)\right)$$

$$\forall A \in \mathcal{A},\ t \in \mathbb{R}.$$

Clearly: if ω is (τ^t, β)-KMS, then so is $\omega \circ \alpha$.
Reciprocally: we will look for conditions s.t. all (τ^t, β)-KMS states are α-invariant:
usually called: absence of symmetry breaking.

- (AI): $\exists$ a sequence $U_n \in \mathcal{A}$ of unitaries s.t.

$$\lim_{n \to \infty} \| \alpha(A) - U_n^\dagger A U_n \| = 0 \qquad \forall A \in \mathcal{A}.$$

and $U_n \in D(\delta)$

(AII): Version (i) $\exists M$ st! $\|\delta(U_n)\| \leq M$

Version (ii) all β-KMS states are α^2-invariant and $\exists M$ st.

$$\| U_n^\dagger \delta(U_n) + U_n \delta(U_n^\dagger) \| \leq M.$$

Theorem: Let α be a symmetry of $(\mathcal{A}, \tau^t)$ s.t. (AI) and [(AII)(i) or AII(ii)] are satisfied. Then all (τ^t, β)-KMS states are α-invariant, for all $\beta \in [0, \infty)$.

Note:
- The case of temperature 0 is left open: essentially transitions in the ground state.
- AII(i) is useful for discrete symmetries
  (ii) —— " —— continuous symmetry groups.

We will prove that $\exists C = C(\beta, \Pi)$ s.t.

$$\omega \circ \tau(A^*A) \leq C\, \omega(A^*A)$$

i.e. "uniform absolute continuity" of $\omega \circ \tau$ w.r.t. ω. By general results (not done here) $\omega \circ \tau$ is ω-normal, and because extremal KMS state are factor states $(\pi_\omega(\mathcal{A})' \cap \pi_\omega(\mathcal{A})'' = \mathbb{C} \cdot \mathbb{1})$, ~~then~~ ω and $\omega \circ \tau$ must be equal.

Key element: EEB inequality.

<u>Proof</u>: In the GNS rep. of ω: Hamiltonian H s.t. $H\Omega = 0$, Spectral resolution $H = \int \lambda\, dE(\lambda)$.

i) For any bounded interval $I \subset \mathbb{R}$, let h_n be a sequence of positive C^∞ functions, h_n compactly supported on an interval of most 1, $\sum_n h_n^2 \big|_I = \mathbb{1}$, the constant function 1.

Define

$$A_n = \int dt\, \hat{h}_n(t)\, \tau_t(A)$$

$$A_n^* = \int dt\, \hat{h}_n(-t)\, \tau_t(A) \qquad (h_n \text{ real})$$

Now:

$$-i\omega(A_n^* \delta(A_n)) = \int \lambda\, \hat{h}_n(-t)\, \hat{h}_n(s) \langle \Omega, \pi(\tau_t(A^*))\, dE(\lambda)\, \pi(\tau_s(A))\Omega \rangle$$

$$= \int \lambda\, \hat{h}_n(t)\, \hat{h}_n(s)\, e^{-it\lambda} e^{is\lambda} \langle \Omega, \pi(A^*)\, dE(\lambda)\, \pi(A)\Omega \rangle$$

$$= \int \lambda\, (h_n(\lambda))^2\, d\mu_A(\lambda)$$

Similarly: $\omega(A_n^* A_n) = \int (h_n(\lambda))^2\, d\mu_A(\lambda)$

hence, if h_n is supported in $[a_n, b_n]$, then

$$-i\omega(A_n^* \delta(A_n)) \leq b_n\, \omega(A_n^* A_n) \qquad (1)$$

Finally: $\omega(A_n A_n^*) = \int (h_n(\lambda))^2\, d\nu_A(\lambda)$

Recalling inequality (II) on page (S27):

$$\omega(A_n A_n^*) \leq e^{-\beta a_n} \omega(A_n^* A_n)$$

so that

$$\omega(A_n^* A_n) \log \frac{\omega(A_n^* A_n)}{\omega(A_n A_n^*)} \geq \beta a_n \, \omega(A_n^* A_n)$$

Now: write EEB for the observable $U_m A_n$:

$$\omega(A_n^* A_n) \log \frac{\omega(A_n^* A_n)}{\omega(U_m A_n A_n^* U_m^*)} \leq -i\beta\, \omega(A_n^* U_m^* \delta(U_m) A_n) - i\beta\, \omega(A_n^* \delta(A_n))$$

L.H.S: divide and multiply in the log by $\omega(A_n A_n^*)$ ~~to get~~ ~~R.H.S.~~ and use (1) and (2) on the R.H.S. to get

$$\omega(A_n^* A_n) \log \frac{\omega(A_n A_n^*)}{\omega(U_m A_n A_n^* U_m^*)} \leq -i\beta\, \omega(A_n^* U_m^* \delta(U_m) A_n) + \beta \underbrace{(b_n - a_n)}_{\leq 1 \text{ by assumption on } h_n} \omega(A_n^* A_n) \quad (3)$$

- With (AII)(i): $|\omega(A_n^* U_m^* \delta(U_m) A_n)| \leq \|\delta(U_m)\| \|U_m^*\| \omega(A_n^* A_n)$

yields

$$\log \frac{\omega(A_n A_n^*)}{\omega(U_m A_n A_n^* U_m^*)} \leq \beta(\Pi + 1)$$

i.e.

$$\omega(A_n A_n^*) \leq e^{\beta(\Pi+1)} \omega(U_m A_n A_n^* U_m^*) \xrightarrow{(m\to\infty)} e^{\beta(\Pi+1)} \omega(\alpha^{-1}(A_n A_n^*)) \quad (*i)$$

Finally: $\omega(A_\varphi A_\varphi^*) = \int d\nu_A(\lambda) = \sum_n \int d\nu_A(\lambda) (h_n(\lambda))^2 = \sum_n \omega(A_n A_n^*)$

and since $[\alpha, \tau_t] = 0$, we have that, by definition, $\alpha(A_n) = (\alpha(A))_n$. Summing $(*i)$ over n yields the claim.

- With (AII)(ii): use (3) and a similar bound for $U_m \leftrightarrow U_m^*$, and summing:

$$\omega(A_L^* A_L) \log \frac{\omega(A_L A_L^*)^2}{\omega(U_m A_L A_L^* U_m^*)\,\omega(U_m^* A_L A_L^* U_m)}$$
$$\leq -i\beta\,\omega\left[A_L^* \left(U_m^* \delta(U_m) + U_m \delta(U_m^*)\right) A_L\right] + 2\beta\,\omega(A_L^* A_L)$$

proceeding as before:

$$\omega(A_L A_L^*)^2 \leq e^{\beta(M+2)}\,\omega\left(\alpha(A_L A_L^*)\right)\omega\left(\alpha^{-1}(A_L A_L^*)\right)$$

the same could have been carried out for $\omega\circ\alpha$ to get

$$\omega\left(\alpha(A_L A_L^*)\right)^2 \leq e^{\beta(M+2)}\,\underbrace{\omega\left(\alpha^2(A_L A_L^*)\right)\omega\left(A_L A_L^*\right)}_{\cdot\,\omega(A_L A_L^*)^2}$$

$\square$

- Remark: This is an abstract property of the KMS state.
  In order to prove absence of symmetry breaking in a concrete system, we only need to check the assumptions.
  $\leadsto$ see later for compact Lie groups acting on quantum spin systems in $d \leq 2$: Mermin-Wagner theorem.

Remarks on the set of KMS states

- $(\mathcal{A}, \tau^t)$: C^*-dynamical system, $\mathcal{A}$ has a identity.
  The structure of the set of (τ^t, β)-KMS states, as a mathematical object allows for a definition of the "phases" of the physical system.
  Let K_β be the set of (τ^t, β)-KMS states for any fixed $\beta \in (0, \infty)$.
- Physical expectation: at high temperature, i.e. small β, K_β has a unique element having all the symmetries of the dynamics,
  but at low temperatures, i.e. large β, K_β has many elements corresponding the the thermodynamic phases and mixtures thereof.

Theorem: Let $\beta \in (0, \infty)$ (here in fact $\beta \in \mathbb{R}$).

(i) K_β is convex and weak-$*$ compact

(ii) K_β is a simplex

(iii) $\omega \in K_\beta$ is extremal $\Leftrightarrow$ ω is a factor state

(iv) if ω_1 and ω_2 are extremal, then they are either equal or disjoint

(v) if ~~$\ker(\omega) = \{0\}$~~ ω is faithful, i.e. $\omega(A^*A) > 0$ for all $A \neq 0$, there exists a unique time-evolution for which ω is a KMS state.

Remarks: * From (ii): there are $(n+1)$ states $\omega_0, \dots, \omega_n$ that are affinely independent $[(\omega_i - \omega_0)_{i=1}^{n}$ are indep.$]$ such that

$$K_\beta = \left\{ \sum_{i=0}^{n} \alpha_i \omega_i \text{ with } \sum_{i=0}^{n} \alpha_i = 1 \right\}.$$

in particular: each $\omega \in K_\beta$ has a <u>unique</u> decomposition in <u>extremal</u> elements ω_i, the <u>pure phases</u>

* (iv) disjointness: complement of quasi-equivalence:
  ω_1 and ω_2 are disjoint if no ω_1-normal state is ω_2-normal.
  in particular: if $\omega_2 \leq C\omega_1$ and ω_1, ω_2 are extremal KMS states, then they are not disjoint, hence they are equal

* About (i): K_β is a convex subset of $\mathcal{E}(\mathcal{A})$.
  Moreover, the EEB yields:
  $$K_\beta = \left\{ \omega \in \mathcal{E}(\mathcal{A}) : \omega(A^*A)\log\frac{\omega(A^*A)}{\omega(AA^*)} - i\beta\omega\left(A^*\delta(A)\right) \geq 0 \right\}$$
  which shows that K_β is closed in the weak-* topology.
  Hence K_β is weakly-* compact.

* About (v): Uniqueness: let $\tau, \tilde{\tau}$ be two such dynamics.
  $$\begin{aligned} F(t) :&= \omega\left(\tilde{\tau}_{-t}(\tau_t(A))B\right) = \\ &= \omega\left(\tau_t(A)\tilde{\tau}_t(B)\right) && (\tilde{\tau}\text{-invariance}) \\ &= \omega\left(\tilde{\tau}_t(B)\tau_{t+i\beta}(A)\right) && (\tau\text{-KMS condition}) \\ &= \omega\left(\tau_{t+i\beta}(A)\tilde{\tau}_{t+i\beta}(B)\right) && (\tilde{\tau}\text{- — " —}) \\ &= F(t+i\beta) \end{aligned}$$
  hence $z \mapsto F(z)$ is a bounded periodic function that is holomorphic $\Rightarrow F$ is constant.
  since $\ker(\omega) = \{0\}$,
  $$\tilde{\tau}_{-t}(\tau_t(A))B = AB \quad \text{for all } t \in \mathbb{R}, \ A, B \text{ analytic elements of } \tau, \text{ resp. } \tilde{\tau}.$$
  choosing $B = \mathbb{1}$:
  i.e. $\cancel{\tau_t(A)B}$ $\tilde{\tau}_{-t}(\tau_t(A)) = A$, on a dense subalgebra
  $\Rightarrow \tilde{\tau}_{-t} = (\tau_t)^{-1} = \tau_{-t}$

* If ω is a (τ^t, β)-KMS state and α a $*$-automorphism, then $\omega \circ \alpha$ is a $(\alpha^{-1}\tau^t\alpha, \beta)$-KMS state. Indeed,

$$(\omega\circ\alpha)(A(\alpha^{-1}\tau^{t+i\beta}\alpha)(B)) = \omega\left(\alpha(A)\,\tau^{t+i\beta}(\alpha(B))\right)$$

$$\overset{KMS}{=} \omega\left(\tau^t(\alpha(B))\,\alpha(A)\right)$$

$$= \omega\circ\alpha\left((\alpha^{-1}\tau^t\alpha)(B)\,A\right)$$

Now if $\omega\circ\alpha = \omega$, by uniqueness above,

$$\alpha^{-1}\tau^t\alpha = \tau^t \quad \text{or} \quad \tau^t\circ\alpha = \alpha\circ\tau^t$$

i.e. if ω is KMS and invariant, then the dynamics has the symmetry α

reciprocally, if α is a symmetry, then $\omega\circ\alpha$ is (τ^t,β)-KMS.

i.e.

Proposition: $(\mathcal{A}_q, \tau^t)$: C^*-dynamical system, $\beta \in (0,\infty)$, ω a (τ^t, β)-KMS state. Then

(i) $\omega\circ\alpha$ is a $(\alpha^{-1}\tau^t\alpha, \beta)$-KMS state.

(ii) if $\omega\circ\alpha = \omega$ (*), then $\tau^t\circ\alpha = \alpha\circ\tau^t$

(iii) if $\tau^t\circ\alpha = \alpha\circ\tau^t$, then $\omega\circ\alpha$ is a (τ^t,β)-KMS state

(*) and ω is faithful,

The free Bose gas & BEC (R. Helling)

Renormalization, Monte-Carlo simulations
& the Metropolis algorithm (R. Helling)

I Phase transitions in Q.S.S.

i) Generalities

Recall: $x \in \Gamma$ $\dim \mathcal{H}_x < \infty$;

$\Lambda \in \mathcal{F}(\Gamma)$: $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x$

Observables: $\mathcal{A}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda) = \bigotimes_{x \in \Lambda} \mathcal{B}(\mathcal{H}_x)$

$\Lambda_1 \subseteq \Lambda_2 \to \mathcal{A}_{\Lambda_1} \simeq \mathcal{A}_{\Lambda_1} \otimes \mathbb{1} \subseteq \mathcal{A}_{\Lambda_2}$

Quasi-local algebra: $\mathcal{A}_\Gamma := \overline{\bigcup_{\Lambda \in \mathcal{F}(\Gamma)} \mathcal{A}_\Lambda}^{\|\cdot\|}$

- Dynamics : $H_\Lambda = \sum_{X \subset \Lambda} \Phi(X)$

where $\Phi(X) = \Phi(X)^* \in \mathcal{A}_X$

various norms: usually of the form:

$$\|\Phi\|_\xi = \sup_{x \in \Gamma} \sum_{X \ni x} \|\Phi(X)\| \, \xi(X)$$

where $\xi : \mathcal{F}(\Gamma) \to \mathbb{R}_+$

with some decay in the size of X.

$\hookrightarrow$ Banach space: $\mathcal{B}_\xi = \{ \Phi : \|\Phi\|_\xi < \infty \}$.

Precisely: Let N be the maximal number of neighbours in Γ.

Let $\lambda > 0$, and define

$$\|\Phi\|_\lambda := \|\Phi\|_{\xi_\lambda} \quad \text{with} \quad \xi_\lambda(X) = |X| N^{2|X|} e^{\lambda D(X)}$$

where $D(X) = \max \{ d(x,y) : x, y \in X \}$.

i.e. $\Phi \in \mathcal{B}_\lambda$ has exponential decay.

- For $\Phi \in \mathcal{B}_\lambda$, let $\tau_t^{\Phi}{}_\Lambda(A) = e^{itH_\Lambda^\Phi} A e^{-itH_\Lambda^\Phi}$, $A \in \mathcal{A}_\Lambda$.

Theorem: Let $\lambda > 0$ and $\Phi \in \mathcal{B}_\lambda$. There exists a strongly continuous, one-parameter group of automorphisms on $\mathcal{A}$, $\{\tau_t^{\Phi}\}_{t\in\mathbb{R}}$ s.t.

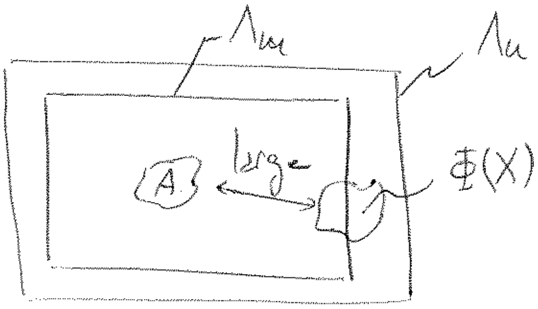
$$\lim_{n\to\infty} \left\| \tau^{\Phi}_{\Lambda_n,t}(A) - \tau_t^{\Phi}(A) \right\| = 0$$

for all $A \in \mathcal{A}_{loc}$. The convergence is uniform on compact sets, and the limit independent of the sequence $\{\Lambda_n\}$

Sketch of proof: $n > m$, $\Lambda_m \subset \Lambda_n$, $A \in \mathcal{A}_{loc}$:

$$\tau^{\Phi}_{\Lambda_n,t}(A) - \tau^{\Phi}_{\Lambda_m,t}(A) = \int_0^t \frac{d}{ds}\left(\tau^{\Phi}_{s,\Lambda_n} \circ \tau^{\Phi}_{t-s,\Lambda_m}(A)\right) ds$$

$$\Rightarrow \left\| \tau^{\Phi}_{t,\Lambda_n}(A) - \tau^{\Phi}_{t,\Lambda_m}(A) \right\| \leq \sum_{x\in\Lambda_n\setminus\Lambda_m} \sum_{X \ni x} \int_0^{|t|} \left\| \left[\Phi(X), \tau^{\Phi}_{s,\Lambda_m}(A)\right] \right\| ds$$



For a local dynamics as above: Lieb-Robinson bound.
$A \in \mathcal{A}_X$; $B \in \mathcal{A}_Y$, $X \cap Y = \emptyset$:

$$\left\| \left[\tau^{\Phi}_{s,\Lambda}(A), B\right] \right\| \leq C(A,B)\, e^{-\lambda(d(X,Y) - v|t|)}$$

$\Rightarrow \left\| \left[\Phi(X), \tau^{\Phi}_{s,\Lambda_m}(A)\right] \right\|$ is exponentially small for fixed t, as $\Lambda_m, \Lambda_n \to \Gamma$

$\Rightarrow \left\{ \tau^{\Phi}_{t,\Lambda_n}(A) \right\}_{n\in\mathbb{N}}$ is a Cauchy sequence $\Rightarrow \exists\, \tau_t^{\Phi}(A)$

$\square$

- Examples: Let $\mathcal{E}_\Gamma$ be the set of edges of Γ, i.e. $(x,y) \in \mathcal{E}_\Gamma$ is a nearest-neighbour pair.

Ising model (quantum). σ_x^i, $i=1,2,3$: Pauli matrices in $\mathcal{A}_x$.

$$H_{\Lambda,h} = -\sum_{(x,y)\in\mathcal{E}_\Lambda} \sigma_x^1 \sigma_y^1 - h \sum_{x\in\Lambda} \sigma_x^3, \quad h>0.$$

Heisenberg model: Let S_x^i, $i=1,2,3$ be the generators of the spin s, unitary irreducible representation of $SU(2)$

$$H_{\Lambda,h}^{F,AF} = \mp \sum_{(x,y)\in\mathcal{E}_\Lambda} \underbrace{\vec{S}_x \cdot \vec{S}_y}_{= S_x^1 S_y^1 + S_x^2 S_y^2 + S_x^3 S_y^3} - h \sum_{x\in\Lambda} S_x^3 \quad , h>0.$$

variants: + space-dependent coefficients J_{xy}
+ anisotropic interactions: $J_{xy}^{(1)}, J_{xy}^{(2)}, J_{xy}^{(3)}$

Often used notation for two-body Hamiltonians: (not N.N.!)

$$H_\Lambda = \sum_{x,y\in\Lambda} J(x,y)\, \Phi_{x,y}$$

with $\Phi_{x,y} \in \mathcal{A}_{\{x,y\}}$, $\|\Phi_{x,y}\| \leq 1$

and $J : \Lambda^2 \to \mathbb{R}$

describes the decay of the interaction.

The Mermin-Wagner Theorem for Q.S.S.

- There are various versions of MW, and various degrees of rigour/vagueness in its statement. In general, it refers to
  + absence of symmetry breaking, or
  + absence of long-range-order, or
  + absence of phase transitions (bad statement!)

  in low-dimensional systems, $d \leq 2$.
- Here: we will see it as an application of the EEB inequality.

  Recall: Need: i) $\{U_n\}_{n\in\mathbb{N}}$, $U_n \in \mathcal{A}$.

$$\| \alpha(A) - U_n^* A U_n \| \to 0 \quad (A \in \mathcal{A})$$

and $U_n \in \mathcal{D}(\delta)$.

ii) all β-KMS states are α^2-invariant, and $\exists M$ s.t.

$$\| U_n^* \delta(U_n) + U_n \delta(U_n^*) \| \leq M \qquad \forall n \in \mathbb{N}$$

- What symmetries? G: compact, connected Lie group.

  Consider: $\mathcal{H}_x = \mathcal{H}$ carries a unitary representation $g \mapsto U_g^x$.

  $\leadsto$ on the algebra: automorphism.

$$A \in \mathcal{A}_{\{x\}} : \quad \alpha_g^{\{x\}}(A) = \left(U_g^x\right)^* A \, U_g^x$$

extends by tensor product to $\mathcal{A}_{loc}$

$\leadsto$ Since $\| \alpha_g(A) \| = \|A\|$, α_g extends to all of $\mathcal{A}$.

Typical example: $G = SU(2)$, $\dim \mathcal{H} = 2s+1$,

$$U_g = e^{i\vec{g}\,\vec{S}} \qquad , \; g \in S^2$$

quantum mechanical rotations around the axis given by g.

check. $\left(U_g^x \otimes U_g^y\right)^* \vec{S}_x \cdot \vec{S}_y \left(U_g^x \otimes U_g^y\right) = \vec{S}_x \cdot \vec{S}_y$

the isotropic Heisenberg interaction is invariant under $SU(2)$.

- Theorem: Let $\mathcal{A}$ be the algebra of a Q.S.S. on $\mathbb{Z}^2$, and let α_g be the above action of G on $\mathcal{A}$, where G is a compact, connected Lie group.
  Let Φ be a two-body interaction s.t.

$$\alpha_g(\Phi_{x,y}) = \Phi_{x,y} \quad \forall g \in G, \ x,y \in \Gamma$$

Assume:

$$\sup_{x \in \Gamma} \sum_{y \in \mathbb{Z}^2} |x-y|^2 \, |J(x,y)| < \infty \qquad (*)$$

For any $\beta < \infty$, if ω is a (τ^Φ, β)-KMS state, then

$$\omega \circ \alpha_g = \omega \quad \forall g \in G.$$

$\rightsquigarrow$ No spontaneous breaking of the symmetry.

- Notes: * A similar theorem (different condition on decay rate) holds in $d=1$, i.e. on $\mathbb{Z}$.
  - * No statement about $T = 0$
  - * There can be phase transitions: "Kosterlitz-Thouless" where correlation functions go from exponential decay to polynomial decay at high β.

- Key of proof: Choose U_i's that smoothly (continuous symmetry!) interpolates between a fixed rotation in finite volume and no rotation at spatial infinity

$\rightsquigarrow$ allow for a uniform bound on the energy of the configuration.

- First: a lemma that allows for the application of the KMS-symmetry-Theorem to continuous symmetries.

<u>Lemma</u>: Let $\{\alpha_\phi : \phi \in S^1\}$ be a compact, connected, continuous one-parameter group of automorphisms of $\mathcal{A}$. Let
$$\mathcal{K} := \{\omega \in \mathcal{E}(\mathcal{A}) : \omega \circ \alpha^2_\phi = \omega \text{ implies } \omega \circ \alpha_\phi = \omega \text{ for all } \phi \in S^1\}$$

$$\text{Then } \omega \in \mathcal{K} \Rightarrow \omega \circ \alpha_\phi = \omega \quad \forall \phi \in S^1.$$

Note: the Theorem, with hypothesis (ii), gives precisely the condition for ω to be in $\mathcal{K}$; the lemma concludes.

<u>Proof</u>: Let $\omega \in \mathcal{K}$. Since $\alpha_\pi^2 = \alpha_{\pi+\pi} = id$, $\omega \circ \alpha_\pi = \omega$.
Repeating the argument n times:
$$\omega \circ \alpha_{\pi/2^n} = \omega \quad , \quad \forall n \in \mathbb{N}.$$
Now: $D = \{\phi \in S^1 : \phi = \sum_{n=0}^{N} a_n \pi/2^n , N \in \mathbb{N}\}$ is dense in S^1.

For $A \in \mathcal{A}$: $\xi_\phi(A) = \omega(\alpha_\phi(A) - A)$ is s.t.
$\xi_\phi(A) = 0$ if $\phi \in D$, $\phi \mapsto \xi_\phi(A)$ is continuous
hence $\xi_\phi(A) = 0$ for all $\phi \in S^1$. $\square$

The lemma must be applied for a generating set of one-dimensional compact subgroups. In the case of $SU(2)$:
$$e^{i\phi S^1} , \quad e^{i\phi S^2} , \quad e^{i\phi S^3}$$
In general, we'll write
$$U^x_\phi = e^{i\phi X_x} , \quad \phi \in S^1$$
$$X_x = X_x^* \in \mathcal{A}_{\{x\}} , \quad x \in \mathbb{Z}^2.$$

<u>Note</u> translation invariance not required, we would need $\|X_x\| \leq \cancel{\alpha} \;\; \forall x \in \mathbb{Z}^2$.

Proof of Theorem. $\Lambda_m = [-m, m]^2 \subset \mathbb{Z}^2$. Let

$$U_m(\phi) := \bigotimes_{x \in \Lambda_{2m}} U^x(\phi_m)$$

⚠

with $\phi_m(x) = \phi$, $x \in \Lambda_m$, arbitrary if $x \in \Lambda_{2m} \setminus \Lambda_m$.

Condition i). $A \in \mathcal{A}_{loc}$ $\exists \Lambda \in \mathcal{F}(\Gamma)$: $A \in \mathcal{A}_\Lambda$. Take $m_0 := \min\{m. \Lambda_m \supset \Lambda\}$.

Then $U_m^+(\phi) A U_m(\phi) = \alpha_\phi(A) \quad \forall m \geq m_0$.

and $U_m(\phi) \subset \mathcal{A}_{\Lambda_{2m}} \subset \mathcal{A}_{loc} \subset D(\delta)$ ✓

Condition ii) Now, choose, for $x = (x_1, x_2) \in \mathbb{Z}^2$

$$\phi_m(x) = \left(2 - \frac{\cancel{\min}\ \max\{|x_1|, |x_2|\}}{m}\right)\phi \quad , x \in \Lambda_{2m} \setminus \Lambda_m$$

note: $\min\{|x_1|, |x_2|\} \leq 2m$ by def,
and $\phi_m(x)$ interpolates between ϕ on $\partial\Lambda_m$ and 0 on $\partial\Lambda_{2m}$.

$$U_m(\phi)\,\delta\big(U_m(\phi)^+\big) = i U_m(\phi) \sum_{x,y} J(x,y) \Big[\Phi_{x,y}, U^x(\phi_m) \otimes U^y(\phi_m)\Big] \bigotimes_{z \neq x,y} U^z(\phi_m)$$

$$= i \sum_{x,y} J(x,y) \Big[U^x(\phi_m) \otimes U^y(\phi_m)\, \Phi_{x,y} \big(U^x(\phi_m) \otimes U^y(\phi_m)\big)^+ - \Phi_{x,y} \Big]$$

so that $\| U_m(\phi)\,\delta(U_m(\phi)^+) + U_m(\phi)^+ \delta(U_m(\phi)) \|$

$$\leq \sum_{x,y} |J(x,y)|\, \|\Delta^m_{x,y}\| \qquad + \; U \leftrightarrow U^+$$

Note: $\phi_m(x) X_x + \phi_m(y) X_y = \frac{1}{2}(\phi_m(x) + \phi_m(y))(X_x + X_y) + \frac{1}{2}(\phi_m(x) - \phi_m(y))(X_x - X_y)$

i.e. $\phi_m(x) X_x + \phi_m(y) X_y =: S^m_{x,y} + D^m_{x,y}$

and (check!) $[S^m_{x,y}, D^m_{x,y}] = 0$.

with this,

$$U^x(\phi_m) \otimes U^y(\phi_m) = e^{i\phi_m(x) X_x} \otimes e^{i\phi_m(y) X_y} = e^{i(S^m_{x,y} + D^m_{x,y})}$$
$$= e^{iS^m_{x,y}} e^{iD^m_{x,y}}$$

Note: $$e^{iS^m_{x,y}} \Phi_{x,y} e^{-iS^m_{x,y}} = \alpha_{\frac{1}{2}(\phi_m(x)+\phi_m(y))}(\Phi_{x,y})$$
$$= \Phi_{x,y} \text{ by assumption.}$$

and expand $e^{iD^m_{x,y}}$:

multiple commutator.

$$e^{iD^m_{x,y}} \Phi_{x,y} e^{-iD^m_{x,y}} = \Phi_{x,y} + \sum_{k\geq 1} \frac{i^k}{k!} \mathrm{ad}^k_{D^m_{x,y}}(\Phi_{x,y})$$

in $\Delta^m_{x,y}$: sum $U\delta(U^*) + U^*\delta(U)$, the two terms differing in the above expansion by the sign of $i \Rightarrow$ all odd terms vanish to yield:

$$\|\Delta^m_{x,y}\| \leq 2 \sum_{k\geq 1} \frac{1}{(2k)!} \left(\tfrac{1}{2}|\phi_m(x) - \phi_m(y)|\right)^{2k} \|\mathrm{ad}^{2k}_{(X_x - X_y)}(\Phi_{x,y})\|$$

importantly, $|\phi_m(x) - \phi_m(y)| \leq |\phi| \min\{1, \frac{|x-y|}{m}\}$

so that $|\cdot|^{2k} \leq \left(\frac{|x-y|}{m}\right)^2 \cdot |\phi|^{2k}$. Hence,

$$\sum_{x,y} |J(x,y)| \, \|\Delta^m_{x,y}\| \leq 4 \sum_{\substack{x\in\Lambda_m \\ y\in\mathbb{Z}^2}} |J(x,y)| \left(\frac{|x-y|}{2m}\right)^2 .$$
$$\cdot \sum_{k\geq 1} \frac{1}{(2k)!} |\phi|^{2k} \underbrace{(2\|X_x - X_y\|)}_{=4X}{}^{2k} \underbrace{\|\Phi_{x,y}\|}_{\leq 1}$$

$$\leq 4 \sum_{\substack{x\in\Lambda_{2m} \\ y\in\mathbb{Z}^2}} \frac{1}{(2m)^2} |J(x,y)| \, |x-y|^2 \exp(4X|\phi|)$$

in 2d: $|\Lambda_{2m}| = (4m+1)^2$ so that

$$\sum_{x,y} |J(x,y)| \, \|\Delta^m_{x,y}\| \le 4 e^{4X|\phi|} \sup_x \frac{(4m+1)^2}{(2m)^2} \sum_{y \in \mathbb{Z}^2} |J(x,y)| \, |x-y|^2$$

$$\le M.$$

i.e. $\| U_m \delta(U_m^*) + U_m^* \delta(U_m) \| \le M$, uniformly in m

$\Rightarrow$ if ω is α_ϕ^2 invariant, then ω is α_ϕ invariant

$\Rightarrow$ by Lemma: $\omega \circ \alpha_\phi = \omega$ $\square$

- Consequence: For $H^{F, AF}_{h=0}$, the Heisenberg model without external field, any β-KMS state, $\beta < \infty$, is invariant under the action of $SU(2)$, or $SO(3)$, $d \le 2$.
  In particular,
  $$\omega(\sigma_x^3) = 0 \qquad \forall x \in \mathbb{Z}^d.$$

- Remark: $(*)$ in 2d, the condition of the theorem can be written as
  $$J_{x,y} \le C(1 + |x-y|)^{-4}.$$
  There are systems with $J_{x,y} \sim C(1+|x-y|)^{-4+\varepsilon}$ for which spontaneous symmetry breaking occurs i.e. the condition $(*)$ is sharp.

vii c) Long-range order in the 3D Heisenberg model

- Dyson-Lieb-Simon, 1978: First rigorous proof of the existence of a phase transition in an interacting system with a continuous symmetry group.
Proof relies on a special property of the Hamiltonian, called <u>reflection positivity</u>: it applies only to the antiferromagnet

$\rightsquigarrow$ <u>Open</u> problem: prove symmetry breaking for the Heisenberg ferromagnet at finite temperature.

Recent step: Correggi, Giuliani, Seiringer, 2013: the free energy behaves at low temperatures like free bosons (validity of the spin wave approximation) but: nothing about its derivative (magnetization)

- Consider: S^1, S^2, S^3: spin matrices, any magnitude S.

$$H_\Lambda^{(u)} := -2 \sum_{(x,y) \in \mathcal{E}_\Lambda} \left(S_x^1 S_y^1 + u S_x^2 S_y^2 + S_x^3 S_y^3 \right)$$

on $\Lambda = \{-\frac{L}{2}+1, \dots, \frac{L}{2}\}^d$, $L \in 2\mathbb{N}$, periodic B.C.

* $u=1$: Heisenberg ferromagnet
* $u=0$: XY-model
* $u=-1$: unitarily equivalent to the Heisenberg antiferromagnet.

Bipartite lattice: $\Lambda = \Lambda_A \cup \Lambda_B$

with $(x,y) \in \mathcal{E}_\Lambda \Rightarrow \begin{pmatrix} x \in \Lambda_A \\ y \in \Lambda_B \end{pmatrix}$ or $\begin{pmatrix} x \in \Lambda_B \\ y \in \Lambda_A \end{pmatrix}$

let $e^{i\pi S_x^2}$ (rotation by π around 2^{nd} axis):

$$e^{-i\pi S_x^2} S_x^{1,3} e^{i\pi S_x^2} = -S_x^{1,3} \quad ; \quad S_x^2 \text{ invariant}$$

For $U_\Lambda := \prod_{x\in\Lambda_A} \exp\left(i\pi S_x^2\right)$:

$$U_\Lambda^{-1} H_\Lambda^{(u)} U_\Lambda = +2 \sum_{(x,y)\in\mathcal{E}_\Lambda} \left(S_x^1 S_y^1 + S_x^3 S_y^3 - u\, S_x^2 S_y^2 \right)$$

Define the average magnetization: $M_\Lambda := \frac{1}{|\Lambda|} \sum_{x\in\Lambda} S_x^3$

and we are interested in $\omega_{\beta,\Lambda}^{(u)}\left((M_\Lambda)^2\right)$. By translation invariance:

$$\omega_{\beta,\Lambda}^{(u)}\left((M_\Lambda)^2\right) = \frac{1}{|\Lambda|}\, \omega_{\beta,\Lambda}^{(u)}\left(\sum_{x\in\Lambda} S_0^3 S_x^3\right)$$

Necessary condition for $\lim_{\Lambda\to\mathbb{Z}^d} \omega_{\beta,\Lambda}^{(u)}\left((M_\Lambda)^2\right) \neq 0$ is that

$$\omega_{\beta,\Lambda}^{(u)}\left(S_0^3 S_x^3\right) \not\to 0 \quad \text{as } |x|\to\infty.$$

long-range order. (LRO)

Theorem: Consider $H_\Lambda^{(u)}$ for $u\in[-1,0]$. Then:

$$\frac{1}{|\Lambda|}\sum_{x\in\Lambda} \omega_{\beta,\Lambda}^{(u)}\left(S_0^3 S_x^3\right) \geq \frac{1}{3} S(S+1)$$

$$- \frac{1}{\sqrt{2}|\Lambda|} \sum_{k\in\Lambda^*\setminus\{0\}} \left(\frac{E_u(k)}{E(k)}\right)^{1/2}$$

$$- \frac{1}{2\beta|\Lambda|} \sum_{k\in\Lambda^*\setminus\{0\}} \frac{1}{E(k)}$$

where: $E(k) = 2\sum_{i=1}^{d} (1-\cos(k_i))$

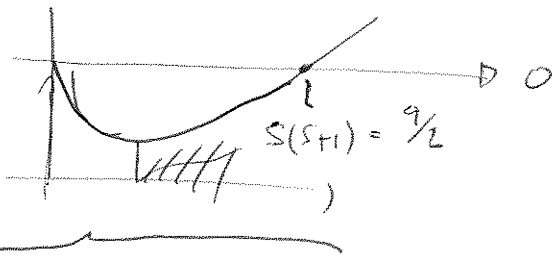
$$E_u(k) = \sum_{i=1}^{d} \Big[(1-u\cos(k_i))\, \omega_{\beta,\Lambda}^{(u)}\left(S_0^1 S_{e_i}^1\right) + (u-\cos(k_i))\, \omega_{\beta,\Lambda}^{(u)}\left(S_0^2 S_{e_i}^2\right) \Big]$$

Note:

$$|\omega^{(u)}_{\beta,\Lambda}(S_0^i S_{e_i}^i)|^2 \leq \omega^{(u)}_{\beta,\Lambda}(|S_0^i|^2)\, \omega^{(u)}_{\beta,\Lambda}(|S_{e_i}^i|^2) \leq \omega^{(u)}_{\beta,\Lambda}((\vec{S})^2)^2 = (S(S+1))^2$$

Hence: $$|E_u(k)| \leq S(S+1) \sum_{i=1}^{d} (1+u)(1-\cos(k_i))$$

i.e. $$\left|\frac{E_u(k)}{E(k)}\right| \leq S(S+1)$$



Hence, from Theorem:

$$\liminf_{\Lambda \to \mathbb{Z}^d} \omega^{(u)}_{\beta,\Lambda}((\Pi_\Lambda)^2) \geq \overbrace{\frac{1}{3} S(S+1) - \frac{1}{\sqrt{2}}\sqrt{S(S+1)}}$$

$$- \frac{1}{(2\pi)^d\, 2\beta} \int_{[-\pi,\pi]^d} dk \, \frac{1}{E(k)}$$

but $1-\cos(x) \leq \frac{1}{2}x^2 \Rightarrow E(k) \leq |k|^2$ and the integral is <u>convergent for $d \geq 3$</u>

$\Rightarrow$ for $S \geq 2$ and $d \geq 3$, for β large enough,

$$\liminf \omega^{(u)}_{\beta,\Lambda}((\Pi_\Lambda)^2) \geq C > 0$$

The estimates can be improved to obtain $S \geq \frac{1}{2}$, $d \geq 3$

- <u>Remarks</u>: i) Free energy density with a magnetic field:

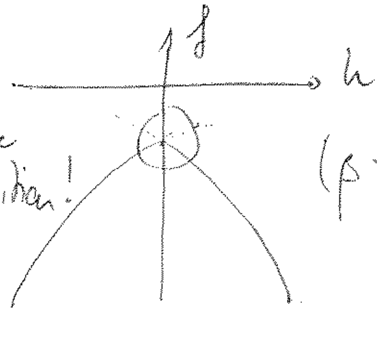
$$f(\beta,h) = \lim_{\Lambda \to \mathbb{Z}^d} |\Lambda|^{-1} \ln\left[\mathrm{Tr}\left(\exp\left(-\beta H^{(u)}_\Lambda + h \sum_{x\in\Lambda} S_x^3\right)\right)\right]$$

Corollary of the Theorem:

$$\lim_{h \downarrow 0} \frac{d}{dh} f(\beta,h) > 0$$

for β large enough

phase transition!



$(\beta > \beta_c)$

ii) The case $u=-1$.

$$\omega_{\beta,\text{Heisenberg}}\left(S_0^3 S_x^3\right) = \omega_{\beta,\Lambda}^{(u=-1)}\left(U_\Lambda^{-1} S_0^3 S_x^3 U_\Lambda\right)$$
$$= (-1)^{d(0,x)}\, \omega_{\beta,\Lambda}^{(u=-1)}\left(S_0^3 S_x^3\right)$$

antiferromagnetic ordering, "Neel ordering":

$$\cdots \begin{matrix}\uparrow&\downarrow&\uparrow&\downarrow&\uparrow\\ \downarrow&\uparrow&\downarrow&\uparrow&\downarrow\end{matrix} \cdots$$

- Keywords for the proof:
  - i) Reflection positivity
  - ii) Gaussian domination
  - iii) Bound on Duhamel's two-point function
  - iv) Infrared bound

and i $\Rightarrow$ ii $\Rightarrow$ iii $\Rightarrow$ iv $\rightsquigarrow$ Theorem

Last step: Let $C_\Lambda(x) := \omega_{\beta,\Lambda}^{(u)}\left(S_0^3 S_x^3\right)$, and note,

$$\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\omega_{\beta,\Lambda}^{(u)}\left(S_0^3 S_x^3\right) = \hat{C}_\Lambda(0) = C_\Lambda(0) - \sum_{k\in\Lambda^*\setminus\{0\}}\hat{C}_\Lambda(k)$$

and the theorem follows from a good control of $\hat{C}_\Lambda(k)$, in particular for small k's i.e. infrared bounds ~~(large wavelengths)~~. (control of long wavelength fluctuations).

- (i) & (ii).

We consider the model with a (fictitious) external field. Let $v=(v_x)_{x\in\Lambda}$, with $v_x\in\mathbb{R}$, and let $h=\Delta v$, i.e.

$$h_x = \sum_{y:(x,y)\in\mathcal{E}_\Lambda}(v_y - v_x)$$

as a matrix in, say $\ell^2(\Lambda)$:

$$\Delta_{xy} = \begin{cases} -2d & \text{if } x=y \\ 1 & \text{if } (x,y)\in\mathcal{E}_\Lambda \\ 0 & \text{otherwise}\end{cases}$$

check: in the $\ell^2(\Lambda)$ scalar product: (cf integration by parts!)

$$\langle f, -\Delta g\rangle = \sum_{(x,y)\in\mathcal{E}_\Lambda} (f_x - f_y)(g_x - g_y)$$

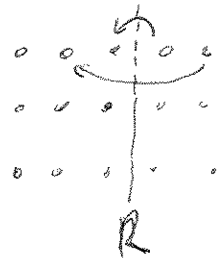
Now: let

$$H_\Lambda(v) = H_\Lambda^{(u)} - \sum_{x\in\Lambda} h_x S_x^3$$

and $$Z_{\beta,\Lambda}^{(u)}(v) = \mathrm{Tr}\left(\exp\left(-\beta H_\Lambda(v)\right)\right).$$

and $$\tilde{Z}_{\beta,\Lambda}^{(u)}(v) = Z_{\beta,\Lambda}^{(u)}(v)\, e^{\frac{1}{4}\beta\langle v,\Delta v\rangle}$$

Consider a reflection R across a hyperplane of the lattice:



and write a field v as $v = v_1|v_2$

<u>Lemma 1</u> (Reflection positivity) ~~the~~ If $u \le 0$, then for any reflection R,

$$\tilde{Z}_{\beta,\Lambda}^{(u)}(v_1|v_2)^2 \le \tilde{Z}_{\beta,\Lambda}^{(u)}(v_1|Rv_1)\,\tilde{Z}_{\beta,\Lambda}^{(u)}(Rv_2|v_2)$$

<u>Lemma 2</u> (Gaussian domination) If $u \le 0$,

$$\frac{\mathrm{Tr}\left(\exp\left(-\beta\left[H_\Lambda^{(u)} - \sum_{x\in\Lambda} h_x S_x^3\right]\right)\right)}{\mathrm{Tr}\left(\exp\left(-\beta H_\Lambda^{(u)}\right)\right)} \le e^{-\frac{1}{4}\beta\langle v,\Delta v\rangle}$$

Note that $-\langle v,\Delta v\rangle = \|h\|^2$, hence the term G.D.

Proof of Lemma 1: If $\mathcal{H} = K \otimes K$, $\dim K < \infty$, $A, B, C_1 \dots C_\ell \in \mathcal{B}(K)$ are <u>real</u> matrices, $h_1, \dots h_\ell \in \mathbb{R}$. Then:

$$\mathrm{Tr}\left(e^{A\otimes 1 + 1\otimes B - \sum_{k=1}^{\ell} (C_k \otimes 1 - 1 \otimes C_k - h_k)^2}\right)^2$$

$$\leq \mathrm{Tr}\left(e^{A\otimes 1 + 1 \otimes A - \sum_{k=1}^{\ell} (C_k \otimes 1 - 1 \otimes C_k)^2}\right) \cdot \mathrm{Tr}\,(A \leftrightarrow B)$$

This follows from

i) Trotter's product formula: $e^{X+Y} = \lim_{n\to\infty} \left(e^{\frac{X}{n}} e^{\frac{Y}{n}}\right)^n$

ii) The operator identity

$$e^{-D^2} = \frac{1}{\sqrt{4\pi}} \int e^{ikD} e^{-\frac{1}{4}k^2} dk$$

to obtain a linear expression for the "C-term"

iii) Cauchy-Schwarz, using the reality of the matrices.

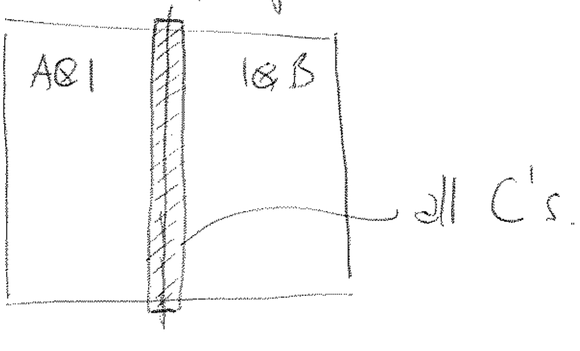
Lemma 1 follows from the claim by using:

$A \otimes 1 = -\beta H_{\Lambda_1}(V_1)$; $1 \otimes B = -\beta H_{\Lambda_2}(V_2)$

$C_{k,1} = -\sqrt{\beta}\, S^1_{z_k}$; $C_{k,2} = \sqrt{\beta u}\, S^2_{z_k}$; $C_{k,3} = \sqrt{\beta}\, S^3_{z_k}$

and $h_{k,i} = V_{z_k} \delta_{i,3}$

z_k is either x_k or y_k for each pair $(x,y) \in \mathcal{E}_\Lambda$ s.t. $x \in \Lambda_1$; $y \in \Lambda_2$



indeed:

$$H_\Lambda(v) = \sum_{(x,y)\in\mathcal{E}_\Lambda} \Big((S_x^1 - S_y^1)^2 + (\sqrt{u}\, S_x^2 - \sqrt{u}\, S_y^2)^2 + \big((S_x^3 + \tfrac{v_x}{2}) - (S_y^3 + \tfrac{v_y}{2})\big)^2 \Big) + \Big(-2\sum_{x\in\Lambda}\big((S_x^1)^2 + u(S_x^2)^2 + (S_x^3)^2\big)\Big) - \underbrace{\frac{1}{4}\sum_{(x,y)\in\mathcal{E}_\Lambda}(v_x - v_y)^2}_{\text{is substracted in } \tilde{Z}}$$

Importantly: S_x^1 and S_x^3 are real matrices, $\sqrt{u}\,S_x^2$ is real iff $\underline{u \leq 0}$ □

Proof of Lemma 2: by a simple calculation, the statement of the lemma is equivalent to

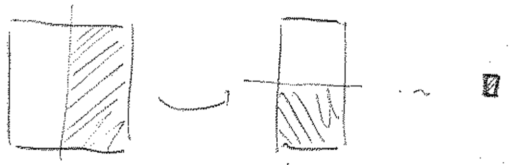
$$\tilde{Z}^{(u)}_{\beta,\Lambda}(v) \leq \tilde{Z}^{(u)}_{\beta,\Lambda}(0)$$

i.e. the zero field $v \equiv 0$ is a maximizer for the functional $\tilde{Z}^{(u)}_{\beta,\Lambda}(v)$.

i) Existence of a maximizer: $v \mapsto \tilde{Z}(v)$ is continuous and goes to zero as $v_x \to \infty$ $\Rightarrow$ $\exists$ maximizer $\underline{v}$.
Let $\Pi = \tilde{Z}(\underline{v})$.

ii) if $\tilde{Z}(\underline{v}_1 | R\underline{v}_1) < \Pi$, then, by Lemma 1,
$\Pi^2 < \Pi\, \tilde{Z}(R\underline{v}_2 | \underline{v}_2)$ is a contradiction
$\Rightarrow$ $\underline{v}_1 | R\underline{v}_1$ is also a maximizer, i.e. the field is symmetric across R

repeating ... $\underline{v}$ is constant

 (recall periodic BC!)

Conclude by noting that

$$\tilde{Z}(v + \text{const}) = \tilde{Z}(v) \Rightarrow \underline{v} \equiv 0 \text{ is a max.} \quad \square$$

- Def Duhamel's two-point function

$$(A, B)_\beta := \frac{1}{Z^{(u)}_{\beta,\Lambda}(0)} \frac{1}{\beta} \int_0^\beta \mathrm{Tr}\left(e^{-sH^{(u)}_\Lambda} A e^{-(1-s)H^{(u)}_\Lambda} B\right) ds$$

All kinds of nice properties and relations to the KMS condition, see exercises.

Lemma 3 : If G.D. holds, then

$$\widehat{(S_0^3, S_x^3)}(k) \leq \frac{1}{2\beta E(k)}, \quad k \in \Lambda^* \setminus \{0\}.$$

not (yet!) the correlation function $C_\Lambda(x)$.

Proof: We write

$$\tilde{Z}^{(u)}_{\beta,\Lambda}(v) = \mathrm{Tr}\left(e^{A + B(h)}\right) f(h)$$

where $A = -\beta H^{(u)}_\Lambda$; $B(h) = \beta \sum_x h_x S_x^3$

and $f(h) = \exp\left(\frac{1}{4}\beta \langle v, \Delta v \rangle\right) = \exp\left(-\frac{1}{4}\beta \langle h, h \rangle\right)$

Now, we choose $v_x^{(k)} = \varepsilon \cos kx$, $k \neq 0$, $\varepsilon > 0$.

and note that $h = \Delta v = -E(k) v$

and G.D yields

$$\nabla_{h_x} \tilde{Z}\Big|_{\varepsilon=0} = 0 \qquad \left(\frac{\partial^2 \tilde{Z}}{\partial h_x \partial h_y}\right)_{h_x, h_y}\Big|_{\varepsilon=0} \leq 0$$

as a matrix

i.e.

$$\frac{\partial^2}{\partial h_x \partial h_y} \mathrm{Tr}\left(e^{A + B(h)}\right)\Big|_{\varepsilon=0} + \mathrm{Tr}\left(e^{A}\right) \frac{\partial^2}{\partial h_x \partial h_y} f\Big|_{\varepsilon=0} \leq 0$$

First of all: $\frac{\partial^2}{\partial h_x \partial h_y} f \big|_{\varepsilon=0} = -\frac{\beta}{2} \delta_{x,y}$

On the other hand: recall Duhamel's formula:

$$e^{t(X+Y)} = e^{tX} + \int_0^t e^{(t-s)X} Y e^{s(X+Y)} ds$$

and after a bit of algebra:

$$\frac{\partial^2}{\partial \lambda \partial \mu} \mathrm{Tr}\left(e^{-\beta H + \mu X + \lambda Y}\right)_{\mu=\lambda=0}$$

$$= \int_0^1 dr \, \mathrm{Tr}\left(e^{-r\beta H} X e^{-(1-r)\beta H} Y\right) = Z_\beta (X, Y)_\beta$$

i.e. $\frac{\partial^2}{\partial h_x \partial h_y} \mathrm{Tr}\left(\exp(A + B(h))\right)_{\varepsilon=0} = \tilde{Z}^{(u)}_{\beta,\Lambda}(0) \left(S^3_x, S^3_y\right)_\beta \beta^2$

writing $\langle h, \circ h \rangle \leq 0$ and noting that $\mathrm{Tr} A = \tilde{Z}^{(u)}_{\beta,\Lambda}(0) > 0$,

$$\beta^2 \sum_{x,y} h_x h_y \left(S^3_x, S^3_y\right)_\beta \leq \frac{\beta}{2} \sum_x h_x^2$$

i.e. $E(k) \sum_{x,z} \cos kx \cos k(x+z) \left(S^3_o, S^3_z\right)_\beta \leq \frac{1}{2\beta} \sum_x (\cos kx)^2$

using $\sum_z \mathrm{Re}\, e^{ikx} e^{ikz} \left(S^3_o, S^3_z\right)_\beta \underset{\uparrow}{=} \cos kx \, \widehat{\left(S^3_o, S^3_\cdot\right)}_\beta (k)$

real F.T. as $(S_o, S_z) = (S_o, S_{-z})$,

we obtain:

$$\widehat{\left(S^3_o, S^3_\cdot\right)}_\beta (k) \leq \frac{1}{2\beta E(k)} \quad , \quad k \in \Lambda^* \setminus \{0\}. \quad \square$$

- Note: in a commutative algebra, Duhamel's two-point function reduces to the thermal two-point function and we would be done.
- Connection: Falk-Bruch's inequality:

$$\frac{1}{2} \omega_\beta\left(A^+ A + A A^+\right) \leq \frac{1}{2} \sqrt{(A^+, A)_\beta \, \omega_\beta\left([A^+, [H, A]]\right)} + (A^+, A)_\beta$$

Explicit calculation for $A = \sum_x e^{-ikx} S_x^3$:

* $\omega_\beta\left([A^\dagger, [H, A]]\right) = 4\beta |\Lambda| E_u(k)$
* $(A^\dagger, A)_\beta = \sum_{x,y} \int \mathrm{Tr}\left(e^{-sH} S_x^3 e^{-(1-s)H} S_y^3\right) e^{ik(x-y)}$

$= |\Lambda| \widehat{(S_0^3, S_0^3)}(k)$

* $\omega_\beta(\{A^\dagger, A\}) = 2|\Lambda| \widehat{\omega_\beta(S_0^3 S_0^3)}(k)$

All in all: Lemma 4 : If $\widehat{(S_0^3, S_0^3)}(k) \leq \frac{1}{2\beta E(k)}$, then

$$\widehat{C}_\Lambda(k) \leq \sqrt{\frac{E_u(k)}{E(k)}} + \frac{1}{2\beta E(k)}$$

- finally, we need: $(S_0^3)^2 = \frac{1}{3} S(S+1) \Rightarrow C_\Lambda(0) = \omega_{\beta,\Lambda}^{(u)}((S_0^3)^2) = \frac{1}{3} S(S+1)$
- Recap: proof of LRO :

Step 1 : Lemma 1 : consider $Z_{\beta,\Lambda}^{(u)}(v) = \mathrm{Tr}\left(H_\Lambda^{(u)} - \sum_{x\in\Lambda} h_x S_x^3\right)$

which is reflection positive for $u \in [-1, 0]$

Lemma 2 : implies Gaussian domination.

$$\frac{Z_{\beta,\Lambda}^{(u)}(h)}{Z_{\beta,\Lambda}^{(u)}(0)} \leq e^{\frac{1}{4}\beta\|h\|^2}$$

Step 2 : Lemma 3 : use G.D. as a variational principle and a good variational vector to get an infrared bound for Duhamel's 2-point function :

$$\widehat{(S_0^3, S_0^3)}_\beta(k) \leq 1/2\beta E(k)$$

Step 3 : Lemma 4 : relate $\omega_\beta(A^\dagger B)$ to $(A^\dagger, B)_\beta$ to get an IRB

- Notes: * Reflection positivity does not seem to be the "right" method, limited to certain lattices, of course: restricted to $u \in [-1, 0]$
  * But: it is a useful property of quantum field theory (Osterwalder-Schröder theory)
  * The whole proof is heavily inspired by its classical analog: Fröhlich-Simon-Spencer 1976.
  * the theorem implies the existence of at least two extremal KMS states ("pure phases"):

    Assume that there exists an $SU(2)$-invariant, extremal KMS limiting state $\omega_\beta^{(u)}$ on $\mathcal{A}_{\mathbb{Z}^d}$. Then $\omega_\beta^{(u)}(S_x^3) = 0 \ \forall x$.

    It turns out that an extremal KMS state that is asymptotically abelian w.r.t. space translations is clustering, i.e.:

$$\lim_{|x|\to\infty} \left| \omega(A\tau_x(B)) - \omega(A)\omega(\tau_x(B)) \right| = 0$$

here:

$$\lim_{|x|\to\infty} \omega_\beta^{(u)}(S_0^3 S_x^3) = \lim_{|x|\to\infty} \omega_\beta^{(u)}(S_0^3 S_x^3) - \omega_\beta^{(u)}(S_0^3)\,\omega_\beta^{(u)}(S_x^3) = 0$$

which is a contradiction with the theorem

~~*~~ Hence: * broken symmetry in the extremal KMS states,
* Since the set of KMS states at β is invariant, there must be more than one.

v d) Ground states and Goldstone's theorem

- **Def**: Let $(\mathcal{A}, \tau^t)$ be a C^*-dynamical system, δ the generator of τ^t. A state ω is a <u>ground state</u> if

$$-i\omega(A^* \delta(A)) \geq 0 \qquad (*)$$

for all $A \in D(\delta)$

- Note: Formally, this is the $\beta = \infty$-limit of the EEB inequality.
- Also: the inequality $(*)$ implies in particular $\omega(A^*\delta(A))$ is purely imaginary. Let $A = A^* \in D(\delta)$:
  - $\omega(\delta(A)A) = \overline{\omega(A\delta(A))} = -\omega(A\delta(A))$ so that
  - $\omega(\delta(A^2)) = 0$, hence
  - $\omega(\tau^t(A^2)) - \omega(A^2) = \int_0^t ds\, \omega(\delta(\tau_s(A^2))) = 0$

  which shows that ω is τ^t-invariant.

Furthermore: τ^t is implemented in $\mathcal{H}_\omega$ by e^{itH_ω}, and

$$\langle \Pi_\omega(A)\Omega_\omega, H_\omega \Pi_\omega(A)\Omega_\omega \rangle = -i \langle \Pi_\omega(A)\Omega_\omega, \Pi_\omega(\delta(A))\Omega_\omega \rangle$$
$$= -i\omega(A^*\delta(A)) \geq 0$$

$\forall A \in D(\delta)$, hence $H_\omega \geq 0$

Hence:

Proposition: ω is a τ^t-ground state iff ω is τ^t-invariant, and $H_\omega \geq 0$

- Remark: another equivalent condition: for any $A, B \in \mathcal{A}$, $\exists F_{A,B}$, continuous in $\operatorname{Im} z \geq 0$, analytic and bounded in $\operatorname{Im} z > 0$ s.t.

$$F_{AB}(t) = \omega(A\tau^t(B)) \qquad t \in \mathbb{R}.$$

- For a translation-invariant interaction of a Q.S.S., a ground state is a minimizer of the mean energy:
  Let Φ be a translation invariant interaction s.t.

$$\|\Phi\|_\lambda = \sum_{X \ni 0} \|\Phi(X)\| \exp(\lambda |X|) < \infty$$

for some $\lambda > 0$. Let

$$E(\Phi) := \inf_{\omega \in \underbrace{E_0(A)}_{\text{set of transl. inv. states}}} \sum_{X \ni 0} \frac{\omega(\Phi(X))}{|X|} = \inf_{\omega \in E_0(A)} H^\Phi(\omega) \quad \cancel{\omega(H(\Phi))}$$

Then: if ω is translation invariant;
ω is a τ^t-ground state iff ω is a minimizer of $H^\Phi(\cdot)$.

- Finally, relation to finite volume?
  Let Ψ_Λ be a G.S. of H_Λ for the eigenvalue E_Λ. Then:

$$\omega_\Lambda(A^* [H_\Lambda, A]) = \underbrace{\langle \Psi_\Lambda, A^* H_\Lambda A \Psi_\Lambda \rangle}_{\geq E_\Lambda \langle \Psi_\Lambda, A^* A \Psi_\Lambda \rangle} - \underbrace{\langle \Psi_\Lambda, A^* A H_\Lambda \Psi_\Lambda \rangle}_{= E_\Lambda \langle \Psi_\Lambda, A^* A \Psi_\Lambda \rangle} \geq 0.$$

and ω_Λ is τ^t_Λ-invariant. Hence it is a ground state.
Moreover, if ω is a weak-* limit of ω_Λ as $\Lambda \to \Gamma$, then

$$\begin{aligned} \omega(A^* \delta(A)) &= \omega(A^* [\delta(A) - \delta_\Lambda(A)]) \\ &\quad + (\omega - \omega_\Lambda)(A^* \delta_\Lambda(A)) \\ &\quad + \omega_\Lambda(A^* \delta_\Lambda(A)) \end{aligned}$$

since the two first terms vanish as $\Lambda \to \Gamma$, and the last one is ≥ 0, we have that weak-* limit points of ground states are ground states.

<u>But</u>, $\omega(A^* \delta(A)) \geq 0$ may have more solutions than the limiting ones!

• Def. Let ω be a ground state of $(\mathcal{A}, \tau^t)$.

$$\gamma := \sup \{ \delta > 0 : (0, \delta) \cap \mathrm{spec}(H_\omega) = \phi \}$$

is called the <u>spectral gap</u> above the ground state.

Note. "excitation spectrum" = $\mathrm{spec}(H_\omega) \setminus \{0\}$.

• Goldstone's thm gives a relation between symmetry breaking and spectral quantities. Informally:

i) Ground states: continuous symmetry breaking implies a gapless excitation spectrum.

ii) KMS states: continuous symmetry breaking implies a slow decay of correlations.

Precisely: ◦ $(\mathcal{A}, \tau^t)$ a quantum spin system, with an interaction Φ which is

* translation invariant
* $\sum_{X \ni 0} |X| \, \|\Phi(X)\| < \infty$.

◦ G: compact connected Lie group and its action $g \mapsto \alpha_g$ on $\mathcal{A}$, s.t.

$$\alpha_g(\Phi(X)) = \Phi(X) \qquad \forall g \in G, X.$$

Theorem (Landau-Perez-Wreszinski 1981)

i) Let ω_0 be a translation-invariant ground state. If $\gamma > 0$, then
$$\omega_0 \circ \alpha_g = \omega_0 \qquad \forall g \in G.$$
(i.e. symmetry breaking $\Rightarrow$ gapless excitations)

ii) Let ω_β be a (τ^t, β)-KMS state, $\beta \in (0, \infty)$. If ω_β is L^1-clustering,

$$\sum_{x \in \Gamma} \left| \omega_\beta\left(A^* \tau_x(A)\right) - \omega_\beta(A^*)\, \omega_\beta\left(\tau_x(A)\right) \right| < \infty,$$

then $\omega_\beta \circ \alpha_g = \omega_\beta \quad \forall\, g \in G$

(i.e. clustering $\Rightarrow$ no symmetry breaking)

- Remark: The theorem can be extended to continuous systems.