Mathematisches Institut
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SS 2002
Set 10

Problem set for
Quantum Groups and Noncommutative Geometry

(37) We have seen that in representation theory and in corepresentation theory of quantum groups H such as $\mathbb{K}G$, $U(\mathfrak{g})$, $SL_q(2)$, $U_q(sl(2))$ the ordinary tensor product (in $\mathbb{K}$-$\mathcal{M}od$) of two (co-)reprensentations is in a canonical way again a (co-)reprensentation. For two H-modules M and N describe the module structure on $M \otimes N$ if

(a) $H = \mathbb{K}G$: $g(m \otimes n) = \ldots$ for $g \in G$;
(b) $H = U(\mathfrak{g})$: $g(m \otimes n) = \ldots$ for $g \in \mathfrak{g}$;
(c) $H = U_q(sl(2))$:
  (i) $E(m \otimes n) = \ldots$,
  (ii) $F(m \otimes n) = \ldots$,
  (iii) $K(m \otimes n) = \ldots$

for the elements $E, F, K \in U_q(sl(2))$.

(38) Let G be a monoid.

(a) The *category of G-families of vector spaces* $\mathcal{M}^G = \prod_{g \in G} \mathcal{V}ec$ has families of vector spaces $(V_g | g \in G)$ as objects and families of linear maps $(f_g : V_g \to W_g | g \in G)$ as morphisms. The composition is $(f_g | g \in G) \circ (h_g | g \in G) = (f_g \circ h_g | g \in G)$. Show that $\mathcal{M}^G$ is a monoidal category with the tensor product

$$(V_g | g \in G) \otimes (W_g | g \in G) := (\oplus_{h,k \in G, hk=g} V_h \otimes W_k | g \in G).$$

Where do unit and associativity laws of G enter the proof?

(b) A vector space V together with a family of subspaces $(V_g \subseteq V | g \in G)$ is called *G-graded*, if $V = \oplus_{g \in G} V_g$ holds. Let $(V, (V_g | g \in G))$ and $(W, (W_g | g \in G))$ be G-graded vector spaces. A linear map $f : V \longrightarrow W$ is called *G-graded*, if $f(V_g) \subseteq W_g$ for all $g \in G$. The G-graded vector spaces and G-graded linear maps form the category $\mathcal{M}^{[G]}$ of *G-graded vector spaces*. Show that $\mathcal{M}^{[G]}$ is a monoidal category with the tensor product $V \otimes W$, where the subspaces $(V \otimes W)_g$ are defined by

$$(V \otimes W)_g := \sum_{h,k \in G, hk=g} V_h \otimes W_k.$$

(c) Show that the monoidal category $\mathcal{M}^G$ of G-families of vector spaces is monoidally equivalent to the monoidal category $\mathcal{M}^{[G]}$ of G-graded vector spaces.

(d) Show that the monoidal category $\mathcal{M}^{[G]}$ of G-graded vector spaces is monoidally equivalent to the monoidal category of $\mathbb{K}G$-comodules $\mathcal{M}^{\mathbb{K}G}$. (Hint: Use the following constructions. For a G-family $(V_g|g \in G)$ construct a G-graded vector space $\hat{V} := \oplus_{g\in G} V_g$ (exterior direct sum) with the subspaces $\hat{V}_g := \mathrm{Im}(V_g)$ in the direct sum. Conversely if $(V, (V_g|g \in G))$ is a G-graded vector space then $(V_g|g \in G)$ is a G-family of vector spaces. For a G-graded vector space $(V, (V_g|g \in G))$ construct the $\mathbb{K}G$-comodule V with the structure map $\delta : V \longrightarrow V \otimes \mathbb{K}G$, $\delta(v) := v \otimes g$ for all (homogeneous elements) $v \in V_g$ and for all $g \in G$. Conversely let $(V, \delta : V \longrightarrow V \otimes \mathbb{K}G)$ be a $\mathbb{K}G$-comodule. Then construct the vector space V with den graded (homogenous) components $V_g := \{v \in V | \delta(v) = v \otimes g\}$).

(39) Let $(\mathcal{D}, \omega)$ be a diagram in $\mathcal{V}ec$. Let $\mathcal{D}$ be a monoidal category and ω be a monoidal functor. Then $(\mathcal{D}, \omega)$ is called a *monoidal diagram.*

Let $(\mathcal{D}, \omega)$ be a monoidal diagram $\mathcal{V}ec$. Let $A \in \mathcal{V}ec$ be an algebra. A natural transformation $\varphi : \omega \longrightarrow \omega \otimes B$ is called monoidal *monoidal* if the diagrams

$$\begin{tikzcd}
\omega(X) \otimes \omega(Y) \arrow[r, "\varphi(X)\otimes\varphi(Y)"] \arrow[d, "\rho"'] & \omega(X) \otimes \omega(Y) \otimes B \otimes B \arrow[d, "\rho\otimes m"] \\
\omega(X \otimes Y) \arrow[r, "\varphi(X\otimes Y)"] & \omega(X \otimes Y) \otimes B
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{K} \arrow[r, "\cong"] \arrow[d] & \mathbb{K} \otimes \mathbb{K} \arrow[d] \\
\omega(I) \arrow[r, "\varphi(I)"] & \omega(I) \otimes B
\end{tikzcd}$$

commute.

We denote the set of monoidal natural transformations by $\mathrm{Nat}^{\otimes}(\omega, \omega \otimes B)$. Show that $\mathrm{Nat}^{\otimes}(\omega, \omega \otimes B)$ is a functor in B.

(40) Show that the *adjoint action* $H \otimes H \ni h \otimes a \mapsto \sum h_{(1)} a S(h_{(2)}) \in H$ makes H an H-module algebra.

Due date: Tuesday, 02.07.2002, 16:15 in Lecture Hall E41