Complements and the Krull-Schmidt Theorem in Arbitrary Categories

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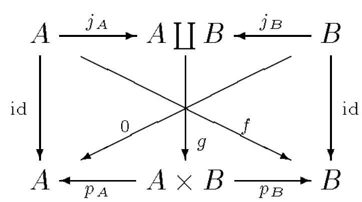
Abstract. We study direct product decompositions of objects in a finitely complete and cocomplete category with zero object and certain axioms for a coimage factorization of morphisms. Direct products $C = A \times B$ can be characterized by "inner" properties of C and its subobjects A and B. We also show that the Fitting Lemma and the Krull-Schmidt Theorem hold.

Key words: Complements, internal direct products, summable morphisms and idempotent morphisms, Fitting Lemma, Krull-Schmidt Theorem.

1 Introduction

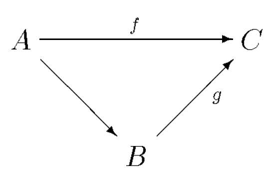
Let $\mathcal{C}$ be a category with finite limits, finite colimits, and a 0-object such that the following axioms are satisfied:

(I) for every morphism $f : A \longrightarrow B$, the induced (and uniquely determined) morphism g in the commutative diagram



is a difference cokernel;

(II) if, in the commutative diagram



f is a difference cokernel and g has kernel 0 then g is an isomorphism.

Axiom I guarantees that $A \times B$ is the "span" of the subobjects $g(A)$ and $g(B)$. In the category of sets $A \coprod B = A\dot{\cup}B$ is too small for this purpose, but in the presence of algebraic structures very often $A \coprod B$ will become large enough. If f is the zero morphism, then we shall use the notation $A \cup B = A \times B$ to denote that $A \times B$ is spanned by A and B.

As we shall see later on axiom I has the consequence that certain morphisms in $\mathcal{C}$ can be added. Axiom II means that $\mathcal{C}$ is in some sense balanced (cf. a mono- and epimorphism is an isomorphism).

Two fundamental examples are **Gr** the category of groups and **Mo** the category of monoids. **Gr** satisfies (I) and (II) as will be seen soon in more general contexts, whereas **Mo** does not satisfy any of (I) or (II). A counterexample for (I) is $A = B = (N_0, +)$ and $f = \mathrm{id}$. It is easy to see that $(1, 0) \notin \mathrm{Im}(g)$, and g is surjective if and only if g is a difference cokernel. To give a counterexample for (II) adjoin an additional element ∞ to $(N_0, +)$ such that $n+\infty = \infty+n = n+1$ and $\infty+\infty = 2$. Then $N_0\cup\{\infty\}$ is a monoid and $f : N_0\cup\{\infty\} \longrightarrow N_0$ with $f(n) = n$ and $f(\infty) = 1$ is surjective (a difference cokernel) with kernel 0. But f is not bijective.

To find a larger class of categories $\mathcal{C}$ which satisfy the hypotheses, let us consider an (equationally defined) algebraic category $\mathcal{C}$. The final object E is always the set with one element with the unique algebra structure on it. In order to be a 0-object in $\mathcal{C}$ it is necessary and sufficient that, for any algebra A in $\mathcal{C}$, there is a unique algebra morphism $E \longrightarrow A$; this means that there must be precisely one 0-ary operation in the theory for $\mathcal{C}$ (and it must be compatible with all other operations on A in the obvious sense). Let us call the distinguished element 0 for every algebra A. Assume that there is an m-ary operation $\omega : A \times \ldots \times A \longrightarrow A$ for some $m \geq 2$ and $i \leq m$, such that for every algebra A

1. $\forall a \in A : \omega(0, \ldots, a, \ldots, 0) = a$ (a is in the i-th place),
2. $\forall a, a' \in A \exists a_1, \ldots, a_m \in A : \omega(a_1, \ldots, a', \ldots, a_m) = a$ (a' is in the i-th place).

Then axiom I is satisfied. To prove this let $f : A \longrightarrow B$ be an algebra morphism and let $(a, b) \in A\times B$ be given. By definition of g we have $gj_A(a) = (a, f(a)) \in A\times B$ and $gj_B(b) = (0, b) \in A \times B$. Pick $b_1, \ldots, b_m$ such that $\omega(b_1, \ldots, f(a), \ldots, b_m) = b$, then $g(\omega(j_B(b_1), \ldots, j_A(a), \ldots, j_B(b_m))) = \omega((0, b_1), \ldots, (a, f(a)), \ldots, (0, b_m)) = (\omega_A(0, \ldots, a, \ldots, 0), \omega_B(b_1, \ldots, f(a), \ldots, b_m)) = (a, b)$. Thus g is surjective which in an algebraic category means the same as g is a difference cokernel ([2], 3.4 Cor. 4).

If there is a binary operation ν such that $\nu(a, b) = 0$ if and only if $a = b$, then $\mathcal{C}$ also satisfies (II). To prove this let $g : B \longrightarrow C$ have kernel zero then $g(b) = g(b') \Rightarrow 0 = \nu(g(b), g(b')) = g(\nu(b, b')) \Rightarrow \nu(b, b') = 0 \Rightarrow b = b'$, hence g is injective. However, in an algebraic category a bijective morphism is an isomorphism.

In particular all algebraic categories where the objects have an underlying group structure and no further distinguished elements satisfy our conditions for $\mathcal{C}$, e.g. rings (without unit elements), associative rings, Lie rings etc. The binary operation

ν can be chosen to be $\nu(a,b) = a - b$. Similarly the category of loops ("non-associative" groups) satisfy the conditions for $\mathcal{C}$.

We will show that certain generalizations of the Fitting Lemma and the Krull-Schmidt Theorem hold in our categories. Instead of considering congruence relations as in [3] or a modular lattice of subobjects as in [1] we use specific subobjects together with certain endomorphisms to prove these theorems.

2 Complements and internal direct products

Let $\mathcal{C}$ satisfy the axioms discussed in section 1. We are interested in the question, whether inside an object $A \times B$ in $\mathcal{C}$ there can be other "subobjects" X of $A \times B$ such that $X \times B = A \times B$. For this we have to make precise what the equality means and how A and B are subobjects of $A \times B$. A subobject will be used in the sense of [2], that is as a representative of the usual equivalence class of monomorphisms.

In the following definition $p_A : A \times B \longrightarrow A$ (resp. $p_B : A \times B \longrightarrow B$) denotes the canonical projection.

Definition 2.1 1. A subobject $\iota_X : X \hookrightarrow A \times B$ is called a *weak complement of* B, if the morphism $X \xrightarrow{\iota_X} A \times B \xrightarrow{p_A} A$ has kernel $0 \longrightarrow X$.

2. A subobject $\iota_X : X \hookrightarrow A \times B$ is called a *complement of* B, if the morphism $X \xrightarrow{\iota_X} A \times B \xrightarrow{p_A} A$ is an isomorphism.

3. U is called *internal direct product* of the subobjects $A \hookrightarrow U$ and $B \hookrightarrow U$ if

   a) $A \hookrightarrow U$ and $B \hookrightarrow U$ are kernels,

   b) the intersection of A and B is $A \cap B = 0$,

   c) the canonical morphism $A \coprod B \longrightarrow U$ is a difference cokernel (a fact which we abbreviate by $A \cup B = U$).

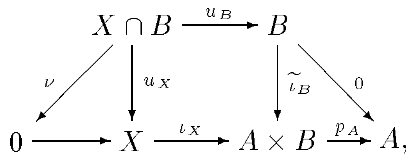
Let A and B be objects in $\mathcal{C}$. We consider B as a subobject of $A \times B$ via the canonical morphism $\widetilde{\iota}_B : B \longrightarrow A \times B$ induced by $\mathrm{id} : B \longrightarrow B$ and $0 : B \longrightarrow A$.

Lemma 2.2 $B \xrightarrow{\widetilde{\iota}_B} A \times B \xrightarrow{p_A} A$ *is a kernel diagram.*

Proof: If $g : X \longrightarrow A \times B$ is given with $p_A g = 0$ then $p_A g = p_A \widetilde{\iota}_B p_B g = 0$ and $p_B g = p_B \widetilde{\iota}_B p_B g$ implies $g = \widetilde{\iota}_B p_B g$, a factorization of g through $\widetilde{\iota}_B$. Since $\widetilde{\iota}_B$ is a section this factorization is unique. □

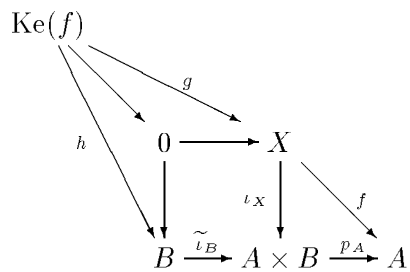
Proposition 2.3 $\iota_X : X \hookrightarrow A \times B$ *is a weak complement of* B *if and only if* $X \cap B = 0$.

Proof: Let X be a weak complement of B. In the commutative diagram



with u_B and u_X the canonical morphisms, we have $p_A \iota_X u_X = 0$ hence there is a factorization ν of u_X through 0 by the property of weak complements. Since u_X is a zero-morphism and a monomorphism we get $X \cap B = 0$.

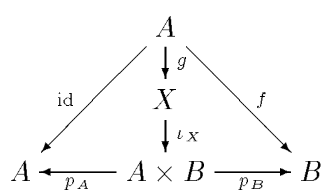
Let $X \cap B = 0$. In the commutative diagram



$B \xrightarrow{\widetilde{\iota}_B} A \times B \xrightarrow{p_A} A$ is a kernel diagram by Lemma 2.2. Hence h can be constructed uniquely from g such that $\widetilde{\iota}_B h = \iota_X g$. Now g can be factored through 0 and thus must be zero. This means that X is a weak complement of B. □

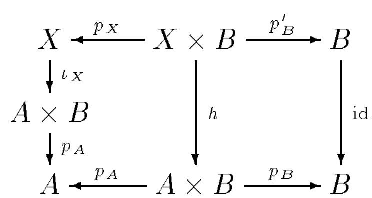
Proposition 2.4 *Let $\iota_X : X \hookrightarrow A \times B$ be a subobject. The following are equivalent:*

1. *X is a complement of B.*
2. *There is a unique morphism $f : A \longrightarrow B$ and an epimorphism $g : A \longrightarrow X$ such that the diagram*



   commutes.

3. *The induced morphism h in the commutative diagram*

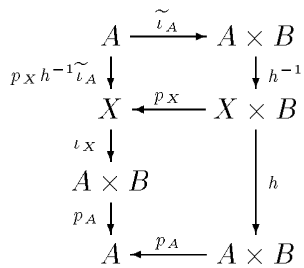


is an isomorphism.

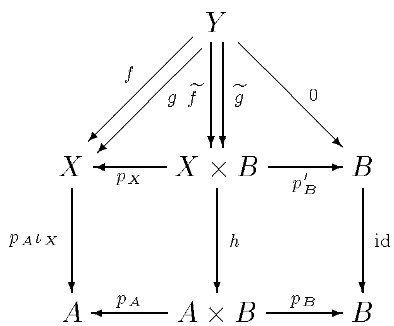
Proof: (1) $\Leftrightarrow$ (2): $p_A \iota_X$ is an isomorphism if and only if there is an epimorphism g with $(p_A \iota_X) g = \mathrm{id}$. Thus f can be constructed uniquely such that $f = p_B \iota_X g$.

(1) $\Rightarrow$ (3): If $p_A \iota_X$ is an isomorphism then, obviously, h is an isomorphism.

(3) $\Rightarrow$ (1): Let h be an isomorphism. In the commutative diagram



we have $p_A \widetilde{\iota}_A = \mathrm{id}$. Hence $(A \longrightarrow X \longrightarrow A) = \mathrm{id}$ in the above diagram. Thus it suffices to show that $p_A \iota_X$ is a monomorphism. Suppose that $p_A \iota_X f = p_A \iota_X g$. Then we have the commutative diagram



Define $\widetilde{f}$ and $\widetilde{g}$ by the universal property of the product $X \times B$. Then $p_A h \widetilde{f} = p_A h \widetilde{g}$ and $p_B h \widetilde{f} = 0 = p_B h \widetilde{g}$ and hence $h \widetilde{f} = h \widetilde{g}$ and $\widetilde{f} = \widetilde{g}$ since h is an isomorphism. So $f = g$ and $p_A \iota_X$ is a monomorphism. □

Remark 2.5 *1) In some sense 2.4, (3), means that $A \times B$ is generated by the subobjects X and B. Observe, however, that there is also a canonical morphism $X \coprod B \longrightarrow A \times B$ which, in general, does not factor through $X \times B$ in the canonical way (e.g.* ***Gr****) nor is it an epimorphism (e.g. commutative monoids).*

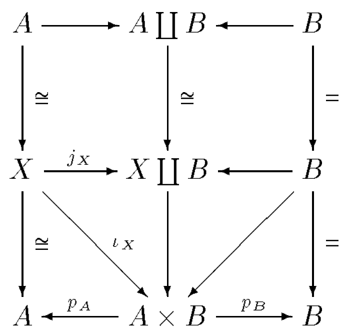
2) In 2.4, (2), one can consider X as the graph of the morphism $f : A \longrightarrow B$. So this part of Proposition 2.4 may be rephrased as:

there is a bijection between the complements of B in $A \times B$ and the morphisms $f : A \longrightarrow B$.

To show that each f determines a subobject X of $A \times B$, let $\widetilde{f} : A \longrightarrow A \times B$ be the morphism with $p_A \widetilde{f} = \mathrm{id}$ and $p_B \widetilde{f} = f$. Then $(A, \widetilde{f})$ is a subobject of $A \times B$, namely the graph of f.

Proposition 2.6 *Let $\iota_X : X \longrightarrow A \times B$ be a complement of B. Then we have $X \cap B = 0$ and $X \cup B = A \times B$.*

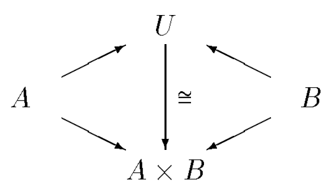
Proof: Since a complement is a weak complement we get $X \cap B = 0$ by Proposition 2.3. In the commutative diagram



the morphism $A \coprod B \longrightarrow A \times B$ is a difference cokernel by axiom (I) and so is $X \coprod B \longrightarrow A \times B$. Hence $X \cup B = A \times B$. □

Theorem 2.7 *1. $A \times B$ is an internal direct product of the subobjects A and B.*

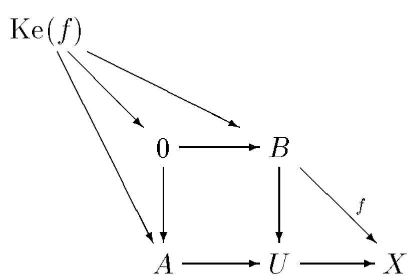
2. If A and B are subobjects of U such that U is an internal direct product of A and B, then there is an isomorphism $U \cong A \times B$ such that



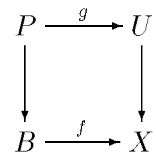
commutes.

Proof: (1) Since $p_A \tilde{\iota}_A = \mathrm{id}$ is an isomorphism, A is a complement of B in $A \times B$. Thus by Lemma 2.2 and Proposition 2.6 we get that $A \times B$ is an internal direct product of A and B.

(2) Given an internal direct product U of A and B. Let $U \longrightarrow X$ be the cokernel of $A \longrightarrow U$. Then $A \longrightarrow U$ is the kernel of $U \longrightarrow X$ since it was a kernel. In the commutative diagram

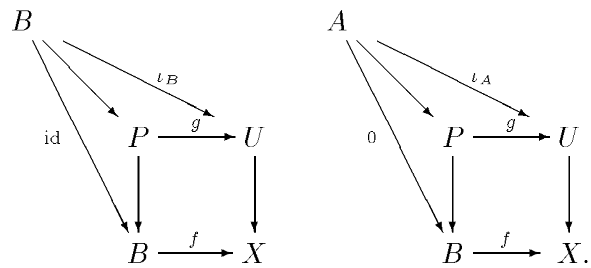


the morphism $\mathrm{Ke}(f) \longrightarrow A$ and $\mathrm{Ke}(f) \longrightarrow 0$ exist since $A \longrightarrow U$ is the kernel of $U \longrightarrow X$ and $0 = A \cap B$ is a pull-back. Hence $\mathrm{Ke}(f) = 0$. Now let

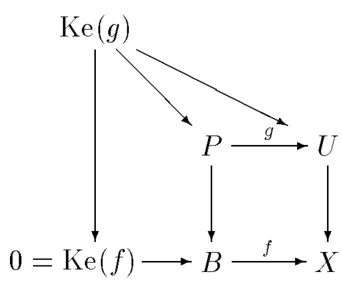


be a pull-back and consider the commutative diagrams

(*)



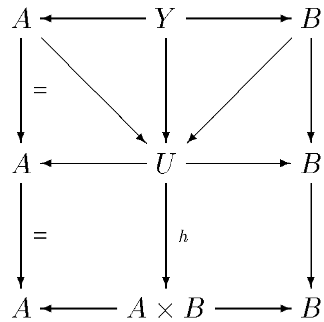
They induce the canonical morphism $A \coprod B \longrightarrow P \longrightarrow U$, which is a difference cokernel. Since $\mathrm{Ke}(f) = 0$ the diagram



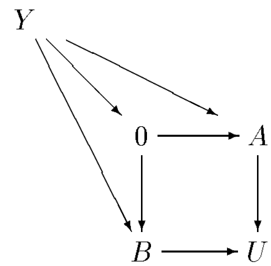
commutes and the canonical morphism $\mathrm{Ke}(g) \longrightarrow P$ is the zero morphism. By axiom (II) g is an isomorphism. Thus we can replace P by U and g by id so that there is a factorization $U \longrightarrow B \xrightarrow{f} X$ of $U \longrightarrow X$, which is the cokernel of $A \longrightarrow U$.

Since $(A \longrightarrow U \longrightarrow X) = 0$ and $\mathrm{Ke}(f) = 0$ we get $(A \longrightarrow U \longrightarrow B) = 0$ and hence a factorization $(U \longrightarrow X \longrightarrow B) = (U \longrightarrow B)$. Since $U \longrightarrow X$ is the cokernel of $A \longrightarrow U$ we get $(X \longrightarrow B \xrightarrow{f} X) = \mathrm{id}$. Now f has kernel zero and is a difference cokernel of $(B \xrightarrow{f} X \longrightarrow B, B \xrightarrow{\mathrm{id}} B)$. Hence by axiom (II) we obtain that $f : B \longrightarrow X$ is an isomorphism. Therefore we may replace X by B and consider $U \longrightarrow B$ as cokernel of $A \longrightarrow U$. Furthermore we have $(B \longrightarrow U \longrightarrow B) = \mathrm{id}$ by diagram $(*)$. Analogously we get a morphism $U \longrightarrow A$, which is a cokernel of $B \longrightarrow U$, such that $(A \longrightarrow U \longrightarrow A) = \mathrm{id}$.

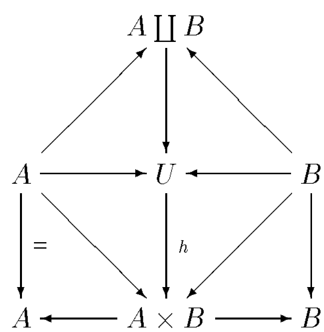
Now we prove $U \cong A \times B$. Let Y be the kernel of h in the following commutative diagram



To prove the existence of $Y \longrightarrow A$ observe that $A \longrightarrow U$ is the kernel of $U \longrightarrow B$ and that $(Y \longrightarrow U \longrightarrow B) = (Y \longrightarrow U \longrightarrow A \times B \longrightarrow B) = 0$. Thus we get a commutative diagram



as $A \cap B = 0$. This shows that $(Y \longrightarrow U) = 0$. Thus $h : U \longrightarrow A \times B$ has kernel 0. On the other hand we have a commutative diagram

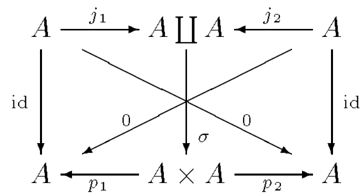


which implies by axiom (I) that $A \coprod B \longrightarrow U \longrightarrow A \times B$ is a difference cokernel. By axiom (II) h is an isomorphism and the above diagram proves that the diagram in the theorem commutes. □

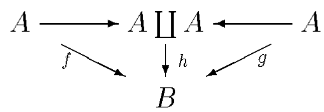
3 Summable morphisms

In this paragraph we shall introduce an addition of certain morphisms. One of the aims is the proof of a formula $\mathrm{id} = \tilde{\iota}_A p_A + \tilde{\iota}_B p_B$ for $A \times B$. Let $\mathcal{C}$ be as in sections 1 and 2.

Definition 3.1 Let $\sigma : A \coprod A \longrightarrow A \times A$ be the canonical morphism defined by the commutative diagram



By axiom (I) σ is a difference cokernel. Let $f, g \in \mathcal{C}(A,B)$. f and g are called *summable* if the canonical morphism h of



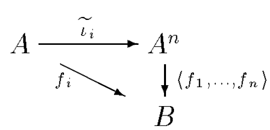
factors (necessarily uniquely) through σ. The factorization morphism $A \times A \longrightarrow B$ will be written as $\langle f, g\rangle$.

A similar definition can be given for n morphisms. The family $(f_i | i = 1, \ldots, n)$ is *summable* if for all i, j with $1 \leq i < j \leq n$ the morphism induced by the $f_i, f_{i+1}, \ldots, f_j$ factors through $\coprod_{k=i}^{j} A \longrightarrow \prod_{k=i}^{j} A$. The factorization is denoted by $\langle f_i, f_{i+1}, \ldots, f_j\rangle : A \times \ldots \times A \longrightarrow B$.

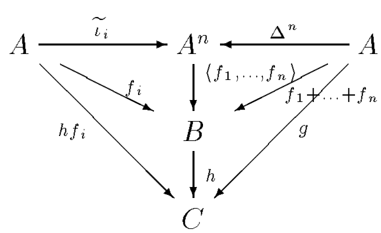
If $(f_i | i = 1, \ldots, n)$ is summable then the *sum* $f_1 + \ldots + f_n$ is defined as the morphism $A \stackrel{\Delta^n}{\longrightarrow} A \times \ldots \times A \stackrel{\langle f_1, \ldots, f_n\rangle}{\longrightarrow} B$.

Lemma 3.2 *Let $f_i \in \mathcal{C}(A,B)$, $i = 1, \ldots, n$, $h \in \mathcal{C}(B,C)$ and $k \in \mathcal{C}(D,A)$. Let $(f_i | i = 1, \ldots, n)$ be summable. Then $(f_i k | i = 1, \ldots, n)$ and $(hf_i | i = 1, \ldots, n)$ are summable and we have $h(f_1 + \ldots + f_n) = hf_1 + \ldots + hf_n$ and $(f_1 + \ldots + f_n)k = f_1 k + \ldots + f_n k$.*

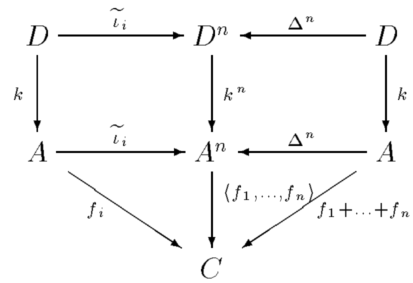
Proof: It is sufficient to prove that factorizations $\langle hf_1, \ldots, hf_n\rangle$ resp. $\langle f_1 k, \ldots, f_n k\rangle$ exist. Observe that the factorization $\langle f_1, \ldots, f_n\rangle$ is the only morphism which makes all the diagrams



commute. Thus the diagrams



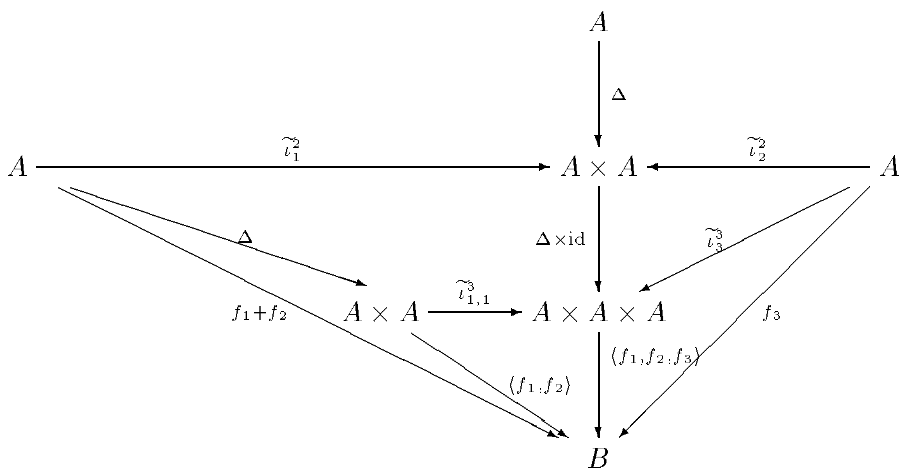
commute with $\langle hf_1, \ldots, hf_n\rangle = h\langle f_1, \ldots, f_n\rangle$ and $hf_1 + \ldots + hf_n = g = h(f_1 + \ldots + f_n)$. The second part follows from the commutative diagrams



by $\langle f_1, \ldots, f_n\rangle k^n = \langle f_1k, \ldots, f_nk\rangle$ and $(f_1 + \ldots + f_n)k = f_1k + \ldots + f_nk$. □

Lemma 3.3 *Sums of summable morphisms satisfy the associative law. In particular if* $(f_1, f_2, f_3) \in \mathcal{C}(A, B)^3$ *is summable then* $(f_1 + f_2) + f_3 = f_1 + f_2 + f_3 = f_1 + (f_2 + f_3)$.

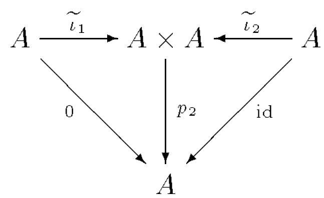
Proof: We prove only the second statement. The first follows by standard reasoning. Consider the following commutative diagram



where the composite vertical morphism denotes $(f_1 + f_2) + f_3$ (and also $f_1 + f_2 + f_3$ in case the diagram commutes). The morphisms $\widetilde{\iota}^2$ are morphisms into A^2 and $\widetilde{\iota}^3$ into A^3. The only commutativity which is not immediately clear is $\langle f_1, f_2, f_3\rangle\widetilde{\iota}_{1,1}^3 = \langle f_1, f_2\rangle$, but we have $\langle f_1, f_2, f_3\rangle\widetilde{\iota}_{1,1}^3\widetilde{\iota}_1^2 = \langle f_1, f_2, f_3\rangle\widetilde{\iota}_1^3 = f_1 = \langle f_1, f_2\rangle\widetilde{\iota}_1^2$ and similarly $\langle f_1, f_2, f_3\rangle\widetilde{\iota}_{1,1}^3\widetilde{\iota}_2^2 = \langle f_1, f_2\rangle\widetilde{\iota}_2^2$. By the uniqueness of the factorization $\langle f_1, f_2\rangle$ we get the required commutativity. □

Lemma 3.4 *For each morphism* $f \in \mathcal{C}(A, B)$ *the morphisms* 0 *and* f *are summable and we have* $0 + f = f = f + 0$.

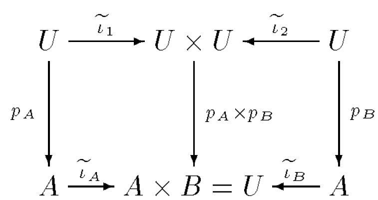
Proof: It is sufficient to prove $0+\mathrm{id} = \mathrm{id}$ for then $0+f = 0f+\mathrm{id}\, f = (0+\mathrm{id})f = \mathrm{id}\, f = f$ by Lemma 3.2. But the factorization $\langle 0, \mathrm{id}\rangle$ is $p_2 : A \times A \longrightarrow A$ since



commutes by definition of $\widetilde{\iota}_1$ and $\widetilde{\iota}_2$. Hence $0 + \mathrm{id} = \langle 0, \mathrm{id}\rangle\Delta = p_2\Delta = \mathrm{id}$. □

Proposition 3.5 *Let* $U = A_1 \times \ldots \times A_n$. *Then* $\mathrm{id}_U = \widetilde{\iota}_1 p_1 + \ldots + \widetilde{\iota}_n p_n$.

Proof: To begin the proof by induction assume $U = A \times B$. The diagram



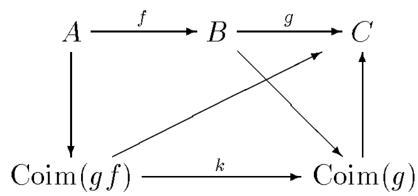
commutes, whence $p_A \times p_B = \langle \widetilde{\iota}_A p_A, \widetilde{\iota}_B p_B\rangle$ and $\mathrm{id}_U = (p_A \times p_B)\Delta = \widetilde{\iota}_A p_A + \widetilde{\iota}_B p_B$. To indicate the induction step assume $V = A \times B \times C = U \times C$. Then $\mathrm{id}_V = \widetilde{\iota}_U p_U + \widetilde{\iota}_C p_C = \widetilde{\iota}_U \,\mathrm{id}_U\, p_U + \widetilde{\iota}_C p_C = \widetilde{\iota}_U(\widetilde{\iota}^U_A p^U_A + \widetilde{\iota}^U_B p^U_B)p_U + \widetilde{\iota}_C p_C = \widetilde{\iota}_U \widetilde{\iota}^U_A p^U_A p_U + \widetilde{\iota}_U \widetilde{\iota}^U_B p^U_B p_U + \widetilde{\iota}_C p_C = \widetilde{\iota}_A p_A + \widetilde{\iota}_B p_B + \widetilde{\iota}_C p_C$. □

Example 3.6 *We want to prove in the case* $\mathcal{C} = \boldsymbol{Gr}$, *the category of groups, that two morphisms* $f, g : A \longrightarrow B$ *are summable if and only if* $f(x)g(y) = g(y)f(x)$ *for all* $x, y \in A$. *Given* $\langle f, g\rangle$ *we have* $\langle f, g\rangle(x, y) = \langle f, g\rangle((x, e) \cdot (e, y)) = \langle f, g\rangle(x, e) \cdot \langle f, g\rangle(e, y) = f(x) \cdot g(y)$. *Since* (x, e) *and* (e, y) *commute we get* $f(x)g(y) = g(y)f(x)$. *Conversely it is a well-known exercise that this condition implies that* $\langle f, g\rangle$ *is a homomorphism. The sum* $f + g$ *is then defined by* $(f + g)(x) = f(x)g(x)$.

4 Idempotent morphisms and the Fitting Lemma

First we need some facts about coimages. The coimage of $f : A \longrightarrow B$ is defined as the difference cokernel of the kernel pair of f ([2] p.70, Lemma 4a)). In the canonical factorization $A \xrightarrow{f'} \mathrm{Coim}(f) \xrightarrow{\iota} B$ of f, ι fails to be a monomorphism in general.

Lemma 4.1 *Given* $f : A \longrightarrow B$ *and* $g : B \longrightarrow C$, *then there is a unique morphism* $k : \mathrm{Coim}(gf) \longrightarrow \mathrm{Coim}(g)$ *such that*

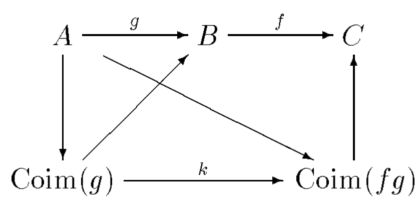


commutes. If f *is an isomorphism then so is* k.

Proof: This is an easy exercise in universal properties of difference cokernels and kernel pairs. □

One proves just as easily

Lemma 4.2 *Given* $g : A \longrightarrow B$ *and* $f : B \longrightarrow C$, *there is a unique morphism* $k : \mathrm{Coim}(g) \longrightarrow \mathrm{Coim}(fg)$ *such that*



commutes. If f *is an isomorphism then so is* k.

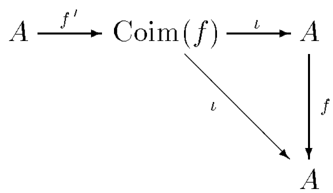
Definition 4.3 Let $f : A \longrightarrow A$ be idempotent. We say that f satisfies condition (G_f) if the canonical morphism $\mathrm{Ke}(f) \coprod \mathrm{Coim}(f) \longrightarrow A$ is a difference cokernel and $\mathrm{Coim}(f) \longrightarrow A$ is a kernel.

In the category of groups **Gr** the first condition is always satisfied, indeed $a \mapsto af(a^{-1}) * f(a) \in \mathrm{Ke}(f) \coprod \mathrm{Coim}(f)$ is a section for the given morphism. The second condition is in **Gr** equivalent to f being normal (see 4.7). In **Mo**, the category of monoids, let $M = (Z/3Z)^\times$ be the multiplicative monoid of $Z/3Z$ and $f : M \longrightarrow M$ be given by $f(\overline{0}) = \overline{0}$, $f(\overline{1}) = f(\overline{2}) = \overline{1}$. Then f is idempotent and $\mathrm{Ke}(f) \coprod \mathrm{Coim}(f) \longrightarrow M$ is surjective. But $\mathrm{Coim}(f) \longrightarrow M$ fails to be a kernel. To see this observe that $\mathrm{Coim}(f) = \{\overline{0}, \overline{1}\}$ and the kokernel of $\mathrm{Coim}(f) \longrightarrow M$ is $M \longrightarrow \{\overline{1}\}$. The kernel of this morphism is $\mathrm{id} : M \longrightarrow M$, but not $\mathrm{Coim}(f) \longrightarrow M$.

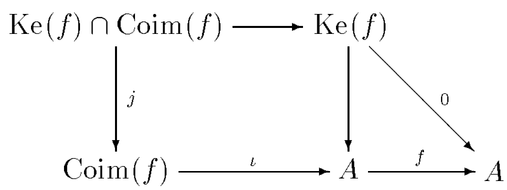
Now let $\mathcal{C}$ be again as in section 1.

Lemma 4.4 *Let* $f : A \longrightarrow A$ *be idempotent with* (G_f). *Then* $A = \mathrm{Ke}(f) \times \mathrm{Coim}(f)$.

Proof: $\mathrm{Ke}(f) \longrightarrow A$ and $\mathrm{Coim}(f) \longrightarrow A$ are kernels. Furthermore (G_f) implies $A = \mathrm{Ke}(f) \cup \mathrm{Coim}(f)$. Next we show $\mathrm{Ke}(f) \cap \mathrm{Coim}(f) = 0$. In the diagram



we have $\iota f' = f = f^2 = f\iota f'$ hence $\iota = f\iota$, since f' is an epimorphism. In the commutative diagram



we have $\iota j = f\iota j = 0$ hence $\mathrm{Ke}(f) \cap \mathrm{Coim}(f) = 0$. Hence Theorem 2.7 finishes the proof. $\square$

Let $f : A \longrightarrow A$ be an endomorphism. Then f has a factorization $A \xrightarrow{f'} \mathrm{Coim}(f) \xrightarrow{\iota} A$. By Lemma 4.1 there is also a canonical morphism $\iota_n : \mathrm{Coim}(f^n) \longrightarrow \mathrm{Coim}(f^{n-1})$ and by Lemma 4.2 there is a canonical morphism $f'_n : \mathrm{Coim}(f^{n-1}) \longrightarrow \mathrm{Coim}(f^n)$.

Definition 4.5 An endomorphism $f : A \longrightarrow A$ is called *bounded*, if the families $(\iota_n : \mathrm{Coim}(f^n) \longrightarrow \mathrm{Coim}(f^{n-1}))$ and $(f'_n : \mathrm{Coim}(f^{n-1}) \longrightarrow \mathrm{Coim}(f^n))$ become stationary (i.e. there is n_0 such that for all $n \geq n_0$ both ι_n and f'_n are isomorphisms), and if for each n there is $r \geq n$ such that $\mathrm{Ke}(f^r) \coprod \mathrm{Coim}(f^r) \longrightarrow A$ is a difference cokernel and $\mathrm{Coim}(f^r) \longrightarrow A$ is a kernel.

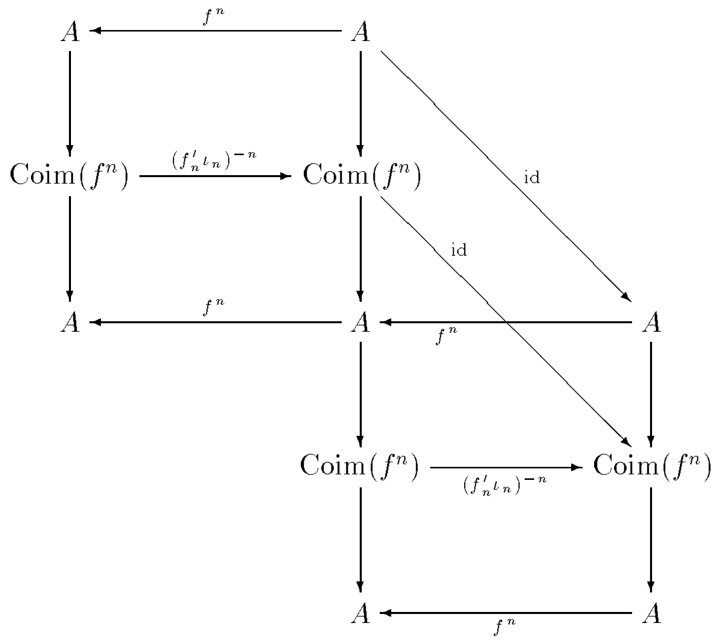
Proposition 4.6 (Fitting Lemma) *Let $f : A \longrightarrow A$ be bounded. Then for every $n_0 \in N$ there is an $n \geq n_0$ such that*

$$A = \mathrm{Ke}(f^n) \times \mathrm{Coim}(f^n).$$

Proof: The chains (ι_n) and (f'_n) become stationary for all $n \geq n_0$. In particular we have an inverse $(f'_n \iota_n)^{-n}$ of $(f'_n \iota_n)^n$. The morphism

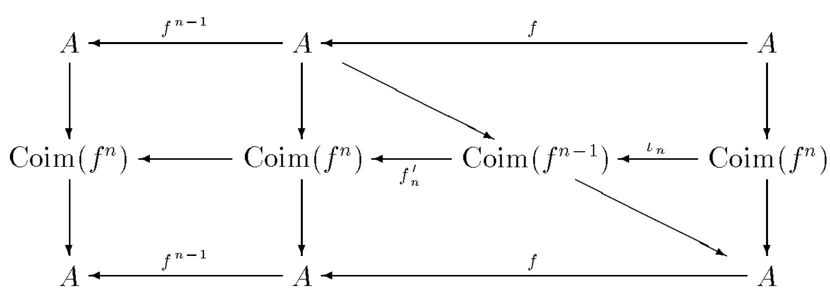
$$\varphi : A \longrightarrow \mathrm{Coim}(f^n) \xrightarrow{(f'_n\iota_n)^{-n}} \mathrm{Coim}(f^n) \longrightarrow A$$

is idempotent which follows from the commutative diagram



in which $\varphi^2 : A \longrightarrow A$ is easily identified. The diagonal $\mathrm{Coim}(f^n) \longrightarrow \mathrm{Coim}(f^n)$ is the identity and gives a commutative lower triangle since $A \longrightarrow \mathrm{Coim}(f^n)$ is an epimorphism. Thus $A \longrightarrow \mathrm{Coim}(f^n) \longrightarrow \mathrm{Coim}(f^n) \xrightarrow{\mathrm{id}} \mathrm{Coim}(f^n) \longrightarrow A$ in the diagram is φ.

The only remaining problem is the commutativity of the rectangles. It follows from the diagram



where the left part is an $(n-1)$-fold repetition of the right part and the right part commutes by Lemma 4.1 and Lemma 4.2 with $g = f^{n-1}$.

For suitably large n we have also $\mathrm{Coim}(f^n) \longrightarrow A$ a kernel. Define $\varphi' := (A \longrightarrow \mathrm{Coim}(f^n) \xrightarrow{(f_n'\iota_n)^{-n}} \mathrm{Coim}(f^n))$. Since $(f_n'\iota_n)^{-n}$ is an isomorphism, we have that $\varphi' : A \longrightarrow \mathrm{Coim}(f^n)$ is a coimage of f^n as well as of $\varphi = (A \xrightarrow{\varphi'} \mathrm{Coim}(f^n) \longrightarrow A)$. Thus $\mathrm{Coim}(\varphi) \longrightarrow A$ is a kernel and $\mathrm{Ke}(\varphi) = \mathrm{Ke}(A \longrightarrow \mathrm{Coim}(\varphi)) = \mathrm{Ke}(A \longrightarrow$

$\mathrm{Coim}(f^n)) = \mathrm{Ke}(f^n)$. Since f is bounded, φ satisfies (G_φ) and Lemma 4.4 holds for φ. Translated back into terms of f gives the required result. □

Example 4.7 *Consider the category of groups* $\boldsymbol{Gr}$. *A morphism* $f : G \longrightarrow G$ *is called normal if* $f(aba^{-1}) = af(b)a^{-1}$ *for all* $a, b \in G$. *If* f *is normal and* G *has a.c.c. and d.c.c. then for all* n *we have* $f^n(aba^{-1}) = af^n(b)a^{-1}$ *and thus* $\mathrm{Coim}(f^n) = \mathrm{Im}(f^n)$ *normal in* G. *The chains of subgroups* $\mathrm{Ke}(f^n) \subseteq \mathrm{Ke}(f^{n+1})$ *and* $\mathrm{Im}(f^n) \supseteq \mathrm{Im}(f^{n+1})$ *become stationary. Finally* $a \mapsto a(f'_n\iota_n)^{-1}f^n(a^{-1}) * (f'_n\iota_n)^{-1}f^n(a)$ *is a section for the canonical map* $\mathrm{Ke}(f^n)\coprod\mathrm{Im}(f^n) \longrightarrow G$. *Thus* f *is bounded and the Fitting Lemma holds.*

5 The Krull-Schmidt-Theorem

Definition 5.1 Let $A \neq 0$ be in $\mathcal{C}$. We call A *indecomposable* if $A = X \times Y$ implies $X = 0$ or $Y = 0$.

Lemma 5.2 *Let* A *be indecomposable and* $f : A \longrightarrow A$ *a bounded endomorphism. Then* f *is either nilpotent or an automorphism.*

Proof: By the Fitting Lemma there is an $n \in N$ such that $A = \mathrm{Ke}(f^n) \times \mathrm{Coim}(f^n)$. If $\mathrm{Coim}(f^n) = 0$ then $f^n = 0$ and f is nilpotent. If $\mathrm{Ke}(f^n) = 0$ then $\mathrm{Coim}(f^n) \longrightarrow A$ must be the identity. Hence $f^n : A \longrightarrow A$ has kernel zero and is a difference cokernel. By axiom (II) for $\mathcal{C}$ we get that f^n and also f are automorphisms. □

Definition 5.3 Given $A = B \times C$. We define the subset $X \subseteq \mathrm{End}(B)$ of *A-productive* endomorphisms as follows:

1. If $A = B' \times C'$ then $(B \longrightarrow A \longrightarrow B' \longrightarrow A \longrightarrow B) \in X$.
2. If $f, g \in X$, then $fg \in X$.
3. If $f \in X \cap \mathrm{Aut}(B)$, then $f^{-1} \in X$.
4. If $f, g \in X$ are summable then $f + g \in X$.
5. $X =$ the smallest set satisfying $(1), \ldots, (4)$.

Definition 5.4 A is *bounded* if for all $A = B \times C$ all A-productive endomorphisms f are bounded.

Lemma 5.5 *Let* B *be indecomposable and* $A = B \times C$ *be bounded. Let* f, g *be* A*-productive and summable. If* f *and* g *are nilpotent, then so is* $f + g$.

Proof: Without loss of generality we assume $f \neq 0 \neq g$. Since $f+g$ is A-productive it is bounded. By Lemma 5.2 $f+g$ is either nilpotent or an automorphism. Assume $f+g \in \mathrm{Aut}(B)$. Then $h=(f+g)^{-1}$ is A-productive and so are hf and hg. Thus $\mathrm{id} = h(f+g) = hf+hg$ with hf and hg A-productive, hence bounded. Since f is nilpotent, there is an n such that $f^n \neq 0$ and $f \cdot f^n = 0$. So f^n factors through $\mathrm{Ke}(f) \longrightarrow B$ and we get $\mathrm{Ke}(f) \neq 0$. This implies that $\mathrm{Ke}(hf) \neq 0$ and thus hf is not an automorphism. By Lemma 5.2 hf is then nilpotent. Replacing f by hf and g by hg we can assume without loss of generality that f and g are summable, A-productive and nilpotent with $\mathrm{id} = f+g$.

Let $f^n = 0 = g^n$. We prove by induction on k that $f^{t_1} g f^{t_2} g \ldots g f^{t_k} = 0$ for $t_i \geq 0$ and $\sum_{i=1}^{k} t_i = n$. For $k=1$ this is trivial. Using Lemma 3.2 and Lemma 3.4 the induction step is $f^{t_1} g \ldots g f^{t_{k+1}} = f^{t_1} g \ldots g f^{t_k} f f^{t_{k+1}} + f^{t_1} g \ldots g f^{t_k} g f^{t_{k+1}} = f^{t_1} g \ldots g f^{t_k} (f+g) f^{t_{k+1}} = f^{t_1} g \ldots g f^{t_k} f^{t_{k+1}} = 0$. With this remark we get $(f+g)^{2n} = 0$ since in the expansion each summand contains at least n factors f or n factors g, so it is zero. Thus $f+g = \mathrm{id}$ cannot hold with $B \neq 0$. □

Corollary 5.6 *Let $A = B \times C$ be bounded and B be indecomposable. Let $f_1, \ldots, f_n \in \mathrm{End}(B)$ be summable and A-productive. If $f_1 + \ldots + f_n = \mathrm{id}$ then one of the f_i is an automorphism.*

Proof: If all f_i are nilpotent then a simple induction proof shows that $f_1 + \ldots + f_n$ is nilpotent. So one of the f_i cannot be nilpotent, and hence it must be an automorphism. □

Theorem 5.7 (Krull-Schmidt) *Let*

$$A = A_1 \times \ldots \times A_m = B_1 \times \ldots \times B_n$$

be two decompositions of A into internal direct products of indecomposable subobjects A_i resp. B_j. Let A be bounded. Then $m = n$ and $A_i \cong B_i$ for all i and a suitable reordering of the B_is.

Proof: We prove the following statement by induction for $t \leq \min(m,n)$.
$P(t)$: there is a reordering of $B_1, \ldots, B_n$ such that $A_i \cong B_i$ for $i = 1, \ldots, t$ and
$A = A_1 \times \ldots \times A_t \times B_{t+1} \times \ldots \times B_n$.
$P(0)$ holds by hypothesis. Assume that $P(t-1)$ holds. Then $A = A_1 \times \ldots \times A_{t-1} \times B_t \times \ldots \times B_n$ with suitable indexing and $A_i \cong B_i$ for $1 \leq i \leq t-1$. Let p'_i and ι'_i be the corresponding projections resp. injections. Furthermore we have $A = A_1 \times \ldots \times A_m$ with the projections p_i and the injections ι_i. Observe that $\iota_i = \iota'_i$ for $i = 1, \ldots, t-1$, and that we use the same subobjects $A_1, \ldots, A_{t-1}$, but the projections may be different. By Proposition 3.5 we have $\mathrm{id} = \iota'_1 p'_1 + \ldots + \iota'_n p'_n$. Since $p_t \iota'_i = p_t \iota_i = 0$ for $1 \leq i \leq t-1$ by definition of the injections, we get

$$\mathrm{id}_{A_t} = p_t \iota_t = p_t \,\mathrm{id}\, \iota_t = p_t \iota'_1 p'_1 \iota_t + \ldots + p_t \iota'_n p'_n \iota_t = p_t \iota'_t p'_t \iota_t + \ldots + p_t \iota'_n p'_n \iota_t.$$

By the Corollary 5.6 one of the $p_t \iota'_i p'_i \iota_t$ must be an automorphism. After reindexing we may assume that $p_t \iota'_t p'_t \iota_t$ is an automorphism of A_t. Since $(p_t \iota'_t p'_t \iota_t)^{r+1} = p_t \iota'_t (p'_t \iota_t p_t \iota'_t)^r p'_t \iota_t$ is also an automorphism, we get $(p'_t \iota_t p_t \iota'_t)^r \neq 0$, hence $p'_t \iota_t p_t \iota'_t$ is also an automorphism due to Lemma 5.2. Thus $p'_t \iota_t : A_t \longrightarrow B_t$ is an isomorphism.

It remains to show that $A = A_1 \times \ldots \times A_t \times B_{t+1} \times \ldots \times B_n$. We have $A = B_t \times (A_1 \times \ldots \times A_{t-1} \times B_{t+1} \times \ldots \times B_n)$. Let $X = A_1 \times \ldots \times A_{t-1} \times B_{t+1} \times \ldots \times B_n$ as subobject of A. Then A_t is a complement for X in $A = B_t \times X$ since $p'_t \iota_t : A_t \longrightarrow A \longrightarrow B_t$ is an isomorphism. By Proposition 2.6 we get $A_t \cap X = 0$ and $A_t \cup X = A$. Furthermore $A_t \longrightarrow A$ and $X \longrightarrow A$ are kernels. Hence A is an internal direct product $A = A_t \times X$ of A_t and X. □

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