ON LIE ALGEBRAS IN THE CATEGORY OF YETTER-DRINFELD MODULES

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ABSTRACT. The category of Yetter-Drinfeld modules $\mathcal{YD}_K^K$ over a Hopf algebra K (with bijektive antipode over a field k) is a braided monoidal category. If H is a Hopf algebra in this category then the primitive elements of H do not form an ordinary Lie algebra anymore. We introduce the notion of a (generalized) Lie algebra in $\mathcal{YD}_K^K$ such that the set of primitive elements $P(H)$ is a Lie algebra in this sense. Also the Yetter-Drinfeld module of derivations of an algebra A in $\mathcal{YD}_K^K$ is a Lie algebra. Furthermore for each Lie algebra in $\mathcal{YD}_K^K$ there is a universal enveloping algebra which turns out to be a Hopf algebra in $\mathcal{YD}_K^K$.

Keywords: braided category, Yetter-Drinfeld module, Lie algebra, universal enveloping algebra.

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1. INTRODUCTION

The concept of Hopf algebras in braided categories has turned out to be very important in the context of understanding the structure of quantum groups and noncommutative noncocommutative Hopf algebras. In particular the work of Radford [7], Majid [3], Lusztig [2], and Sommerhäuser [8] show the importance of the decomposition of quantum groups into a product of ordinary Hopf algebras and of Hopf algebras in braided categories.

Since by the work of Yetter [9] Hopf algebras in braided categories that are defined on an underlying (finite-dimensional) vector space can be considered as Hopf algebras in some category of Yetter-Drinfeld modules, we will restrict our attention to Hopf algebras H in a category of Yetter-Drinfeld modules $\mathcal{YD}_K^K$ over a Hopf algebra K with bijective antipode.

There are two structurally interesting and important concepts that survive in this generalized situation, the concept of group-like elements

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($\Delta(g) = g\otimes g$, $\varepsilon(g) = 1$) and the concept of primitive elements ($\Delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$).

For ordinary Hopf algebras H the set of primitive elements $P(H)$ of H forms a Lie algebra. This result (in a somewhat generalized form) still holds for Hopf algebras in a symmetric monoidal category. This is, however, not true for braided monoidal categories.

There have been various attempts to generalize the notion of Lie algebras to braided monoidal categories. The main obstruction for such a generalization is the assumption that the category is only braided and not symmetric. One of the most important examples of such braided categories is given by the category of Yetter-Drinfeld modules $\mathcal{YD}_K^K$ over a Hopf algebra K with bijective antipode which is always properly braided (except for $K = k$, the base field) [6].

We introduce a concept of Lie algebras in $\mathcal{YD}_K^K$ that generalizes the concepts of ordinary Lie algebras, Lie super algebras, Lie color algebras, and (G, χ)-Lie algebras as given in [5].

The Lie algebras defined on Yetter-Drinfeld modules have *partially defined n-ary* bracket operations for every $n \in \mathbb{N}$ and every primitive n-th root of unity. They satisfy generalizations of the (anti-)symmetry and Jacabi identities.

Our main aim is to show that these Lie algebras have universal enveloping algebras which turn out to be Hopf algebras in $\mathcal{YD}_K^K$. Conversely the set of primitive elements of a Hopf algebra in $\mathcal{YD}_K^K$ is such a generalized Lie algebra. We also give an example that generalizes the concept of orthogonal or symplectic Lie algebras.

2. Braid Symmetrization

We begin with two simple module theoretic observations. The following is well known: if A, B are algebras and M is an A-B-bimodule, then $\mathrm{Hom}_A(.P, .M)$ is a right B-module for every A-module P. We need a comodule analogue of this.

Let A be an algebra, C be a coalgebra, and ${}_AM^C$ be an A-C-dimodule, i.e. a left A-module and a right C-comodule such that $\delta(am) = (a \otimes 1)\delta(m)$.

Proposition 2.1. *Let P be a finitely generated left A-module. Then $\mathrm{Hom}_A(.P, .M)$ is a right C-comodule with the canonical comodule structure such that*

$$\left(\mathrm{Hom}_A(P, M) \xrightarrow{\delta} \mathrm{Hom}_A(P, M)\otimes C \longrightarrow \mathrm{Hom}_A(P, M\otimes C)\right) = \mathrm{Hom}_A(P, \delta).$$

Proof. Let $p_1, \dots, p_n$ be a generating set of P and let $f \in \mathrm{Hom}_A(.P, .M)$. Let $m_i := f(p_i)$. Then by the structure theorem on comodules the m_i

are contained in a finite dimensional subcomodule $M_0 \subseteq M$ which is even a comodule over a finite dimensional subcoalgebra $C_0 \subseteq C$, i.e. the diagram

$$\begin{tikzcd} M_0 \arrow[r, "\delta"] \arrow[d] & M_0 \otimes C_0 \arrow[d] \\ M \arrow[r, "\delta"] & M \otimes C \end{tikzcd}$$

commutes. Furthermore $M_1 := AM_0$ is a C_0-comodule contained in M, since M is a dimodule, and $f : P \longrightarrow M$ obviously factors through M_1. Since M and M_1 are dimodules the diagram

$$\begin{tikzcd} \mathrm{Hom}_A(.P,.M_1) \arrow[r, "\delta_*"] \arrow[d] & \mathrm{Hom}_A(.P,.M_1 \otimes C_0) \arrow[d] & \mathrm{Hom}_A(.P,.M_1) \otimes C_0 \arrow[l, "\cong"'] \arrow[d] \\ \mathrm{Hom}_A(.P,.M) \arrow[r, "\delta_*"] & \mathrm{Hom}_A(.P,.M \otimes C) & \mathrm{Hom}_A(.P,.M) \otimes C \arrow[l] \end{tikzcd}$$

commutes, so each f has a uniquely defined image $\delta_*(f) \in \mathrm{Hom}_A(.P,.M) \otimes C$. Now it is easy to check that this map induces a comodule structure on $\mathrm{Hom}_A(P, M)$. □

The second observation is the following. We consider k-algebras A and B. Let $\alpha : B \longrightarrow A$ be an algebra homomorphism. α induces an underlying functor $V_\alpha : A\text{-Mod} \longrightarrow B\text{-Mod}$ with right adjoint $\mathrm{Hom}_B(A, \text{-}) : B\text{-Mod} \longrightarrow A\text{-Mod}$. If $\alpha : B \longrightarrow A$ is surjective then $\mathrm{Hom}_B(A, M) \longrightarrow \mathrm{Hom}_B(B, M) \cong M$ is injective, so that we can identify $\mathrm{Hom}_B(A, M) = \{m \in M | \mathrm{Ker}(\alpha)m = 0\}$.

Let B_n be the Artin braid group with generators τ_i, $i = 1, \ldots , n-1$ and relations

$$\begin{aligned} \tau_i\tau_j &= \tau_j\tau_i \qquad \text{if } |i-j| \geq 2; \\ \tau_i\tau_{i+1}\tau_i &= \tau_{i+1}\tau_i\tau_{i+1}. \end{aligned} \tag{1}$$

Let $\zeta \in k$ be invertible. Then $kB_n \ni \tau_i \mapsto \zeta\tau_i \in kB_n$ (for the generators τ_i of B_n) is an algebra automorphism denoted again by $\zeta : kB_n \longrightarrow kB_n$. This holds true since the relations for B_n are homogeneous.

(Observe that this construction can be performed for every group algebra if the group is given by generators and homogeneous relations. The given construction of an automorphism for every $\zeta \in U(k)$ defines a group homomorphism $U(k) \longrightarrow \mathrm{Aut}(kB_n) \longrightarrow \mathrm{Aut}(kB_n\text{-Mod})$.)

Now consider the canonical quotient homomorphism $B_n \longrightarrow S_n$ from the braid group onto the symmetric group. It induces a surjective homomorphism $\gamma : kB_n \longrightarrow kS_n$ with kernel

$$\mathrm{Ker}(\gamma) = \langle \psi(\tau_i^2 - 1)\varphi | \varphi, \psi \in B_n, i = 1, \ldots , n-1\rangle.$$

The composition $\alpha : kB_n \xrightarrow{\zeta} kB_n \xrightarrow{\gamma} kS_n$ defines a functor kB_n-Mod $\longrightarrow kS_n$-Mod by

$$\begin{aligned} M(\zeta) : &= \mathrm{Hom}_{kB_n}({}_{\zeta}kS_n, M) \\ &= \{m \in M | \varphi^{-1}\tau_i^2\varphi(m) = \zeta^2 m \quad \forall \varphi \in B_n,\ i = 1, \ldots, n-1\}. \\ &= \{m \in M | \tau_i^2\varphi(m) = \zeta^2\varphi(m) \quad \forall \varphi \in B_n,\ i = 1, \ldots, n-1\} \end{aligned} \tag{2}$$

This holds since the map $\gamma : kB_n \longrightarrow kS_n$ has as kernel the two-sided ideal generated as a k-subspace by $\{\psi(\tau_i^2 - 1)\varphi \,|\psi, \varphi \in B_n, i = 1, \ldots, n-1\}$. So $f \in \mathrm{Hom}_{kB_n}({}_{\zeta}kS_n, M)$ with $f(1) = m \in M$, iff $\zeta^{-1}(\psi(\tau_i^2 - 1)\varphi)m = 0$ for all ψ, φ, i, iff $\zeta^{-1}(\tau_i^2 - 1)\varphi m = 0$ for all φ, i, iff $\tau_i^2\varphi m = \zeta^2\varphi m$ for all φ, i, iff $\varphi^{-1}\tau_i^2\varphi(m) = \zeta^2 m$ for all φ, i.

If the action of B_n on M is given by an action of S_n and the canonical epimorphism $B_n \longrightarrow S_n$, then the construction of the $M(\zeta)$ becomes trivial, since $M(\zeta) = \{m \in M | \tau_i^2\varphi(m) = \zeta^2\varphi(m) = \varphi(m)\} = 0$ if $\zeta^2 \neq 1$ and $M(-1) = M(1) = M$. Observe that the module $M(\zeta)$ depends only on ζ^2, but that the action of kS_n on $M(\zeta)$ depends on ζ.

$M(1)$ gives a solution of the following universal problem.

Proposition 2.2. *For every kB_n-module M the subspace*

$$M(1) := \{m \in M | \varphi^{-1}\tau_i^2\varphi(m) = m \quad \forall \varphi \in B_n,\ i = 1, \ldots, n-1\}$$

is a kS_n-module and the inclusion $M(1) \longrightarrow M$ is a kB_n-module homomorphism, such that for every kS_n-module T and every kB_n-module homomorphism $f : T \longrightarrow M$ there is a unique kS_n-module homomorphism $g : T \longrightarrow M(1)$ such that the diagram

$$\begin{tikzcd} T \arrow[d, "g"'] \arrow[dr, "f"] & \\ M(1) \arrow[r, "\iota"] & M \end{tikzcd}$$

commutes.

Definition 2.3. We call the functor -$(\zeta) : kB_n$-Mod $\longrightarrow kS_n$-Mod the *ζ-symmetrization* of kB_n-modules.

The definition gives

$$M(\zeta) = \{m \in M | \varphi^{-1}\tau_i^2\varphi(m) = \zeta^2 m \quad \forall \varphi \in B_n,\ i = 1, \ldots, n-1\}.$$

The action of S_n on $M(\zeta)$ is given by

$$\sigma_i(m) = \zeta^{-1}\tau_i(m), \tag{3}$$

where σ_i resp. τ_i are the canonical generators of S_n resp. B_n. Thus $M(\zeta)$ is also a kB_n-submodule of M. Since the functor $M \mapsto M(\zeta)$ is a rightadjoint functor, it preserves limits. Like for eigenspaces we have that the sum of the subspaces $M(\zeta)$ for all ζ with different ζ^2 is a direct sum. On $M(\zeta)$ we have two distinct kS_n-structures $\sigma_i(m) = -\zeta^{-1}\tau_i(m)$ and $\sigma_i = \zeta^{-1}\tau_i(m)$, since ζ and $-\zeta$ define the same subspace $M(\zeta) = M(-\zeta) \subseteq M$.

The ζ-symmetrization $M(\zeta)$ of M can also be calculated by

Lemma 2.4.

$$M(\zeta) = \{m \in M | \tau_i^{-1}\tau_{i+1}^{-1} \ldots \tau_{j-1}^{-1}\tau_j^2\tau_{j-1} \ldots \tau_{i+1}\tau_i(m) = \zeta^2 m \quad \forall 1 \leq i \leq j \leq n-1\}$$

which reduces the number of conditions to be imposed on the $m \in M$ in order to be in $M(\zeta)$.

Proof. Given in Appendix. □

One of the interesting kB_n-structures, for which we will apply the previous construction, occurs on n-fold tensor products $M^n := M \otimes \ldots \otimes M$ of an object M in a braided monoidal category of vector spaces.

Let K be a Hopf algebra. Let M be an K-module such that M is a kB_n-K-bimodule. The functoriality of our construction then makes $M(\zeta)$ again an K-module and in fact a kS_n-K-bimodule.

Let M be an K-comodule such that M is a kB_n-K-dimodule. Then by Proposition 2.1 $M(\zeta)$ is an K-comodule and in fact a kS_n-K-dimodule.

Let K be a Hopf algebra with bijective antipode. Let $\mathcal{YD}_K^K$ denote the category of Yetter-Drinfeld modules over K, i.e. of right K-modules and right K-comodules M such that $\sum (x \cdot h)_0 \otimes (x \cdot h)_1 = \sum (x_0 \cdot h_2) \otimes S(h_1)x_1h_3$ for all $x \in M$. The usual tensor product makes $\mathcal{YD}_K^K$ a monoidal category. $\mathcal{YD}_K^K$ has a braiding given by $\tau_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$, $\tau(x \otimes y) = \sum y_0 \otimes xy_1$. We assume that the reader is familiar with the properties of the B_n-action that is induced by the braiding τ on n-fold tensor products ([4] 10.6).

Theorem 2.5. *Let K be a Hopf algebra with bijective antipode. Then for each $\zeta \in k^*$ and each $n \geq 2$ the construction given above defines a (non-additive) functor*

$$\mathcal{YD}_K^K \ni M \mapsto (M \otimes \ldots \otimes M)(\zeta) \in \mathcal{YD}_K^K.$$

Proof. If $M \in \mathcal{YD}_K^K$ then the n-fold tensor product $M \otimes \ldots \otimes M$ is a Yetter-Drinfeld module on which B_n and thus kB_n acts in such a way, that M is a (kB_n, K)-bimodule and a (kB_n, K)-dimodule. The ζ-symmetrization functor -(ζ) preserves the module and comodule structures hence the Yetter-Drinfeld structure.

The functor is not additive since the "diagonal" functor $M \mapsto M \otimes M$ is not additive. □

We abbreviate $M^n(\zeta) := M^{\otimes n}(\zeta) = M \otimes \ldots \otimes M(\zeta)$. Then $M^n(\zeta)$ is a submodule of M^n in the category of Yetter-Drinfeld modules and the elements in $M^n(\zeta)$ are of the form $z = \sum_k x_{k,1} \otimes \ldots \otimes x_{k,n}$. We often suppress the summation index and summation sign and simply write $z = z_1 \otimes \ldots \otimes z_n \in M^n(\zeta)$ although $M^n(\zeta)$ does not decompose into a tensor product.

3. Symmetric Multiplication and Jacobi Identities

For the rest of the paper let $\mathcal{C}$ be the category $\mathcal{YD}_K^K$ of Yetter-Drinfeld modules over a Hopf algebra K with bijective antipode over a field k. We study objects $P \in \mathcal{C}$ together with (partially defined) operations in $\mathcal{C}$

$$[.,.] : P \otimes \ldots \otimes P(\zeta) = P^n(\zeta) \longrightarrow P$$

for all $n \in \mathbb{N}$ and all *primitive n-th roots of unity* ζ.

Occasionally we write $[.,.]_n$ for such an operation $[.,.]$. By composing such operations certain additional operations may be constructed as follows.

Proposition 3.1. *Let ζ be a primitive n-th roots of unity. Then the operations*

$$[.,[.,.]_n]_2 : P^{n+1}(\zeta) \ni x_1 \otimes \ldots \otimes x_{n+1} \mapsto [x_1,[x_2,\ldots,x_{n+1}]] \in P$$

and

$$[[.,.]_n,.]_2 : P^{n+1}(\zeta) \ni x_1 \otimes \ldots \otimes x_{n+1} \mapsto [[x_1,\ldots,x_n],x_{n+1}] \in P$$

are well defined.

Proof. Given in Appendix. □

We will have to consider objects

$$P^{n+1}(-1,\zeta) := P \otimes P^n(\zeta) \cap \{z \in P^{n+1} | \forall \varphi \in S_n : (1 \otimes \varphi)^{-1} \tau_1^2 (1 \otimes \varphi)(z) = z\}.$$

Since this is a kernel (limit) construction in $\mathcal{C}$, $P^{n+1}(-1,\zeta)$ is again an object in $\mathcal{C}$.

Proposition 3.2. *Let ζ be a primitive n-th roots of unity. Then the operations*

$$[.,[.,.]_n]_2 : P^{n+1}(-1,\zeta) \ni x \otimes y_1 \otimes \ldots \otimes y_n \mapsto [x,[y_1,\ldots,y_n]] \in P$$

and

$$[.,[.,.]_2,.]_n \tau_{i-1} \ldots \tau_1 : P^{n+1}(-1,\zeta) \ni x \otimes y_1 \otimes \ldots \otimes y_n \mapsto [y_1,\ldots,[x,y_i],\ldots,y_n] \in P$$

are well defined.

Proof. Given in Appendix. □

We introduce special bracket multiplications which then lead to the definition of a Lie algebra on a Yetter-Drinfeld module.

Definition 3.3. Let A be an algebra in $\mathcal{C} = \mathcal{YD}_K^K$ and let $\nabla^n : A \otimes \ldots \otimes A \longrightarrow A$ denote the n-fold multiplication. We define a *bracket* or *symmetric multiplication*

$$[.,.] : A^n(\zeta) \longrightarrow A \qquad \text{by} \qquad [z] := \sum_{\sigma \in S_n} \nabla^n \sigma(z)$$

where the action of S_n on $A^n(\zeta)$ is given as in (3).

We will only use those bracket operations which are defined with ζ a primitive n-th root of unity (for all $n \in \mathbb{N}$ and all ζ).

We consider these bracket operations as a generalization of the Lie-bracket $[\text{-},\text{-}] : L \times L \longrightarrow L$ or $[-] : L \otimes L \longrightarrow L$. Observe that our bracket operation is only partially defined and should *not* be considered as a multilinear operation, since $A^n(\zeta) \subseteq A^n$ is just a submodule in $\mathcal{C}$ and does not necessarily decompose into an n-fold tensor product. The elements in $A^n(\zeta)$ are, however, of the form $z = \sum_k x_{k,1} \otimes \ldots \otimes x_{k,n}$.

If we suppress the summation index and the summation sign then we may write the bracket operation on $z = x_1 \otimes \ldots \otimes x_n$ also as $[z] = [x_1, \ldots, x_n]$. If we define

$$\sigma(z) =: x_{\sigma^{-1}(1)} \otimes \ldots \otimes x_{\sigma^{-1}(n)} \tag{4}$$

then we get

$$[x_1, \ldots, x_n] = \sum_{\sigma \in S_n} x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(n)}.$$

Observe that the components $x_1, \ldots, x_n$ in this expression are interchanged according to the action of the braid group resp. the symmetric group on $A^n(\zeta)$, so $x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}$ is only a symbolic expression.

The bracket operation obviously satisfies the "anti"-symmetry identity

$$[\sigma(z)] = [z] \qquad\qquad \forall \sigma \in S_n. \tag{5}$$

We apply Proposition 3.1 to an algebra A in $\mathcal{C}$ with the operations given in Definition 3.3 and get

Theorem 3.4. (1. Jacobi identity) *For all $n \in \mathbb{N}$, for all primitive n-th roots of unity ζ, and for all $z \in A^{n+1}(\zeta)$ we have*

$$\sum_{i=1}^{n+1} [x_i, [x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}]] = \sum_{i=1}^{n+1} [.,[.,.]_n]_2 (1 \ldots i)(z) = 0, \tag{6}$$

where we use the notation (4).

Proof. We define $(S_{n+1})_{(i)} := \{\sigma \in S_{n+1} | \sigma(i) = 1\}$. Then $S_{n+1} = \bigcup_i (S_{n+1})_{(i)}$. For $\sigma \in (S_{n+1})_{(i)}$ let $\bar{\rho} := \sigma(i \dots 1)$. Since $\bar{\rho}(1) = 1$ there is a unique $\rho \in S_n$ with $\bar{\rho} = 1 \otimes \rho$ and $\sigma = (1 \otimes \rho)(1 \dots i)$. So we obtain a bijection

$$S_n \ni \rho \mapsto (1 \otimes \rho)(1 \dots i) \in (S_{n+1})_{(i)}.$$

Analogously we define $(S_{n+1})^{(i)} := \{\sigma \in S_{n+1} | \sigma(i) = n+1\}$ and get a bijection

$$S_n \ni \rho \mapsto (\rho \otimes 1)(n+1 \dots i) \in (S_{n+1})^{(i)}.$$

Now observe that $\tau_n \dots \tau_1(z) = \zeta^n \sigma_n \dots \sigma_1(z) = (n+1 \dots 1)(z)$ (by $\zeta^n = 1$) for $z \in P^{n+1}(\zeta)$ to get

$$\begin{aligned}
\textstyle\sum_{i=1}^n & [., [., .]](1 \dots i)(z) = \\
&= \textstyle\sum_{i=1}^n \nabla(1 \otimes [., .])(1 \dots i)(z) - \nabla([., .] \otimes 1)\tau_n \dots \tau_1(1 \dots i)(z) \\
&= \textstyle\sum_{i=1}^n \nabla(1 \otimes [., .])(1 \dots i)(z) - \nabla([., .] \otimes 1)(n+1 \dots i)(z) \\
&= \textstyle\sum_{i=1}^n \sum_{\rho \in S_n} \nabla^{n+1}(1 \otimes \rho)(1 \dots i)(z) - \nabla^{n+1}(\rho \otimes 1)(n+1 \dots i)(z) \\
&= \textstyle\sum_{\sigma \in S_{n+1}} \nabla^{n+1}\sigma(z) - \nabla^{n+1}\sigma(z) = 0.
\end{aligned}$$

□

Theorem 3.5. (2. Jacobi identity) *For all $n \in \mathbb{N}$, for all primitive n-th roots of unity ζ, and for all $z = x \otimes y_1 \otimes \dots \otimes y_n \in A^{n+1}(-1, \zeta)$ we have*

$$[x, [y_1, \dots, y_n]] = \sum_{i=1}^n [y_1, \dots, [x, y_i], \dots, y_n] \tag{7}$$

where $y_1 \otimes \dots \otimes y_{i-1} \otimes x \otimes y_i \otimes \dots \otimes y_n := \tau_{i-1} \dots \tau_1(z)$ and

$$[y_1, \dots, y_{i-1}, [x, y_i], \dots, y_n] = [., [., .]_2, .]_n \tau_{i-1} \dots \tau_1(z). \tag{8}$$

Proof. The equation in the Theorem can also be written as

$$[., [., .]_n]_2(z) = \sum_{i=1}^n [., [., .]_2, .]_n \tau_{i-1} \dots \tau_1(z).$$

Lemma 8.1 together with $\tilde{\varphi}(i) = j$ shows

$$\begin{aligned}
\nabla^n \varphi(1 \otimes \dots \otimes \nabla & \otimes \dots \otimes 1)\tau_{i-1} \dots \tau_1(z) \\
&= \nabla^n(1 \otimes \dots \otimes \nabla \otimes \dots \otimes 1)\varphi_{(i)}\tau_{i-1} \dots \tau_1(z) \\
&= \nabla^{n+1}\tau_{j-1} \dots \tau_1(1 \otimes \varphi)(z); \\
\nabla^n \varphi(1 \otimes \dots \otimes \nabla & \otimes \dots \otimes 1)\tau_i \dots \tau_1(z) \\
&= \nabla^n(1 \otimes \dots \otimes \nabla \otimes \dots \otimes 1)\varphi_{(i)}\tau_i \dots \tau_1(z) \\
&= \nabla^{n+1}\tau_j \dots \tau_1(1 \otimes \varphi)(z);
\end{aligned}$$

hence

$$\begin{aligned}\nabla^n\varphi(1\otimes\ldots\otimes\nabla\otimes\ldots\otimes 1)\tau_{i-1}\ldots\tau_1(z)&\\ &=\nabla^{n+1}\tau_{k-1}\ldots\tau_1(1\otimes\varphi)(z)\\ &=\nabla^{n+1}\tau_l\ldots\tau_1(1\otimes\varphi)(z)\\ &=\nabla^n\varphi(1\otimes\ldots\otimes\nabla\otimes\ldots\otimes 1)\tau_j\ldots\tau_1(z).\end{aligned}\tag{9}$$

for all $i,j=1,\ldots,n$ with $\tilde\varphi(i)=\tilde\varphi(j)+1$, i.e. for all i except $\tilde\varphi^{-1}(1)$ and all j except $\tilde\varphi^{-1}(n)$. The other i's and j's used in (9) are in bijective correspondence.

To prove the equation of the theorem we write each $\sigma\in S_n$ as $\zeta^r\varphi$ with a representative $\varphi\in B_n$ and a suitable power ζ^r according to (3) and use (9). Then we get

$$\begin{aligned}&\textstyle\sum_{i=1}^n[.,[.,.]_2,.]_n\tau_{i-1}\ldots\tau_1(z)=\\ &\quad=\textstyle\sum_{i=1}^n[.,.]_n(1\otimes\ldots\otimes(\nabla-\nabla\tau)\otimes\ldots\otimes 1)\tau_{i-1}\ldots\tau_1(z)=\\ &\quad=\textstyle\sum_{i=1}^n\sum_{\sigma\in S_n}\nabla^n\zeta^r\varphi(1\otimes\ldots\otimes\nabla\otimes\ldots\otimes 1)\tau_{i-1}\ldots\tau_1(z)\\ &\qquad-\textstyle\sum_{j=1}^n\sum_{\sigma\in S_n}\nabla^n\zeta^r\varphi(1\otimes\ldots\otimes\nabla\otimes\ldots\otimes 1)\tau_j\tau_{j-1}\ldots\tau_1(z)\\ &\quad=\textstyle\sum_{\sigma\in S_n}\nabla^{n+1}(1\otimes\zeta^r\varphi)(z)-\sum_{\sigma\in S_n}\nabla^{n+1}\tau_n\ldots\tau_1(1\otimes\zeta^r\varphi)(z)\\ &\quad=\textstyle\sum_{\sigma\in S_n}\nabla^{n+1}(1\otimes\zeta^r\varphi)(z)-\sum_{\sigma\in S_n}\nabla^{n+1}(\zeta^r\varphi\otimes 1)\tau_n\ldots\tau_1(z)\\ &\quad=\textstyle\nabla(1\otimes\sum_{\sigma\in S_n}\nabla^n\sigma)(z)-\nabla(\sum_{\sigma\in S_n}\nabla^n\sigma\otimes 1)\tau_{P,P^n}(z)\\ &\quad=(\nabla-\nabla\tau)(1\otimes[.,.]_n)(z)\\ &\quad=[.,[.,.]_n]_2(z).\end{aligned}$$

$\square$

Clearly there are symmetric right sided identities.

4. Lie Algebras on Yetter-Drinfeld Modules

Now we can define the notion of a Lie algebra in the category of Yetter-Drinfeld modules.

Definition 4.1. A Yetter-Drinfeld module P together with operations in $\mathcal{YD}_K^K$

$$[.,.]:P\otimes\ldots\otimes P(\zeta)=P^n(\zeta)\longrightarrow P$$

for all $n\in\mathbb{N}$ and all primitive n-th roots of unity ζ is called a *Lie algebra* if the following identities hold:

(1) for all $n\in\mathbb{N}$, for all primitive n-th roots of unity ζ, for all $\sigma\in S_n$, and for all $z\in P^n(\zeta)$

$$[z]=[\sigma(z)],$$

(2) for all $n\in\mathbb{N}$, for all primitive n-th roots of unity ζ, and for all $z\in P^{n+1}(\zeta)$

$$\sum_{i=1}^{n+1}[x_i,[x_1,\ldots,\hat{x}_i,\ldots,x_{n+1}]]=\sum_{i=1}^{n+1}[.,[.,.]](1\ldots i)(z)=0,$$

where we use the notation (4),

(3) for all $n \in \mathbb{N}$, for all primitive n-th roots of unity ζ, and for all $z = x \otimes y_1 \otimes \ldots \otimes y_n \in P^{n+1}(-1, \zeta)$ we have

$$[x, [y_1, \ldots, y_n]] = \sum_{i=1}^{n} [y_1, \ldots, [x, y_i], \ldots, y_n]$$

where we use the notation (8).

Corollary 4.2. *Let A be an algebra in $\mathcal{YD}_K^K$. Then A carries the structure of a Lie algebra A^L with the symmetric multiplications*

$$[-] : A^n(\zeta) \longrightarrow A \qquad by \qquad [z] := \sum_{\sigma \in S_n} \nabla^n \sigma(z).$$

for all $n \in \mathbb{N}$ and all roots of unity $\zeta \in k^*$.

Proof. This is a rephrasing of the "anti"-symmetry identity (5) and the Jacobi identities (6) and (7) in Theorems 3.4 and 3.5. □

5. The Lie Algebra of Primitive Elements

Let A be an algebra in $\mathcal{C} = \mathcal{YD}_K^K$. Then $A \otimes A$ is an algebra with the multiplication $A \otimes A \otimes A \otimes A \xrightarrow{1 \otimes \tau \otimes 1} A \otimes A \otimes A \otimes A \xrightarrow{\nabla \otimes \nabla} A \otimes A$. Let $p : A \longrightarrow A \otimes A$ be the map $p(x) := x \otimes 1 + 1 \otimes x$. Then $p(= 1 \otimes \eta + \eta \otimes 1)$ is in $\mathcal{C}$ but p is not an algebra morphism. Let $p^n : A^n \longrightarrow (A \otimes A)^n$ be the n-fold tensor product of p with itself.

Lemma 5.1. *Let H be a Hopf algebra in $\mathcal{C}$. Then $P(H) := \{x \in H | \Delta(x) = x \otimes 1 + 1 \otimes x\}$ is a Yetter-Drinfeld submodule of H in $\mathcal{C}$.*

Proof. $P(H) = \mathrm{Ker}(\Delta - p)$. □

In particular we have $\delta(x) \in P(H) \otimes K$ and $x\lambda \in P(H)$ for all $x \in P(H)$ and all $\lambda \in K$.

Lemma 5.2. $p^n(A^n(\zeta)) \subseteq (A \otimes A)^n(\zeta)$.

Proof. By Theorem 2.5 $p : A \longrightarrow A \otimes A$ induces $p^n : A^n(\zeta) \longrightarrow (A \otimes A)^n(\zeta)$. □

Theorem 5.3. *Let ζ be a primitive n-th root of unity and let $z \in A^n(\zeta)$. Then*

$$[p^n(z)] = p([z]).$$

Proof. If $z = \sum_k x_{k,1} \otimes \ldots \otimes x_{k,n} \in A^n(\zeta)$ then the equation of the theorem reads as

$$\begin{array}{c}[\sum_k (x_{k,1} \otimes 1 + 1 \otimes x_{k,1}) \otimes \ldots \otimes (x_{k,n} \otimes 1 + 1 \otimes x_{k,n})] = \\ [\sum_k x_{k,1} \otimes \ldots \otimes x_{k,n}] \otimes 1 + 1 \otimes [\sum_k x_{k,1} \otimes \ldots \otimes x_{k,n}] .\end{array} \tag{10}$$

We want to evaluate

$$\begin{aligned}&[\textstyle\sum_k (x_{k,1} \otimes 1 + 1 \otimes x_{k,1}) \otimes \ldots \otimes (x_{k,n} \otimes 1 + 1 \otimes x_{k,n})]\\ &\quad = \textstyle\sum_{\sigma \in S_n} \nabla^n \sigma(\sum_i (x_{k,1} \otimes 1 + 1 \otimes x_{k,1}) \otimes \ldots \otimes (x_{k,n} \otimes 1 + 1 \otimes x_{k,n}))\end{aligned}$$

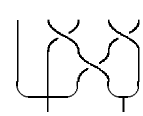
where $\sigma \in S_n$ operates on $p^n(z) \in (A \otimes A)^n(\zeta)$ as described in section 2.

Let $x, y \in A$ and $\tau(x \otimes y) = \sum_i u_i \otimes v_i$. Then $(1 \otimes x) \cdot (y \otimes 1) = (\nabla \otimes \nabla)(\sum_i 1 \otimes u_i \otimes v_i \otimes 1) = \sum_i u_i \otimes v_i = \tau(x \otimes y) = \sum_i (u_i \otimes 1) \cdot (1 \otimes v_i)$. So we have

$$\begin{aligned}(x \otimes 1)(y \otimes 1) &= (xy \otimes 1),\\ (x \otimes 1)(1 \otimes y) &= (x \otimes y),\\ (1 \otimes x)(1 \otimes y) &= (1 \otimes xy),\\ (1 \otimes x)(y \otimes 1) &= \tau(x \otimes y).\end{aligned} \tag{11}$$

We expand a product $(x_1 \otimes 1 + 1 \otimes x_1) \cdot \ldots \cdot (x_n \otimes 1 + 1 \otimes x_n)$. It produces after multiplication 2^n summands, each a product of n terms. A typical product is $(x_1 \otimes 1)(1 \otimes x_2)(x_3 \otimes 1)\ldots$, some of the factors being of the form $x_j \otimes 1$, the others of the form $1 \otimes x_j$. To evaluate such a product we use the rule of multiplication in $A \otimes A$ given by $\nabla_{A \otimes A} = (\nabla \otimes \nabla)(1 \otimes \tau \otimes 1)$.

To explain the following calculation we consider as an example the product $(x_1 \otimes 1)(1 \otimes x_2)(x_3 \otimes 1)(1 \otimes x_4)(x_5 \otimes 1)$. It is calculated with the following braid diagram



The second and fourth factors are pulled over to the right and then all factors are multiplied according to (11). Thus we have $(x_1 \otimes 1)(1 \otimes x_2)(x_3 \otimes 1)(1 \otimes x_4)(x_5 \otimes 1) = (\nabla^3 \otimes \nabla^2)\varphi(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5)$, where $\varphi = \tau_3\tau_4\tau_2$ as defined by the given braid diagram.

We prove now by induction on n that for every product $(x_1 \otimes 1)(1 \otimes x_2)(x_3 \otimes 1)\ldots$ with i factors of the form $x_j \otimes 1$ and $n - i$ factors of the form $1 \otimes x_j$ there is an element $\varphi \in B_n$ such that

$$(x_1 \otimes 1)(1 \otimes x_2)(x_3 \otimes 1)\ldots = (\nabla^i \otimes \nabla^{n-i})\varphi(x_1 \otimes \ldots \otimes x_n).$$

Furthermore if t denotes the number of pairs of factors f_1, f_2 in the product $(x_1 \otimes 1)(1 \otimes x_2)(x_3 \otimes 1)\ldots$ where f_1 is to the left of f_2, f_1 is of the form $(1 \otimes x_j)$ and f_2 is of the form $(x_j \otimes 1)$, or briefly the number of factors in reverse position, then φ is composed of t generators τ_j of B_n. Observe that φ and the number t are uniquely determined by the properties of the multiplication of $A\otimes A$ and the braid group B_n, which has homogeneous relations.

For $n = 1$ we have the trivial cases $x \otimes 1 = (\nabla^1 \otimes \nabla^0)(x)$ and $1 \otimes x = (\nabla^0 \otimes \nabla^1)(x)$, where $\nabla^1 = \mathrm{id}$ and $\nabla^0 = 1$. For the induction nothing is to be proved if $i = n$ or $i = 0$. In these cases we have $t = 0$.

We assume now that the claim is true for n. The induction step for $i \neq 0, n+1$ is given by

$$
\begin{aligned}
(x_1 \otimes 1) \cdot &(1 \otimes x_2) \cdot (x_3 \otimes 1) \cdot \ldots \cdot (1 \otimes x_{n+1}) = \\
&= \{(\nabla^i \otimes \nabla^{n-i})\varphi(x_1 \otimes \ldots \otimes x_n)\} \cdot (1 \otimes x_{n+1}) \\
&= \{(\nabla^i \otimes \nabla^{n-i}) \textstyle\sum_k (u_{k,1} \otimes \ldots \otimes u_{k,n})\} \cdot (1 \otimes x_{n+1}) \\
&= (\textstyle\sum_k u_{k,1} \cdot \ldots \cdot u_{k,i} \otimes u_{k,i+1} \cdot \ldots \cdot u_{k,n}) \cdot (1 \otimes x_{n+1}) \\
&= \textstyle\sum_k (u_{k,1} \otimes 1) \cdot \ldots \cdot (u_{k,i} \otimes 1) \cdot (1 \otimes u_{k,i+1}) \cdot \ldots \cdot (1 \otimes u_{k,n}) \cdot (1 \otimes x_{n+1}) \\
&= (\nabla^i \otimes \nabla^{n-i+1}) \textstyle\sum_k (u_{k,1} \otimes \ldots \otimes u_{k,n} \otimes x_{n+1}) \\
&= (\nabla^i \otimes \nabla^{n-i+1})(\varphi \otimes 1)(x_1 \otimes \ldots \otimes x_{n+1})
\end{aligned}
$$

where t, the number of factors in reverse position, does not change, neither does the number of generators τ_i used in the representation of $\varphi \otimes 1$. The second possibility is

$$
\begin{aligned}
(x_1 \otimes 1) \cdot &(1 \otimes x_2) \cdot (x_3 \otimes 1) \cdot \ldots \cdot (x_{n+1} \otimes 1) \\
&= \{(\nabla^i \otimes \nabla^{n-i})\varphi(x_1 \otimes \ldots \otimes x_n)\} \cdot (x_{n+1} \otimes 1) \\
&= (\textstyle\sum_k u_{k,1} \cdot \ldots \cdot u_{k,i} \otimes u_{k,i+1} \cdot \ldots \cdot u_{k,n}) \cdot (x_{n+1} \otimes 1) \\
&= (\textstyle\sum_k u_{k,1} \cdot \ldots \cdot u_{k,i} \otimes u_{k,i+1} \cdot \ldots \cdot u_{k,n-1}) \cdot (1 \otimes u_{k,n}) \cdot (x_{n+1} \otimes 1)
\end{aligned}
$$

$$
\begin{aligned}
&= (\textstyle\sum_k u_{k,1} \cdot \ldots \cdot u_{k,i} \otimes u_{k,i+1} \cdot \ldots \cdot u_{k,n-1}) \cdot (v_{k,n} \otimes 1) \cdot (1 \otimes v_{k,n+1}) \\
&= (\nabla^{i+1} \otimes \nabla^{n-i-1})\rho(\textstyle\sum_k u_{k,1} \otimes \ldots \otimes u_{k,n-1} \otimes v_{k,n}) \cdot (1 \otimes v_{k,n+1}) \\
&= (\nabla^{i+1} \otimes \nabla^{n-i})(\rho \otimes 1)(\textstyle\sum_k u_{k,1} \otimes \ldots \otimes u_{k,n-1} \otimes v_{k,n} \otimes v_{k,n+1}) \\
&= (\nabla^{i+1} \otimes \nabla^{n-i})(\rho \otimes 1)(1^{n-1} \otimes \tau)(\varphi \otimes 1)(x_1 \otimes \ldots \otimes x_{n+1}).
\end{aligned}
$$

where $\varphi(x_1 \otimes \ldots \otimes x_n) = \sum_k u_{k,1} \otimes \ldots \otimes u_{k,n}$, $\tau(u_{k,n} \otimes x_{n+1}) = \sum v_{k,n} \otimes v_{k,n+1}$, and $(1^{n-1} \otimes \tau)(\varphi \otimes 1)(x_1 \otimes \ldots \otimes x_n \otimes x_{n+1}) = \sum_k u_{k,1} \otimes \ldots \otimes u_{k,n-1} \otimes v_{k,n} \otimes v_{k,n+1}$. We determine the number $t(\psi)$ of generators τ_i occurring in $\psi = (\rho \otimes 1)(1^{n-1} \otimes \tau)(\varphi \otimes 1)$. We have by induction $t(\varphi) = t_n$ the number of factors in $(x_1 \otimes 1) \cdot (1 \otimes x_2) \cdot (x_3 \otimes 1) \cdot \ldots$ in reverse position. Also we have $t_{n+1} = t_n + (n-i)$ the number of factors in $(x_1 \otimes 1) \cdot (1 \otimes x_2) \cdot (x_3 \otimes 1) \cdot \ldots \cdot (x_{n+1} \otimes 1)$ in reverse position. Then $t(\psi) = t((\rho \otimes 1)(1^{n-1} \otimes \tau)(\varphi \otimes 1)) = t(\rho \otimes 1) + t(1^{n-1} \otimes \tau) + t(\varphi \otimes 1) = (n-i-1) + 1 + t_n = t_{n+1}$.

If we sum up we obtain

$$(x_1 \otimes 1 + 1 \otimes x_1) \cdot \ldots \cdot (x_n \otimes 1 + 1 \otimes x_n) = \sum_i \sum_{\varphi_i} (\nabla^i \otimes \nabla^{n-i})\varphi_i(x_1 \otimes \ldots \otimes x_n),$$

for certain $\varphi_i \in B_n$ which arise in the evaluation given above.

Now let $z \in A^n(\zeta)$. We expand the products in $\nabla^n p^n z = \sum_k (x_{k,1} \otimes 1 + 1 \otimes x_{k,1}) \cdot \ldots \cdot (x_{k,n} \otimes 1 + 1 \otimes x_{k,n})$. Each of these products in the sum is treated in the same way as described above. Using (3) we get

$$\begin{aligned} \nabla^n p^n(z) &= \textstyle\sum_k (x_{k,1} \otimes 1 + 1 \otimes x_{k,1}) \cdot \ldots \cdot (x_{k,n} \otimes 1 + 1 \otimes x_{k,n}) \\ &= \textstyle\sum_k \sum_i \sum_{\varphi_i} (\nabla^i \otimes \nabla^{n-i}) \varphi_i (x_{k,1} \otimes \ldots \otimes x_{k,n}) \\ &= \textstyle\sum_i \sum_{\varphi_i} (\nabla^i \otimes \nabla^{n-i}) \varphi_i(z) \\ &= \textstyle\sum_i \sum_{\varphi_i} (\nabla^i \otimes \nabla^{n-i}) \zeta^{-t(\varphi_i)} \rho_i(z) \end{aligned}$$

where $\rho_i \in S_n$ are the canonical images of the $\varphi_i \in B_n$ and $t(\varphi_i)$ is the number of factors τ_j in the representation of φ_i.

This gives us

$$\begin{aligned} [p^n(z)] &= \textstyle\sum_{\sigma \in S_n} \nabla^n p^n \sigma(z) \\ &= \textstyle\sum_\sigma \sum_i \sum_{\varphi_i} \zeta^{-t(\varphi_i)} (\nabla^i \otimes \nabla^{n-i}) \rho_i \sigma(z) \\ &= \textstyle\sum_\sigma \sum_i \left(\sum_{\varphi_i} \zeta^{-t(\varphi_i)} \right) (\nabla^i \otimes \nabla^{n-i}) \sigma(z) \\ &= \textstyle\sum_i c_i (\nabla^i \otimes \nabla^{n-i}) \sum_\sigma \sigma(z) \end{aligned}$$

where the factors $c_i = \sum_{\varphi_i} \zeta^{-t(\varphi_i)} \in k$. We want to show that the c_i are zero for all $0 < i < n$.

So fix n and i. Consider one product $(x_1 \otimes 1) \cdot (1 \otimes x_2) \cdot (x_3 \otimes 1) \cdot \ldots$ in the development of $(x_1 \otimes 1 + 1 \otimes x_1) \cdot \ldots \cdot (x_n \otimes 1 + 1 \otimes x_n) = \sum_i \sum_{\varphi_i} (\nabla^i \otimes \nabla^{n-i}) \varphi_i (x_1 \otimes \ldots \otimes x_n)$ and its corresponding φ_i. The chosen summand is completely determined by giving the positions in $\{1, \ldots, n\}$ of the $n-i$ factors of the form $(1 \otimes x_j)$. The first of these factors has λ_1 factors of the form $(x_j \otimes 1)$ to its right with $0 \leq \lambda_1 \leq i$. So it contributes λ_1 pairs of factors in reverse position. The second factor of the form $(1 \otimes x_j)$ contributes λ_2 (with $0 \leq \lambda_2 \leq \lambda_1 \leq i$) pairs of factors in reverse position, and so on. We obtain $t = \lambda_1 + \lambda_2 + \ldots + \lambda_{n-i}$ pairs in reverse position. If we know the λ_i with $0 \leq \lambda_{n-i} \leq \ldots \leq \lambda_2 \leq \lambda_1 \leq i$ then they also determine uniquely the position of the factors of the form $(1 \otimes x_j)$. Each partition of $t = \lambda_1 + \lambda_2 + \ldots + \lambda_{n-i}$ into (at most) $n - i$ parts each $\leq i$ gives one term ζ^{-t} in $c_i = \sum_{\varphi_i} \zeta^{-t(\varphi_i)}$ and we find $p(i, n-i, t)$ partitions of t into at most $n - i$ parts each $\leq i$. So we get

$$c_i = \sum_{t \geq 0} p(i, n-i, t) \zeta^{-t}.$$

By a theorem of Sylvester ([1] Theorem 3.1) we have

$$\sum_{t \geq 0} p(i, n-i, t) q^t = \frac{(1 - q^n)(1 - q^{n-1}) \ldots (1 - q^{n-i+1})}{(1 - q^i)(1 - q^{i-1}) \ldots (1 - q)}$$

hence $c_i = 0$ for $0 < i < n$ since ζ and also ζ^{-1} are *primitive* n-th roots of unity.

So we have shown

$$\begin{aligned}[p^n(z)] &= \textstyle\sum_{\sigma\in S_n} \nabla^n p^n \sigma(z) \\ &= \textstyle\sum_i c_i(\nabla^i \otimes \nabla^{n-i}) \sum_\sigma \sigma(z) \\ &= \textstyle\sum_\sigma \nabla^n \sigma(z) \otimes 1 + 1 \otimes \sum_\sigma \nabla^n \sigma(z) \\ &= p[z].\end{aligned}$$

□

Corollary 5.4. *Let H be a Hopf algebra in $\mathcal{C}$. Then the set of primitive elements $P(H)$ forms a Lie algebra in $\mathcal{C}$.*

Proof. By Lemma 5.1 $P(H)$ is a Yetter-Drinfeld submodule of H. Let $z \in P(H)^n(\zeta)$. Then $p([z]) = [p^n(z)] = [\Delta^n(z)] = \Delta([z])$ since Δ is an algebra homomorphism. Hence $[z] \in P(H)$. So $P(H)$ is a Lie subalgebra of H^L. □

Definition 5.5. Let A be an algebra in $\mathcal{C}$ and let $\mathrm{end}(A)$ be the inner endomorphism object of A in $\mathcal{C}$, i.e. the Yetter- Drinfeld module $\mathrm{end}(A)$ satisfying $\mathcal{C}(X \otimes A, A) \cong \mathcal{C}(X, \mathrm{end}(A))$ for all $X \in \mathcal{C}$. It can be shown that

$$\begin{aligned}\mathrm{end}(A) := \{f \in \mathrm{Hom}(A,A) | \exists \textstyle\sum f_{(0)} \otimes f_{(1)} \in \mathrm{Hom}(A,A) \otimes K \forall a \in A : \\ \textstyle\sum f_{(0)}(a) \otimes f_{(1)} = \sum f(a_{(0)})_{(0)} \otimes f(a_{(0)})_{(1)} S(a_{(1)})\}\end{aligned}$$

is the Yetter-Drinfeld module with the required universal property. $\mathrm{end}(A)$ operates on A by a canonical map $\mathrm{ev} : \mathrm{end}(A) \otimes A \longrightarrow A$ with $\mathrm{ev}(f \otimes a) = f(a)$.

A *derivation* from A to A is a linear map $(d : A \longrightarrow A) \in \mathrm{end}(A)$ such that

$$d(ab) = d(a)b + (1 \otimes d)(\tau \otimes 1)(d \otimes a \otimes b)$$

for all $a, b \in A$. Observe that in the symmetric situation this means $d(ab) = d(a)b + ad(b)$.

It is clear that all derivations from A to A form an object $\mathrm{Der}(A)$ in $\mathcal{C}$ and that there is an operation $\mathrm{Der}(A) \otimes A \longrightarrow A$.

Corollary 5.6. $\mathrm{Der}(A)$ *is a Lie algebra.*

Proof. Let m denote the multiplication of A. An endomorphism $x : A \longrightarrow A$ in $\mathrm{end}(A)$ is a derivation iff $m(x \otimes 1 + 1 \otimes x) = xm$ where $(x \otimes y)(a \otimes b) = (\mathrm{ev} \otimes \mathrm{ev})(1 \otimes \tau \otimes 1)(x \otimes y \otimes a \otimes b)$ for elements a and b in A and elements x and y in $\mathrm{end}(A)$. So $x \in \mathrm{end}(A)$ is a derivation iff $mp(x) = xm$.

To show that $\mathrm{Der}(A)$ is a Lie algebra it suffices to show that it is closed under Lie multiplication since it is a subobject of $\mathrm{end}(A)$, which is an algebra in the category $\mathcal{C}$. Let ζ be a primitive n-th root of unity. Let $\nabla : \mathrm{end}(A) \otimes \mathrm{end}(A) \longrightarrow \mathrm{end}(A)$ be the multiplication of $\mathrm{end}(A)$.

If $x_1, x_2 \in \mathrm{Der}(A)$ then $mp(x_1)p(x_2) = x_1 mp(x_2) = x_1 x_2 m$ or more generally $m(\nabla^n p^n)(x_1 \otimes \ldots \otimes x_n) = \nabla^n(x_1 \otimes \ldots \otimes x_n)m$ for all $x_1 \otimes \ldots \otimes x_n \in \mathrm{Der}(A)^n$. Thus we get for $z \in \mathrm{Der}(A)^n(\zeta)$

$$\begin{aligned} mp([z]) &= m[p^n(z)] = \sum m(\nabla^n \sigma(p^n(z))) \\ &= \sum m(\nabla^n p^n \sigma(z)) = \sum \nabla^n \sigma(z) m = [z]m \end{aligned}$$

hence $[z] \in \mathrm{Der}(A)$. □

6. The Universal Enveloping Algebra of a Lie Algebra

As in [5] we can now construct the universal enveloping algebra of a Lie algebra P in $\mathcal{C}$ as $U(P) := T(P)/I$ where $T(P)$ is the tensor algebra over P, which lives again in $\mathcal{C}$, and where I is the ideal generated by the relations $[z] - \sum \nabla^n \sigma(z)$ for all $z \in P^n(\zeta)$, for all n and for all primitive n-th roots of unity ζ. Then $U(P)$ clearly is a universal solution for the following universal problem

$$\begin{tikzcd} P \arrow[r, "\iota"] \arrow[dr, "f"'] & U(P) \arrow[d, "g"] \\ & A \end{tikzcd}$$

where for each morphism of Lie-algebras f there is a unique morphism of algebras g such that the diagram commutes.

Theorem 6.1. *Let P be a Lie algebra in $\mathcal{C}$. Then the universal enveloping algebra $U(P)$ is a Hopf algebra in $\mathcal{C}$.*

Proof. It is easily seen that $\delta : P \longrightarrow (U(P) \otimes U(P))^L$ in $\mathcal{M}^{kG}$ given by $\delta(x) := \overline{x} \otimes 1 + 1 \otimes \overline{x}$ where $\overline{x}$ is the canonical image of $x \in P$ in $U(P)$ and the counit $\varepsilon : U(P) \longrightarrow k$ given by the zero morphism $0 : P \longrightarrow k$ define the structure of a bialgebra on $U(P)$ in $\mathcal{C}$.

Now we want to define $S : U(P) \longrightarrow U(P)^{op+}$ by the Lie homomorphism $S : P \longrightarrow U(P)^{op+}$, $S(x) = -\bar{x}$. Here A^{op+} is the algebra obtained from the algebra A by the multiplication $A \otimes A \xrightarrow{\tau} A \otimes A \xrightarrow{\nabla} A$. Then for $z \in P^n(\zeta)$ we have

$$\begin{aligned} S([z]) &= -\overline{[z]} = -\textstyle\sum_\sigma \nabla^n \sigma(\overline{z}) = -\sum_\sigma \nabla^n \pi \pi^{-1} \sigma(\overline{z}) \\ &= -\textstyle\sum_\sigma (\nabla^{op})^n \pi^{-1} \sigma(\overline{z}) \text{ (by (3))} \ = -\sum_\sigma (\nabla^{op})^n \zeta^{\frac{n(n-1)}{2}} \rho^{-1} \sigma(\overline{z}) \\ &= -\zeta^{\frac{n(n-1)}{2}} [\overline{z}] = (-1)^n [\overline{z}] = [\overline{S^n(z)}] \end{aligned}$$

where $\pi \in B_n$ is the braid map given by the twist of all n strands with source $\{1, \ldots, n\}$ and domain $\{n, \ldots, 1\}$, $\pi = (\tau_1) \ldots (\tau_{n-2} \ldots \tau_2 \tau_1)(\tau_{n-1} \ldots \tau_2 \tau_1)$ and
$\zeta^{-\frac{n(n-1)}{2}} \pi = \rho \in S_n$.

Hence S is a Lie homomorphism and factorizes through $U(P)$. Since $U(P)$ is generated as an algebra by P we prove that S is the antipode by complete induction:

$$\nabla(1 \otimes S)\Delta(1) = 1S(1) = 1 = \varepsilon(1),$$

$$\nabla(1 \otimes S)\Delta(\overline{x}) = \overline{x} + S(\overline{x}) = 0 = \varepsilon(\overline{x}).$$

Before we prove the general induction step we observe that $\Delta : U(P) \longrightarrow U(P) \otimes U(P)$ is a morphism in $\mathcal{C} = \mathcal{YD}_K^K$ so that we have in particular

$$\sum (a^0)_1 \otimes (a^0)_2 \otimes a^1 = \sum (a_1)^0 \otimes (a_2)^0 \otimes (a_1)^1(a_2)^1 \in U(P) \otimes U(P) \otimes K$$

for $a \in U(P)$. (Here we use $\delta(a) = \sum a^0 \otimes a^1$ to denote the comodule structure in $\mathcal{YD}_K^K$.) Assume now that a is writte as a product of $n \geq 1$ elements in P and that $\sum a_1 S(a_2) = 0$. Then for all $x \in P$ we have $\sum (a_1)^0 S((a_2)^0) S(x(a_1)^1(a_2)^1) = \sum (a^0)_1 S((a^0)_2) S(xa^1) = 0$ since $\delta(a) = \sum a^0 \otimes a^1 \in \overline{P \otimes \ldots \otimes P} \otimes K \subseteq U(P) \otimes K$. So we have

$$\begin{aligned}
\nabla(1 \otimes S)\Delta(xa) &= \nabla(1 \otimes S) \textstyle\sum (xa_1 \otimes a_2 + (a_1)^0 \otimes (x(a_1)^1)a_2) \\
&= \textstyle\sum xa_1 S(a_2) + \sum (a_1)^0 S((x(a_1)^1)a_2) \\
&= \textstyle\sum (a_1)^0 S((a_2)^0) S(x(a_1)^1(a_2)^1) = 0 = \eta\varepsilon(xa).
\end{aligned}$$

The second condition $\nabla(S \otimes 1)\Delta = \eta\varepsilon$ is proved in a similar way (by using elements of the form ax and the equation $\sum S((a\kappa)_1)(a\kappa)_2 = 0$ for a written as a product of n elements in P and $\kappa \in K$). So S is an antipode and $U(P)$ is a Hopf algebra in $\mathcal{C}$. $\square$

7. (G, χ) Lie algebras

In [5] we introduced and studied the concept of G-graded Lie algebras or (G, χ)-Lie algebras for an abelian group G with a bicharacter χ generalizing the concepts of Lie algebras, Lie super algebras, and Lie color algebras. The reader may find examples of such (G, χ)-Lie algebras in [5]. A generalization of this concept of Lie algebras to the group graded case for a noncommutative group requires the use of Yetter-Drinfeld modules over kG. We show that (G, χ)-Lie algebras are Lie algebras on Yetter-Drinfeld modules in the sense of this paper. We use the notation of [5].

Let G be an abelian group with a bicharacter $\chi : G \otimes_{\mathbb{Z}} G \longrightarrow k^*$. Let P be a kG-comodule. Then P is a Yetter-Drinfeld module over kG [4] with the module structure $x \cdot g = \chi(h, g)x$ for homogeneous elements $x = x_h \in M$ with $\delta(x) = x \otimes h$. The braid map is $\tau(x_h \otimes y_g) = y_g \otimes x_h \cdot g = \chi(h, g) y_g \otimes x_h$, hence the braiding given in [5] after Example 2.3.

Let $\zeta \in k^*$ be given. Let $(g_1, \dots, g_n)$ be a ζ-family, i.e. $\chi(g_i, g_j)\chi(g_j, g_i) = \zeta^2$. Let $Q := \sum_{\sigma \in S_n} P_{g_{\sigma(1)}} \otimes \dots \otimes P_{g_{\sigma(n)}}$.

Lemma 7.1. *Q is a right S_n-module by*

$$(x_1 \otimes \dots \otimes x_n)\sigma = \rho(\sigma, (g_1, \dots, g_n))x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(n)}$$

for $\sigma \in S_n$ and $x_1 \otimes \dots \otimes x_n \in P_{g_1} \otimes \dots \otimes P_{g_n}$.

Proof. We have to show the compatibility of this operation with the composition of permutations. Let $\sigma, \tau \in S_n$. We use Lemma 2.2 of [5]. Then

$$\begin{aligned}
&(x_1 \otimes \dots \otimes x_n)(\sigma\tau) = \\
&\quad = \rho(\sigma\tau, (g_1, \dots, g_n))x_{\sigma\tau(1)} \otimes \dots \otimes x_{\sigma\tau(n)} \\
&\quad = \rho(\sigma, (g_1, \dots, g_n))\rho(\tau, (g_{\sigma(1)}, \dots, g_{\sigma(n)}))x_{\sigma\tau(1)} \otimes \dots \otimes x_{\sigma\tau(n)} \\
&\quad = (\rho(\sigma, (g_1, \dots, g_n))x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(n)})\tau \\
&\quad = ((x_1 \otimes \dots \otimes x_n)\sigma)\tau.
\end{aligned}$$

□

Q becomes a left S_n-module by $\sigma(x_1 \otimes \dots \otimes x_n) = \rho(\sigma^{-1}, (g_1, \dots, g_n))x_{\sigma^{-1}(1)} \otimes \dots \otimes x_{\sigma^{-1}(n)}$. Thus $\bigoplus_{\{(g_1,\dots,g_n)\ \zeta\text{-family}\}} P_{g_1} \otimes \dots \otimes P_{g_n}$ is also a left S_n-module.

This action is connected with the action of B_n on $\bigoplus_{\{(g_1,\dots,g_n)\ \zeta\text{- family}\}} P_{g_1} \otimes \dots \otimes P_{g_n}$ by

$$\zeta^{-1}\tau_i(x_1 \otimes \dots \otimes x_n) = \sigma_i(x_1 \otimes \dots \otimes x_n) \tag{12}$$

for the canonical generators τ_i of B_n resp. σ_i of S_n, since

$$\begin{aligned}
&\zeta^{-1}\tau_i(x_1 \otimes \dots \otimes x_n) = \\
&\quad = \zeta^{-1}\chi(g_i, g_{i+1})x_1 \otimes \dots \otimes x_{i+1} \otimes x_i \otimes \dots \otimes x_n \\
&\quad = \rho(\sigma_i^{-1}, (g_1, \dots, g_n))x_{\sigma_i^{-1}(1)} \otimes \dots \otimes x_{\sigma_i^{-1}(n)} \\
&\quad = \sigma_i(x_1 \otimes \dots \otimes x_n).
\end{aligned}$$

In particular we have

$$\begin{aligned}
&\tau_i^{-1}\tau_{i+1}^{-1} \dots \tau_{j-1}^{-1}\tau_j^2\tau_{j-1} \dots \tau_{i+1}\tau_i(x_1 \otimes \dots \otimes x_n) = \\
&\quad = \zeta^2\sigma_i^{-1}\sigma_{i+1}^{-1} \dots \sigma_{j-1}^{-1}\sigma_j^2\sigma_{j-1} \dots \sigma_{i+1}\sigma_i(x_1 \otimes \dots \otimes x_n) \\
&\quad = \zeta^2(x_1 \otimes \dots \otimes x_n),
\end{aligned}$$

so that $x_1 \otimes \dots \otimes x_n \in P^n(\zeta)$ by Lemma 2.4. Thus we have

$$\bigoplus_{\{(g_1,\dots,g_n)\ \zeta\text{-family}\}} P_{g_1} \otimes \dots \otimes P_{g_n} \subseteq P^n(\zeta).$$

Conversely let $\sum x_1 \otimes \dots \otimes x_n \in P^n = \bigoplus_{\{(g_1,\dots,g_n)\}} P_{g_1} \otimes \dots \otimes P_{g_n}$ with homogeneous summands and assume that one of the summands is non-zero in $P_{g_1} \otimes \dots \otimes P_{g_n}$ where $(g_1, \dots, g_n)$ is not a ζ-family for example by $\chi(g_i, g_{i+1})\chi(g_{i+1}, g_i) \neq \zeta^2$. Then $(\tau_i^2 - \zeta^2)(\sum x_1 \otimes \dots \otimes x_n)$ has a non-zero component in $P_{g_1} \otimes \ldots \otimes P_{g_n}$, hence $\sum x_1 \otimes \ldots \otimes x_n$ cannot be in $P^n(\zeta)$. This proves

Proposition 7.2. *Let $\zeta \in k^*$ be given. Then*

$$P^n(\zeta) = \bigoplus_{\{(g_1,\ldots,g_n)\ \zeta\text{-family}\}} P_{g_1} \otimes \ldots \otimes P_{g_n}.$$

By Lemma 7.1 and (12) the bracket multiplication of [5] is a special case of the bracket multiplication of this paper and (G,χ)-Lie algebras are Lie algebras over Yetter-Drinfeld modules.

Example 7.3. As a new example of Lie algebras we give one family of examples of (G,χ)-Lie algebras. Let $G = C_3 = \{0,1,2\}$ be the cyclic group with 3 elements. Define the structure of a right kG-module on a kG-comodule V (i.e. on a C_3-graded vector space $V = V_0 \oplus V_1 \oplus V_2$) using the bicharacter $\chi : C_3 \otimes_{\mathbb{Z}} C_3 \cong C_3 \longrightarrow k^*$, $\chi(1 \otimes 1) = \xi$ a primitive 3-rd root of unity, by $v \cdot g := \chi(\deg(v) \otimes g)v = \chi(\deg(v), g)v$ for $g \in G$ and homogeneous elements $v \in V$. Then V is a Yetter-Drinfeld module.

Let $A := \mathrm{end}(V)$ be the inner endomorphism object of V in kG-comod. By Corollary 4.2 A is a Lie algebra. One verifies easily (see [5]) that the only non-zero components $A^n(\zeta)$ for the partial Lie multiplication are

$$A^2(-1) = A_0 \otimes (A_1 \oplus A_2) \oplus (A_0 \oplus A_1 \oplus A_2) \otimes A_0$$

and

$$A^3(\xi) = A_1 \otimes A_1 \otimes A_1 \oplus A_2 \otimes A_2 \otimes A_2.$$

Now let $\langle .,. \rangle : V \otimes V \longrightarrow k$ be a bilinear form on V in $\mathcal{C}$. We define

$$\mathfrak{g}(V)_i := \{f \in A_i | \forall v, w \in V, \deg(v) = j :\ \langle f(v), w\rangle = -\chi(i,j)\langle v, f(w)\rangle\}.$$

This space is the homogeneous component of $\mathfrak{g}(V) \subseteq A$ that becomes a Yetter-Drinfeld module.

For $f \in \mathfrak{g}(V)_0$ and $g \in \mathfrak{g}(V)_i$, $i \in C_3$, $v \in V_j$, $w \in V$ we have

$$\begin{aligned}\langle [f,g](v), w\rangle &= \langle (fg - gf)(v), w\rangle \\ &= \langle fg(v), w\rangle - \langle gf(v), w\rangle \\ &= \chi(i,j)\langle v, gf(w)\rangle - \chi(i,j)\langle v, fg(w)\rangle \\ &= -\chi(i,j)\langle v, [f,g](w)\rangle,\end{aligned}$$

hence $[f,g] \in \mathfrak{g}(V)_i$. Analogously one shows $[g,f] \in \mathfrak{g}(V)_i$.

For $k = 1,2,3$ let $f_k \in \mathfrak{g}(V)_i$ ($i = 1$ or $i = 2$). Then

$$\begin{aligned}\langle [f_1,f_2,f_3](v), w\rangle &= \textstyle\sum_{\sigma \in S_3} \langle f_{\sigma(1)} f_{\sigma(2)} f_{\sigma(3)}(v), w\rangle \\ &= (-1) \textstyle\sum_{\sigma \in S_3} \chi(i, i+i+j)\chi(i, i+j)\chi(i,j)\langle v, f_{\sigma(3)} f_{\sigma(2)} f_{\sigma(1)}(w)\rangle \\ &= -\langle v, [f_1,f_2,f_3](w)\rangle,\end{aligned}$$

hence $[f_1, f_2, f_3] \in \mathfrak{g}(V)_0$. Thus we have a Lie algebra $\mathfrak{g}(V)$. Depending on the choice of the bilinear form this is a generalization of the orthogonal or the symplectic Lie algebra.

8. Appendix

Proof. of Lemma 2.4:

Define actions $\pi_{i,j}$ for $1 \leq i < j \leq n$ on M by

$$\pi_{i,j} := \tau_i^{-1}\tau_{i+1}^{-1} \dots \tau_{j-2}^{-1}\tau_{j-1}\tau_{j-2} \dots \tau_{i+1}\tau_i \tag{13}$$

Observe that $\pi_{i,i+1} = \tau_i$. Since $\tau_i\tau_j = \tau_j\tau_i$ if $|i-j| \geq 2$ a simple calculation gives

$$\begin{gathered}\pi_{i,j}\tau_k = \tau_k\pi_{i,j} \text{ for all } k < i-1 \text{ and all } k > j,\\ \pi_{i,j}\tau_{i-1} = \tau_{i-1}\pi_{i-1,j},\\ \pi_{i,j}\tau_k = \tau_k\pi_{i,j} \text{ for all } i < k < j-1,\\ \pi_{i,j}\tau_{j-1} = \tau_{j-1}\pi_{i,j-1} \text{ if } i < j-1 \text{ and}\\ \pi_{i,j}\tau_{j-1} = \tau_i\tau_i = \tau_{j-1}\pi_{i,j} \text{ if } i = j-1.\end{gathered} \tag{14}$$

Let $N \subseteq M$ be a kB_n submodule of M. Assume furthermore that $\tau_i^2\tau_{i+1} = \tau_{i+1}\tau_i^2$ on N for all $i = 1, \dots, n-2$. Then $\tau_{i+1}^2\tau_i = \tau_i^3 = \tau_i\tau_{i+1}^2$. Consequently we have

$$\tau_j^2\tau_i = \tau_i\tau_j^2 \tag{15}$$

on N for all $i, j = 1, \dots, n-1$. Thus the τ_j^2 commute with all $\varphi \in B_n$ if they act on N.

We introduce the vector subspace $M(\zeta) \subseteq \overline{M(\zeta)} \subseteq M$ by

$$\overline{M(\zeta)} := \{z \in M | \forall 1 \leq i < j \leq n : \pi_{i,j}^2(z) = \zeta^2 z\}$$

and show that $\overline{M(\zeta)}$ is invariant under the action of the τ_i and $\tau_i^2\tau_{i+1} = \tau_{i+1}\tau_i^2$ on $M(\zeta)$ for all $i = 1, \dots, n-2$.

For $z \in M(\zeta)$ and $i < j$ we have $\pi_{i,j}^2\tau_k(z) = \tau_k\pi_{i,j}^2(z) = \zeta^2\tau_k(z)$ for all k with $1 \leq k < i-1$ and $j < k \leq n$ by (14) and for all k with $i < k < j-1$ by (14). Furthermore we have $\pi_{i,j}^2\tau_{i-1}(z) = \tau_{i-1}\pi_{i-1,j}^2(z) = \zeta^2\tau_{i-1}(z)$ by (14), $\pi_{i,j}^2\tau_{j-1}(z) = \tau_{j-1}\pi_{i,j-1}^2(z) = \zeta^2\tau_{j-1}(z)$ (for $i < j-1$) by (14), and $\pi_{i,j}^2\tau_{j-1}(z) = \tau_{j-1}\pi_{i,j}^2(z) = \zeta^2\tau_{j-1}(z)$ (for $i = j-1$) by (14). So there remain two cases to investigate for which we use $\pi_{i,j}^2(z) = \zeta^2 z$ and symmetrically $\pi_{i,j}^{-2}(z) = \zeta^{-2}z$ for all $z \in M(\zeta)$.

In the first case we get

$$\begin{aligned}\pi_{i,j}^2\tau_i(z) &= \tau_i^{-1} \dots \tau_{j-1}^2 \dots \tau_i\tau_i(z) = \tau_i^{-1} \dots \tau_{j-1}^2 \dots \tau_{i+1}(\zeta^2 z)\\ &= \zeta^2\tau_i^{-1} \dots \tau_{j-1}^2 \dots \tau_{i+1}(z) = \zeta^2\tau_i^{-1}\pi_{i+1,j}^2(z)\\ &= \zeta^2\tau_i^{-1}(\zeta^2 z) = \zeta^2\tau_i^{-1}\tau_i^2(z) = \zeta^2\tau_i(z)\end{aligned}$$

for $i+1 < j$ and $\pi_{i,i+1}^2\tau_i(z) = \tau_i^3(z) = \zeta^2\tau_i(z)$.

In the second case we get

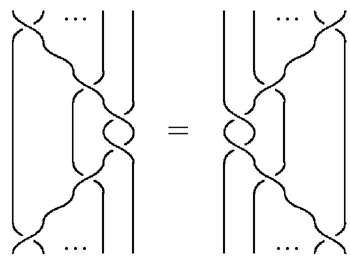
$$\begin{aligned}\pi_{i,j}^2\tau_j(z) &= \tau_i^{-1}\ldots\tau_{j-1}^2\ldots\tau_i\tau_j(z) = \zeta^2\tau_i^{-1}\ldots\tau_{j-1}^2\ldots\tau_i\tau_j\tau_j^{-2}(z)\\ &= \zeta^2\tau_j^{-1}\tau_j\tau_i^{-1}\ldots\tau_{j-1}^2\ldots\tau_i\tau_j^{-1}(z) = \zeta^2\tau_j^{-1}\tau_i^{-1}\ldots\tau_j\tau_{j-1}^2\tau_j^{-1}\ldots\tau_i(z)\\ &= \zeta^2\tau_j^{-1}\tau_i^{-1}\ldots\tau_{j-1}^{-1}\tau_j^2\tau_{j-1}\ldots\tau_i(z) = \zeta^2\tau_j^{-1}\pi i, j+1^2(z)\\ &= \zeta^2\tau_j^{-1}(\zeta^2 z) = \zeta^2\tau_j(z).\end{aligned}$$

Hence we have $\tau_i(z) \in M(\zeta)$ for all $z \in M(\zeta)$ and all $i = 1,\ldots,n-1$.

The claim $\tau_i^2\tau_{i+1} = \tau_{i+1}\tau_i^2$ is clear from the invariance and the fact, that τ_i^2 on $M(\zeta)$ is multiplication by ζ^2.

Since the τ_i^2 commute in their action on $\overline{M(\zeta)}$ with all $\varphi \in B_n$ it is clear that $\overline{M(\zeta)} \subseteq M(\zeta)$. $\square$

We now study specific braids. The following identity



implies

$$\tau_1^{-1}\ldots\tau_{i-1}^{-1}\tau_i^2\tau_{i-1}\ldots\tau_1 = \tau_i\ldots\tau_2\tau_1^2\tau_2^{-1}\ldots\tau_i^{-1} \tag{16}$$

and similarly $\tau_1\ldots\tau_{i-1}\tau_i^2\tau_{i-1}^{-1}\ldots\tau_1^{-1} = \tau_i^{-1}\ldots\tau_2^{-1}\tau_1^2\tau_2\ldots\tau_i$ for all $i = 1,\ldots,n$.

Let $B_n \ni \varphi \mapsto \tilde{\varphi} \in S_n$ denote the canonical epimorphism.

For each braid $\varphi \in B_n$ there exists a braid $\varphi_{(i)} \in B_{n+1}$ such that the diagram

$$\begin{array}{ccc} P\otimes\ldots\otimes \underset{i}{P^2}\otimes\ldots\otimes P & \xrightarrow{1\otimes\ldots\otimes f\otimes\ldots\otimes 1} & P\otimes\ldots\otimes P\\ \Big\downarrow{\scriptstyle\varphi_{(i)}} & & \Big\downarrow{\scriptstyle\varphi}\\ P\otimes\ldots\otimes \underset{j}{P^2}\otimes\ldots\otimes P & \xrightarrow{1\otimes\ldots\otimes f\otimes\ldots\otimes 1} & P\otimes\ldots\otimes P\end{array}$$

commutes for all $f : P^2 \longrightarrow P$ in $\mathcal{C}$ (where $j = \tilde{\varphi}(i)$). The braid $\varphi_{(i)}$ can be given explicitly, but we are only interested in the following special forms

$$\begin{array}{lll} \tau_{j(i)} = \tau_{j+1} & \text{if } j > i; & \tau_{i-1(i)} = \tau_i\tau_{i-1};\\ \tau_{j(i)} = \tau_j & \text{if } j < i-1; & \tau_{i(i)} = \tau_i\tau_{i+1}\end{array}$$

which can be easily verified.

By (16) we have for all $z \in P^{n+1}(-1,\zeta)$

$$\tau_i^2 \tau_{i-1} \dots \tau_1(z) = \tau_{i-1} \dots \tau_1(z). \tag{17}$$

Lemma 8.1. *For $z \in P^{n+1}(-1,\zeta)$, $\varphi \in B_n$ and $j := \tilde{\varphi}(i)$ we have*

$$\begin{aligned} \varphi_{(i)} \tau_{i-1} \dots \tau_1(z) &= \tau_{j-1} \dots \tau_1(1 \otimes \varphi)(z); \\ \varphi_{(i)} \tau_i \dots \tau_1(z) &= \tau_j \dots \tau_1(1 \otimes \varphi)(z). \end{aligned}$$

Proof. To prove this we first observe that these two relations are compatible with the group structure of B_n. For $\tilde{\varphi}\tilde{\psi}(i) = \tilde{\varphi}(j) = k$ we have

$$\begin{aligned} \varphi_{(j)} \psi_{(i)} \tau_{i-1} \dots \tau_1(z) = \varphi_{(j)} \tau_{j-1} \dots \tau_1(1 \otimes \psi)(z) = \tau_{k-1} \dots \tau_1(1 \otimes \varphi\psi)(z); \\ \varphi_{(j)} \psi_{(i)} \tau_i \dots \tau_1(z) = \varphi_{(j)} \tau_j \dots \tau_1(1 \otimes \psi)(z) = \tau_k \dots \tau_1(1 \otimes \varphi\psi)(z) \end{aligned}$$

so we only have to show these relations for the generators $\varphi = \tau_j$, $j = 1, \dots, n-1$. In these cases we have

$$\begin{aligned} \tau_{j(i)} \tau_{i-1} \dots \tau_1(z) &= \tau_{j+1} \tau_{i-1} \dots \tau_1(z) \\ &= \tau_{i-1} \dots \tau_1 \tau_{j+1}(z) \\ &= \tau_{i-1} \dots \tau_1(1 \otimes \tau_j)(z) && \text{for } j > i; \\ \tau_{j(i)} \tau_{i-1} \dots \tau_1(z) &= \tau_j \tau_{i-1} \dots \tau_1(z) \\ &= \tau_{i-1} \dots \tau_j \tau_{j+1} \tau_j \dots \tau_1(z) \\ &= \tau_{i-1} \dots \tau_{j+1} \tau_j \tau_{j+1} \dots \tau_1(z) \\ &= \tau_{i-1} \dots \tau_1 \tau_{j+1}(z) \\ &= \tau_{i-1} \dots \tau_1(1 \otimes \tau_j)(z) && \text{for } j < i-1; \\ \tau_{i-1(i)} \tau_{i-1} \dots \tau_1(z) &= \tau_i \tau_{i-1} \tau_{i-1} \dots \tau_1(z) \\ &= \tau_i \tau_{i-2} \dots \tau_1(z) \\ &= \tau_{i-2} \dots \tau_1 \tau_i(z) \\ &= \tau_{i-2} \dots \tau_1(1 \otimes \tau_{i-1})(z); \\ \tau_{i(i)} \tau_{i-1} \dots \tau_1(z) &= \tau_i \tau_{i+1} \tau_{i-1} \dots \tau_1(z) \\ &= \tau_i \dots \tau_1 \tau_{i+1}(z) \\ &= \tau_i \dots \tau_1(1 \otimes \tau_i)(z); \end{aligned}$$

$$\begin{aligned}
\tau_{j(i)}\tau_i\ldots\tau_1(z) &= \tau_{j+1}\tau_i\ldots\tau_1(z)\\
&= \tau_i\ldots\tau_1\tau_{j+1}(z)\\
&= \tau_i\ldots\tau_1(1\otimes\tau_j)(z) \qquad\qquad \text{for } j>i;\\
\tau_{j(i)}\tau_i\ldots\tau_1(z) &= \tau_j\tau_i\ldots\tau_1(z)\\
&= \tau_i\ldots\tau_j\tau_{j+1}\tau_j\ldots\tau_1(z)\\
&= \tau_i\ldots\tau_{j+1}\tau_j\tau_{j+1}\ldots\tau_1(z)\\
&= \tau_i\ldots\tau_1\tau_{j+1}(z)\\
&= \tau_i\ldots\tau_1(1\otimes\tau_j)(z) \qquad\qquad \text{for } j<i-1;\\
\tau_{i-1(i)}\tau_{i-1}\ldots\tau_1(z) &= \tau_i\tau_{i-1}\tau_i\tau_{i-1}\tau_{i-2}\ldots\tau_1(z)\\
&= \tau_{i-1}\tau_i\tau_{i-1}^2\tau_{i-2}\ldots\tau_1(z)\\
&= \tau_{i-1}\tau_i\tau_{i-2}\ldots\tau_1(z)\\
&= \tau_{i-1}\tau_{i-2}\ldots\tau_1\tau_i(z)\\
&= \tau_{i-1}\ldots\tau_1(1\otimes\tau_{i-1})(z);\\
\tau_{i(i)}\tau_i\ldots\tau_1(z) &= \tau_i\tau_{i+1}\tau_i\ldots\tau_1(z)\\
&= \tau_{i+1}\tau_i\tau_{i+1}\ldots\tau_1(z)\\
&= \tau_{i+1}\tau_i\ldots\tau_1\tau_{i+1}(z)\\
&= \tau_{i+1}\ldots\tau_1(1\otimes\tau_i)(z)
\end{aligned}$$

where we used (17) in the 3. and 7. equations. □

Lemma 8.2. *For all $z\in P^{n+1}(-1,\zeta)$ and all $f:P^2\longrightarrow P$ we have*

$$(P^{i-1}\otimes f\otimes P^{n-i})\tau_{i-1}\ldots\tau_1(z)\in P^n(\zeta).$$

Proof. For all $\varphi\in B_n$ and all $k=1,\ldots,n$ we have

$$\begin{aligned}
&\tau_k^2\varphi\;(P^{i-1}\otimes f\otimes P^{n-i})\tau_{i-1}\ldots\tau_1(z)=\\
&\quad=\tau_k^2(P^{j-1}\otimes f\otimes P^{n-j})\varphi_{(i)}\tau_{j-1}\ldots\tau_1(z)\\
&\quad=(P^{j-1}\otimes f\otimes P^{n-j})\tau_{k(j)}^2\tau_{j-1}\ldots\tau_1(1\otimes\varphi)(z)\\
&\quad=(P^{j-1}\otimes f\otimes P^{n-j})\tau_{j-1}\ldots\tau_1(1\otimes\tau_k^2\varphi)(z)\\
&\quad=\varphi(P^{i-1}\otimes f\otimes P^{n-i})\tau_{i-1}\ldots\tau_1(z)
\end{aligned}$$

hence $(P^{i-1}\otimes f\otimes P^{n-i})\tau_{i-1}\ldots\tau_1(z)\in P^n(\zeta)$. □

Now we can give the

Proof. of Proposition 3.1:

We first show that $P^{n+1}(\zeta)\subseteq P\otimes(P^n(\zeta))\subseteq P^{n+1}$. Let $z=\sum_k z_{k,1}\otimes\ldots\otimes z_{k,n+1}$ be in $P^{n+1}(\zeta)$ with linearly independent $z_{k,1}$. Let $\varphi,\tau_i\in B_n$ be given. Define $1\otimes\varphi\in B_{n+1}$ resp. $1\otimes\tau_i\in B_{n+1}$ by the operation of φ resp. τ_i on the factors $z_{k,2}\otimes\ldots\otimes z_{k,n+1}$, e.g. $1\otimes\tau_i^{(n)}=\tau_{i+1}^{(n+1)}$. Then

$$\begin{aligned}
\textstyle\sum z_{k,1}\otimes\varphi^{-1}\tau_i^2\varphi(\sum z_{k,2}\otimes\ldots\otimes z_{k,n+1}) &= (1\otimes\varphi^{-1}\tau_i^2\varphi)(z)\\
&= \textstyle\sum_k z_{k,1}\otimes\zeta^2\sum z_{k,2}\otimes\ldots\otimes z_{k,n+1}.
\end{aligned}$$

Since the $z_{k,1}$ are linearly independent, the terms $\sum z_{k,2}\otimes\ldots\otimes z_{k,n+1}$ are in $P^n(\zeta)$ hence $z\in P\otimes P^n(\zeta)$.

Now we show that a factorization as given in the following diagram exists

$$\begin{tikzcd}
P^{n+1}(\zeta) \arrow[r, "\iota"] \arrow[d, "1\otimes[.,.]"] & P\otimes P^n(\zeta) \arrow[d, "1\otimes[.,.]"] \\
P^2(-1) \arrow[r, "\iota"] & P\otimes P.
\end{tikzcd}$$

The morphism $1\otimes[.,.] : P\otimes P^n(\zeta) \longrightarrow P\otimes P$ is in $\mathcal{C}$. Consider the braiding $\tau : P\otimes P^n(\zeta) \longrightarrow P^n(\zeta)\otimes P$. Since it is a natural transformation the diagram

$$\begin{tikzcd}
P\otimes P^n(\zeta) \arrow[r, "\tau"] \arrow[d] & P^n(\zeta)\otimes P \arrow[d] \\
P\otimes(P\otimes\ldots\otimes P) \arrow[r, "\varphi"] & (P\otimes\ldots\otimes P)\otimes P
\end{tikzcd}$$

commutes with $\varphi = \tau_n\ldots\tau_1 = \tau_{(P,P^n)}$, so $\tau(\sum_k z_{k,1}\otimes(\sum z_{k,2}\otimes\ldots\otimes z_{k,n+1})) = \tau_n\ldots\tau_1(\sum_k z_{k,1}\otimes z_{k,2}\otimes\ldots\otimes z_{k,n+1})$. Hence we get

$$\tau(1\otimes[.,.])(z) = ([.,.]\otimes 1)\tau_n\ldots\tau_1(z) \tag{18}$$

and similarly

$$\tau([.,.]\otimes 1)(z) = (1\otimes[.,.])\tau_1\ldots\tau_n(z) \tag{19}$$

for $z\in P^{n+1}(\zeta)$. This implies $\tau^2(1\otimes[.,.])(z) = \tau([.,.]\otimes 1)\tau_n\ldots\tau_1(z) = (1\otimes[.,.])\tau_1\ldots\tau_n\tau_n\ldots\tau_1(z) = (1\otimes[.,.])\zeta^{2n}(z) = (-1)^2(1\otimes[.,.])(z)$, so that $(1\otimes[.,.])(z)$ is in $P^2(-1)$ and thus $\sum[z_{k,1},[z_{k,2},\ldots,z_{k,n+1}]]$ is defined.

The second claim of the Proposition is proved in a symmetric way. $\square$

We continue with the

Proof. of Proposition 3.2:

We use (18), (19), and (16) to get

$$\begin{aligned}
\tau^2(1\otimes[.,.]_n)(z) &= \\
&= (1\otimes[.,.]_n)\tau_1\ldots\tau_n\tau_n\ldots\tau_1(z)\\
&= (1\otimes[.,.]_n)\tau_1\ldots\tau_{n-1}\tau_n^2\tau_{n-1}^{-1}\ldots\tau_1^{-1}\tau_1\ldots\tau_{n-1}\tau_{n-1}\ldots\tau_1(z)\\
&= (1\otimes[.,.]_n)\tau_n\ldots\tau_2\tau_1^2\tau_2^{-1}\ldots\tau_n^{-1}\tau_1\ldots\tau_{n-2}\tau_{n-1}^2\tau_{n-2}\ldots\tau_1(z)\\
&= (1\otimes[.,.]_n)(\tau_n\ldots\tau_2\tau_1^2\tau_2^{-1}\ldots\tau_n^{-1})(\tau_{n-1}\ldots\tau_2\tau_1^2\tau_2^{-1}\ldots\tau_{n-1}^{-1})\ldots(\tau_1^2)(z)\\
&= (1\otimes[.,.]_n)(z)
\end{aligned}$$

for all $z\in P^{n+1}(-1,\zeta)$ hence $(1\otimes[.,.]_n)(z)$ is in $P^2(-1)$ and $[.,[.,.]_n]_2(z)$ is defined.

Now we prove that $[.,[.,.]_2,.]_n\tau_{i-1}\dots\tau_1 : P^{n+1}(-1,\zeta)\ni x\otimes y_1\otimes\dots\otimes y_n\mapsto [y_1,\dots,[x,y_i],\dots,y_n]\in P$ is well defined. Let $z\in P^{n+1}(-1,\zeta)$. Then we have $\tau_1^2\tau_2^{-1}\dots\tau_i^{-1}(z)=\tau_2^{-1}\dots\tau_i^{-1}(z)$ since $\tau_2^{-1}\dots\tau_i^{-1}=1\otimes\tau_1^{-1}\dots\tau_{i-1}^{-1}$. If we represent $y=\tau_2^{-1}\dots\tau_i^{-1}(z)=\sum a_i\otimes b_i\in P^2\otimes P^{n-1}$ in shortest form, then the set $\{b_i\}$ is linearly independent, so $\sum a_i\otimes b_i=\tau_1^2(\sum a_i\otimes b_i)=\sum\tau_1^2(a_i)\otimes b_i$, hence $\tau_1^2(a_i)=a_i$ and $y\in P^2(-1)\otimes P^{n-1}$. So we get $\tau_{i-1}\dots\tau_1\tau_i\dots\tau_2(y)=\tau_{i-1}\dots\tau_1(z)\in P^{i-1}\otimes P^2(-1)\otimes P^{n-i}$ and $(1\otimes\dots\otimes[.,.]\otimes\dots\otimes 1)\tau_{i-1}\dots\tau_1(z)\in P^n$ is defined.

By Lemma 8.2 we have $(1\otimes\dots\otimes[.,.]_2\otimes\dots\otimes 1)(z)\in P^n(\zeta)$, so that $[.,[.,.]_2,.]_n\tau_{i-1}\dots\tau_1(z)$ is well defined. □

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