Problem set for Quantum Groups and Noncommutative Geometry

- (37) We have seen that in representation theory and in corepresentation theory of quantum groups H such as $\mathbb{K}G$, $U(\mathfrak{g})$, $SL_q(2)$, $U_q(sl(2))$ the ordinary tensor product (in \mathbb{K} - $\mathcal{M}od$) of two (co-)representations is in a canonical way again a (co-)representation. For two H-modules M and N describe the module structure on $M \otimes N$ if
 - (a) $H = \mathbb{K}G$: $g(m \otimes n) = \dots$ for $g \in G$; (b) $H = U(\mathfrak{g})$: $g(m \otimes n) = \dots$ for $g \in \mathfrak{g}$; (c) $H = U_q(sl(2))$: (i) $E(m \otimes n) = \dots$, (ii) $F(m \otimes n) = \dots$, (iii) $K(m \otimes n) = \dots$, for the elements $E, F, K \in U_q(sl(2))$.
- (38) Let G be a monoid.

(a) The category of *G*-families of vector spaces $\mathcal{M}^G = \prod_{g \in G} \mathcal{V}ec$ has families of vector spaces $(V_g | g \in G)$ as objects and families of linear maps $(f_g : V_g \rightarrow W_g | g \in G)$ as morphisms. The composition is $(f_g | g \in G) \circ (h_g | g \in G) = (f_g \circ h_g | g \in G)$. Show that \mathcal{M}^G is a monoidal category with the tensor product

$$(V_g|g \in G) \otimes (W_g|g \in G) := (\bigoplus_{h,k \in G, hk=g} V_h \otimes W_k|g \in G).$$

Where do unit and associativity laws of G enter the proof?

(b) A vector space V together with a family of subspaces $(V_g \subseteq V | g \in G)$ is called *G-graded*, if $V = \bigoplus_{g \in G} V_g$ holds. Let $(V, (V_g | g \in G))$ and $(W, (W_g | g \in G))$ be *G*-graded vector spaces. A linear map $f : V \to W$ is called *G-graded*, if $f(V_g) \subseteq W_g$ for all $g \in G$. The *G*-graded vector spaces and *G*-graded linear maps form the category $\mathcal{M}^{[G]}$ of *G-graded vector spaces*. Show that $\mathcal{M}^{[G]}$ is a monoidal category with the tensor product $V \otimes W$, where the subspaces $(V \otimes W)_g$ are defined by

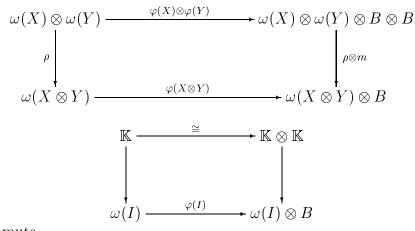
$$(V \otimes W)_g := \sum_{h,k \in G, hk=g} V_h \otimes W_k.$$

(c) Show that the monoidal category \mathcal{M}^G of *G*-families of vector spaces is monoidally equivalent to the monoidal category $\mathcal{M}^{[G]}$ of *G*-graded vector spaces.

(d) Show that the monoidal category $\mathcal{M}^{[G]}$ of *G*-graded vector spaces is monoidally equivalent to the monoidal category of $\mathbb{K}G$ -comodules $\mathcal{M}^{\mathbb{K}G}$. (Hint: Use the following constructions. For a *G*-family $(V_g|g \in G)$ construct a *G*graded vector space $\hat{V} := \bigoplus_{g \in G} V_g$ (exterior direct sum) with the subspaces $\hat{V}_g := \operatorname{Im}(V_g)$ in the direct sum. Conversely if $(V, (V_g|g \in G))$ is a *G*-graded vector space then $(V_g|g \in G)$ is a *G*-family of vector spaces. For a *G*-graded vector space $(V, (V_g|g \in G))$ construct the $\mathbb{K}G$ -comodule *V* with the structure map $\delta : V \to V \otimes \mathbb{K}G$, $\delta(v) := v \otimes g$ for all (homogeneous elements) $v \in V_g$ and for all $g \in G$. Conversely let $(V, \delta : V \to V \otimes \mathbb{K}G)$ be a $\mathbb{K}G$ -comodule. Then construct the vector space *V* with den graded (homogenous) components $V_g := \{v \in V | \delta(v) = v \otimes g\}$).

(39) Let (\mathcal{D}, ω) be a diagram in $\mathcal{V}ec$. Let \mathcal{D} be a monoidal category and ω be a monoidal functor. Then (\mathcal{D}, ω) is called a *monoidal diagram*.

Let (\mathcal{D}, ω) be a monoidal diagram $\mathcal{V}ec$. Let $A \in \mathcal{V}ec$ be an algebra. A natural transformation $\varphi : \omega \to \omega \otimes B$ is called monoidal *monoidal* if the diagrams



commute.

and

We denote the set of monoidal natural transformations by $\operatorname{Nat}^{\otimes}(\omega, \omega \otimes B)$. Show that $\operatorname{Nat}^{\otimes}(\omega, \omega \otimes B)$ is a functor in B.

(40) Show that the *adjoint action* $H \otimes H \ni h \otimes a \mapsto \sum h_{(1)} aS(h_{(2)}) \in H$ makes H an H-module algebra.

Due date: Tuesday, 02.07.2002, 16:15 in Lecture Hall E41