On Braiding and Dyslexia

Bodo Pareigis

Abstract. Braided monoidal categories have important applications in knot theory, algebraic quantum field theory, and the theory of quantum groups and Hopf algebras. We will construct a new class of braided monoidal categories.

Typical examples of braided monoidal categories are the category of modules over a quasitriangular Hopf algebra and the category of comodules over a coquasitriangular Hopf algebra. We consider the notion of a commutative algebra A in such a category. The category of (left and/or right) A-modules with the tensor product over A is again a monoidal category which is not necessarily braided. However, if we restrict this category to a special class of modules which we call *dyslectic* then this new category of dyslectic A-modules turns out to be a braided monoidal category, too, and it is a coreflexive subcategory of all A-modules.

The easiest way to obtain a braided monoidal category is to consider all modules over a quasitriangular Hopf algebra or all comodules over a coquasitriangular Hopf algebra [1, 8, 14]. In particular representation theory of quantum groups is braided. There is a converse to this theorem.

Consider a symmetric monoidal category $\mathcal{M}$ (which is cocomplete such that the tensor product preserves arbitrary colimits in both variables) such as $\mathbb{K}$-Vec. Given a diagram $\mathcal{D}$ of finite objects in $\mathcal{M}$ (objects having right duals, finite dimensional vector spaces) then there is a universal coalgebra C in $\mathcal{M}$ such that all objects of $\mathcal{D}$ are comodules over C. If the diagram is closed w.r.t. tensor products (or the corresponding functor $\omega : \mathcal{D} \longrightarrow \mathcal{M}$ together with $\mathcal{D}$ are monoidal) then the associated universal coalgebra is a bialgebra [11]. Furthermore if the category $\mathcal{D}$ is braided (ω will not preserve the braiding) then the associated bialgebra is coquasitriangular. The bialgebras obtained in this way may be of special interest [12].

Our main object in this paper is to construct a new interesting braided monoidal category.

The property of symmetry of a monoidal category $\mathcal{M}$ easily carries over to many other categories constructed over $\mathcal{M}$. So for example the category of algebras in $\mathcal{M}$ under the same tensor product is symmetric, the category o(monoids)f modules over

1991 *Mathematics Subject Classification.* Primary 16W30, 19D23.

a cocommutative Hopf algebra in $\mathcal{M}$ is symmetric, or the category of A-modules for a commutative algebra A in $\mathcal{M}$ with tensor product over A is symmetric.

This is not the case for braidings. If $\mathcal{M}$ is only braided, then algebras in $\mathcal{M}$ do not form a braided category, or modules over a commutative algebra A will not form a braided category. This is remedied by a new concept of dyslectic modules over a commutative algebra A.

We will show that the category of dyslectic modules over a commutative algebra A, i.e. right modules that do not "see" if the multiplication has been twisted around the module via (a twofold application of) the braiding or not, is a braided monoidal category with tensor product $M \otimes_A N$ taken over A. Furthermore this category is a coreflexive subcategory of all A-modules.

1. Modules over a commutative algebra in a braided monoidal category

As base category we consider a cocomplete braided monoidal category $\mathcal{C}$ such that the tensor product preserves arbitrary colimits in both variables. Using coherence we may assume without loss of generality that $\mathcal{C}$ is a strict monoidal category. A braiding of a monoidal category $(\mathcal{C}, \otimes : \mathcal{C} \times \mathcal{C} \longrightarrow C)$ consists of a natural isomorphism of bifunctors $\sigma_{M,N} : M \otimes N \longrightarrow N \otimes M$ such that

$$\sigma_{M,N\otimes P} = (\mathrm{id}_N \otimes \sigma_{M,P})(\sigma_{M,N} \otimes \mathrm{id}_P)$$

and

$$\sigma_{M\otimes N,P} = (\sigma_{M,P} \otimes \mathrm{id}_N)(\mathrm{id}_M \otimes \sigma_{N,P}).$$

We do not require $\sigma_{N,M}\sigma_{M,N} = \mathrm{id}_{M\otimes N}$. If this also holds, then σ is called a symmetry for $\mathcal{C}$. Observe that the first two conditions generate representations of the braid groups on objects $M \otimes M \ldots \otimes M$; if in addition the last condition holds then we have representations of the symmetric groups.

To obtain an example of such a category one can start with a cocomplete symmetric monoidal category $\mathcal{M}$ such that the tensor product preserves arbitrary colimits. As an example consider $\mathcal{M} = \mathbb{K}$-Mod, the category of $\mathbb{K}$-modules over a commutative ring $\mathbb{K}$. (The reader may assume throughout, that $\mathcal{M} = \mathbb{K}$-Mod. Otherwise he should view the calculations as calculations with generalized elements in the sense of [9].)

Let H be a Hopf algebra in $\mathcal{M}$ with the structure morphisms $\Delta_H, \varepsilon_H, \nabla_H, \eta_H, S_H$. Then the category of right H-comodules $\mathcal{M}^H$ is known to be a monoidal category with tensor product as in $\mathcal{M}$. Observe that colimits in $\mathcal{M}^H$ exist and are formed in $\mathcal{M}$ with a uniquely defined suitable comodule structure. If we assume furthermore that H is *coquasitriangular* or *braided*, then $\mathcal{M}^H$ is braided [14]. All in all we have a cocomplete braided monoidal category $\mathcal{M}^H$ such that the tensor product preserves arbitrary colimits.

Many examples of braided monoidal categories can be found in the literature e.g. [3, 6, 7].

In the base category $\mathcal{C}$ we consider now an algebra A and the category of right A-modules $\mathcal{C}_A$.

For the special example $\mathcal{C} = \mathcal{M}^H$, with H a coquasitriangular Hopf algebra in $\mathcal{M}$, we take A an H-comodule algebra in $\mathcal{M}$ with structure morphisms $\nabla_A : A \otimes A \longrightarrow A$ and $\delta_A : A \longrightarrow A \otimes H$, $\delta(a) = \sum a_{(0)} \otimes a_{(1)}$. We study the category of right (H, A)-Hopf-modules $\mathcal{M}_A^H$ with structure morphisms $\delta_M : M \longrightarrow M \otimes H$ and $\rho_M : M \otimes A \longrightarrow M$, i.e. of right H-comodules and right A-modules M such that

$$\delta_M(ma) = \sum m_{(0)} a_{(0)} \otimes m_{(1)} a_{(1)}.$$

They are the right A-modules in the base category $\mathcal{M}^H$.

Since tensor products in $\mathcal{C}$ preserve colimits, $\mathcal{C}_A$ is cocomplete with colimits formed in $\mathcal{C}$ with a uniquely defined suitable module structure.

Remark 1.1. The category ${}_A\mathcal{C}_A$ of (A, A)-bimodules in $\mathcal{C}$ is a monoidal category with the tensor product $M \otimes_A N$ the cokernel of

$$M \otimes A \otimes N \rightrightarrows M \otimes N \longrightarrow M \otimes_A N.$$

To show this one can apply arguments similar to [10], 1.8 and 1.10, in particular the fact that the tensor product is right exact, to prove the associativity $M \otimes_A (N \otimes_A P) \cong (M \otimes_A N) \otimes_A P$.

Since colimits preserve colimits (in our case the tensor product over A) the category ${}_A\mathcal{C}_A$ is cocomplete and the tensor product preserves arbitrary colimits.

Assume now that A is commutative in $\mathcal{C}$, i.e. the following diagram commutes

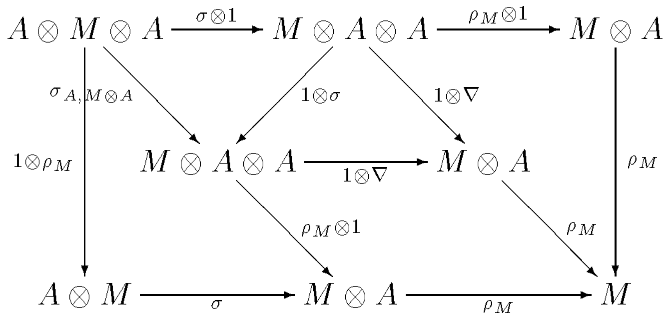
$$\begin{array}{ccc} A \otimes A & \xrightarrow{\sigma} & A \otimes A \\ & {\scriptstyle \nabla} \searrow \quad \swarrow {\scriptstyle \nabla} & \\ & A & \end{array}$$

Remark 1.2. Let $M \in \mathcal{C}_A$. Then it is easy to see that the morphism

$$\lambda_M : A \otimes M \xrightarrow{\sigma} M \otimes A \xrightarrow{\rho_M} M$$

defines an (A, A)-bimodule structure on M. The compatibility of the left and right

A-structures follow for example from the commutative diagram



The category of right A-modules $\mathcal{C}_A$ thus can be viewed as a full subcategory of A-A-bimodules ${}_A\mathcal{C}_A$. Actually this is possible in two distinct ways, namely by defining the left structure by $A \otimes M \xrightarrow{\sigma} M \otimes A \xrightarrow{\rho} M$ or by $A \otimes M \xrightarrow{\sigma^{-1}} M \otimes A \xrightarrow{\rho} M$. So we get the two full embeddings $\Sigma_l : \mathcal{C}_A \longrightarrow {}_A\mathcal{C}_A$ and $\Sigma_r : \mathcal{C}_A \longrightarrow {}_A\mathcal{C}_A$. We shall restrict our considerations to $\Sigma_l : \mathcal{C}_A \longrightarrow {}_A\mathcal{C}_A$ defined by the left action as in remark 1.2.

We now investigate the tensor product over A in ${}_A\mathcal{C}_A$. A straightforward calculation gives

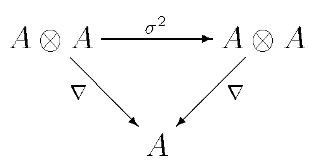
Lemma 1.3. *Let $M, N \in \mathcal{C}_A$. Then the left module structure on $M \otimes_A N$ as defined in 1.2 coincides with the induced left module structure of $M \otimes_A N$ in ${}_A\mathcal{C}_A$, hence $\mathcal{C}_A$ is a monoidal category with tensor product over A and $\Sigma_l : \mathcal{C}_A \longrightarrow {}_A\mathcal{C}_A$ defines a full monoidal embedding.*

We see that for a commutative algebra A the category $\mathcal{C}_A$ is a full monoidal subcategory of ${}_A\mathcal{C}_A$. Actually we have more.

Proposition 1.4. *If A is commutative in $\mathcal{C}$ then $\mathcal{C}_A$ is a cocomplete monoidal category such that the tensor product $M \otimes_A N$ preserves arbitrary colimits in both variables.*

2. Dyslectic algebras and modules

Definition 2.1. Let $\mathcal{C}$ be as before. We call an algebra A in $\mathcal{C}$ *dyslectic* [4] if



commutes or equivalently if $(\nabla\sigma : A \otimes A \longrightarrow A) = (\nabla\sigma^{-1} : A \otimes A \longrightarrow A)$.

A module M in $\mathcal{C}_A$ is called *dyslectic* if the following diagram commutes

$$\begin{array}{ccc} M \otimes A & \xrightarrow{\sigma^2} & M \otimes A \\ & {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle\rho} & \\ & M & \end{array}.$$

A commutative algebra A is clearly dyslectic. However, not all A-modules over a commutative algebra A are dyslectic. In fact the category of dyslectic A-modules $\mathrm{dys}\,\mathcal{C}_A$ is the equalizer of the two embedding functors $\Sigma_l, \Sigma_r : \mathcal{C}_A \longrightarrow {}_A\mathcal{C}_A$.

For a commutative algebra A there is a braiding morphism for the tensor product.

Proposition 2.2. *Let A be commutative and $M \in \mathcal{C}_A$ be dyslectic. Then the following is a commutative diagram of difference cokernels*

$$\begin{array}{ccccc}
M \otimes A \otimes N & \xrightarrow{\rho_M \otimes 1} & & M \otimes N & \xrightarrow{\nu} M \otimes_A N \\
& \xrightarrow{1\otimes\sigma} M \otimes N \otimes A \xrightarrow{1\otimes\rho_N} & & & \\
\downarrow{\scriptstyle 1\otimes\sigma} & & & \downarrow{\scriptstyle\sigma} & \downarrow{\scriptstyle\widetilde{\sigma}} \\
M \otimes N \otimes A & & & & \\
\downarrow{\scriptstyle\sigma_{M,N\otimes A}} & & & & \\
N \otimes A \otimes M & \xrightarrow{1\otimes\sigma} N \otimes M \otimes A \xrightarrow{1\otimes\rho_M} & & N \otimes M & \xrightarrow{\nu} N \otimes_A M. \\
& \xrightarrow{\rho_N \otimes 1} & & &
\end{array}$$

Proof. The lower left hand "square" $(\rho_N \otimes 1)\sigma_{M,N\otimes A}(1 \otimes \sigma) = \sigma(1 \otimes \rho_N)(1 \otimes \sigma)$ commutes by functoriality of σ. Furthermore if M is dyslectic then the following diagram commutes

$$\begin{array}{ccccccc}
M \otimes A \otimes N & \xrightarrow{1\otimes\sigma} & M \otimes N \otimes A & & & & \\
\downarrow{\scriptstyle\rho_M\otimes 1} & \searrow{\scriptstyle\sigma_{M\otimes A,N}} & \downarrow{\scriptstyle\sigma\otimes 1} & \searrow{\scriptstyle\sigma_{M,N\otimes A}} & & & \\
 & & N \otimes M \otimes A & \xrightarrow{1\otimes\sigma} & N \otimes A \otimes M & \xrightarrow{1\otimes\sigma} & N \otimes M \otimes A \\
 & & \downarrow{\scriptstyle 1\otimes\rho_M} & & & & \downarrow{\scriptstyle 1\otimes\rho_M} \\
M \otimes N & \xrightarrow{\sigma} & N \otimes M & & \xrightarrow{1} & & N \otimes M
\end{array}$$

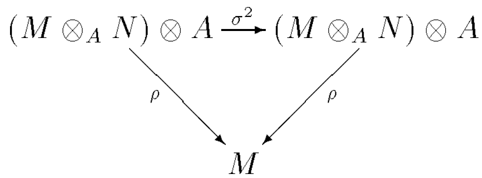
which is the upper left hand "square". $\square$

Remark 2.3. There is a second way to define a quasi-symmetry map in $\mathcal{C}_A$, namely with $\sigma^{-1}_{M,N}$ instead of $\sigma_{M,N}$. A similar proof as for Proposition 2.2 gives:

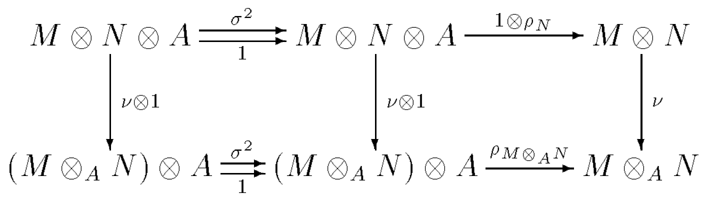
Let A be commutative and $N \in \mathcal{C}_A$ be dyslectic. Then $\sigma^{-1}_{M,N}$ induces a morphism on the difference cokernels: $\widetilde{\sigma} : M \otimes_A N \longrightarrow N \otimes_A M$.

Proposition 2.4. *If M and N in $\mathcal{C}_A$ are dyslectic then so is $M \otimes_A N$.*

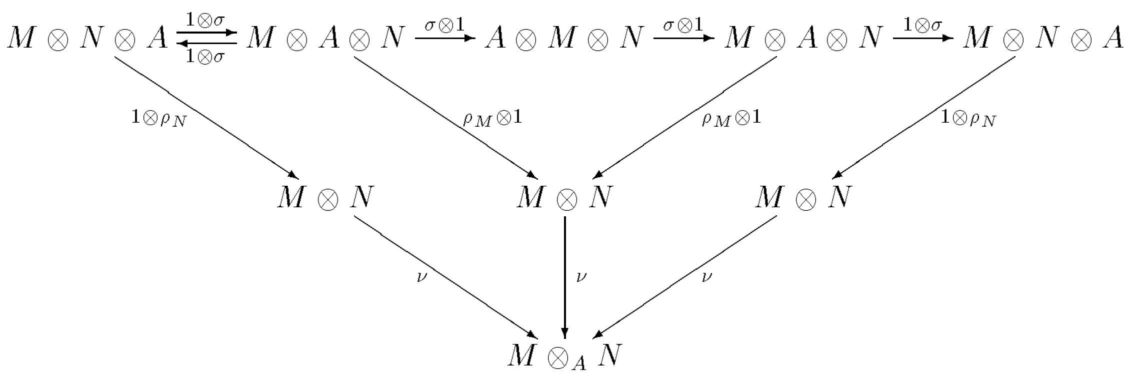
Proof. We have to show that



commutes. Since the following diagram commutes



and the left most $\nu \otimes 1$ is an epimorphism it suffices to show that $\nu(1 \otimes \rho_N)\sigma^2 = \nu(1 \otimes \rho_N)$. Observe that tensor products preserve difference cokernels. If we expand σ^2 we get a commutative diagram



where the left quadrangle commutes by definition of ν and since N is dyslectic. The center triangle commutes since M is dyslectic. The right quadrangle commutes by definition of ν. $\square$

It is now clear that the monoidal structure of $\mathcal{C}_A$ restricts to a monoidal structure on $\mathrm{dys}\,\mathcal{C}_A$ and that $\tilde{\sigma} : M \otimes_A N \longrightarrow N \otimes_A M$ defines a braiding of $\mathrm{dys}\,\mathcal{C}_A$.

Theorem 2.5. *Let $\mathcal{C}$ a cocomplete braided monoidal category such that the tensor product preserves arbitrary colimits. Let A be a commutative algebra in $\mathcal{C}$. Then the category of dyslectic right A-modules $\mathrm{dys}\,\mathcal{C}_A$ is a cocomplete braided monoidal category such that the tensor product $M \otimes_A N$ preserves arbitrary colimits.*

Proof. With the help of the commutative diagram

$$\begin{array}{ccccc} X_i \otimes A & \overset{\sigma^2}{\underset{1}{\rightrightarrows}} & X_i \otimes A & \xrightarrow{\rho} & X_i \\ \downarrow & & \downarrow & & \downarrow \\ \varinjlim X_i \otimes A & \overset{\sigma^2}{\underset{1}{\rightrightarrows}} & \varinjlim X_i \otimes A & \longrightarrow & \varinjlim X_i \end{array}$$

and the fact that tensor products in $\mathcal{C}$ preserve colimits it is easy to see that the colimit in $\mathcal{C}$ of dyslectic right A-modules is again dyslectic, so $\operatorname{dys}\mathcal{C}_A$ is cocomplete, the embedding $\operatorname{dys}\mathcal{C}_A \longrightarrow \mathcal{C}_A$ preserves colimits, and the tensor product $M \otimes_A N$ in $\operatorname{dys}\mathcal{C}_A$ preserves arbitrary colimits. It is an easy exercise to check that the morphism $\tilde{\sigma}$ from 2.2 is functorial and is a braiding for $\mathcal{C}_A$. □

Observe that any commutative algebra A is dyslectic as an A-module. Since $\operatorname{dys}\mathcal{C}_A$ is cocomplete any colimit of a diagram with objects coproducts $A^{(n)}$ of A is dyslectic. So there are many dyslectic modules over a commutative algebra in $\mathcal{C}$.

Remark 2.6. There is an interesting relation between the notion of dyslectic modules and the center of a monoidal category (we owe this remark to the referee). Since each object $M \in \operatorname{dys}\mathcal{C}_A$ comes with a natural transformation $a(M) : M \otimes_A \text{-} \longrightarrow \text{-} \otimes_A M$ of functors on $\mathcal{C}_A$ as defined in 2.2 the category of dyslectic modules is also a braided monoidal subcategory of the center of $\mathcal{C}_A$ in the sense of [5]. Unlike the center, however, it is a full subcategory of $\mathcal{C}_A$.

3. Cofree dyslectic modules

The purpose of this section is to show that there are many examples of dyslectic modules. To this end we have to study inner hom-functors. So we assume now that the cocomplete braided monoidal base category $\mathcal{C}$ has difference kernels (equalizers) and is right closed, i.e. there is a right adjoint functor $[M, \text{-}] : \mathcal{C} \longrightarrow \mathcal{C}$ for every functor "tensor product with M on the right" $\text{-} \otimes M : \mathcal{C} \longrightarrow \mathcal{C}$.

To get examples of such categories we start, as in section 1, with a symmetric monoidal category $\mathcal{M}$ (which is cocomplete such that the tensor product preserves arbitrary colimits). Assume that $\mathcal{M}$ is closed and has difference kernels.

If H is a Hopf algebra in $\mathcal{M}$ and has a dual (see [14] Chap. 2) then we call H a finite Hopf algebra.

Lemma 3.1. *If H is a finite Hopf algebra then $\mathcal{M}^H$ is right closed.*

Proof. Let $\sum h_i^* \otimes h_i$ be a dual basis for H (with $\sum h_i^*(h)h_i = h$).

For $N, P \in \mathcal{M}^H$ define the structure of an H-comodule on $\operatorname{Hom}(N, P)$ by

$$\delta : \operatorname{Hom}(N, P) \longrightarrow \operatorname{Hom}(N, P) \otimes H, \quad \delta(f) = \sum f(-_{(0)})_{(0)} \cdot h_i^*(f(-_{(0)})_{(1)} S(-_{(1)})) \otimes h_i.$$

Then the canonical morphism $\mathcal{M}(M \otimes N, P) \cong \mathcal{M}(M, \mathrm{Hom}(N, P))$ given by $f(m \otimes n) = g(m)(n)$ restricts to

$$\mathcal{M}^H(M \otimes N, P) \cong \mathcal{M}^H(M, \mathrm{Hom}(N, P))$$

since let f satisfy $\sum f(m_{(0)} \otimes n_{(0)}) \otimes m_{(1)}n_{(1)} = \sum f(m \otimes n)_{(0)} \otimes f(m \otimes n)_{(1)}$. Then

$$\begin{aligned} \sum g(m_{(0)})(n) \otimes m_{(1)} &= \sum f(m_{(0)} \otimes n) \otimes m_{(1)} \\ &= \sum f(m_{(0)} \otimes n_{(0)}) \otimes m_{(1)}\varepsilon(n_{(1)}) \\ &= \sum f(m_{(0)} \otimes n_{(0)}) \otimes m_{(1)}n_{(1)}S(n_{(2)}) \\ &= \sum f(m \otimes n_{(0)})_{(0)} \otimes f(m \otimes n_{(0)})_{(1)}S(n_{(1)}) \\ &= \sum f(m \otimes n_{(0)})_{(0)} \otimes h_i^*(f(m \otimes n_{(0)})_{(1)}S(n_{(1)}))h_i \\ &= \sum f(m \otimes n_{(0)})_{(0)} \cdot h_i^*(f(m \otimes n_{(0)})_{(1)}S(n_{(1)})) \otimes h_i \\ &= \sum g(m)(n_{(0)})_{(0)} \cdot h_i^*(g(m)(n_{(0)})_{(1)}S(n_{(1)})) \otimes h_i \\ &= \sum g(m)_{(0)}(n) \otimes g(m)_{(1)}. \end{aligned}$$

In a similar way one shows that the inverse map also restricts to morphisms in $\mathcal{M}^H$. □

Remark 3.2. In general $\mathcal{M}^H$ will not be left closed.

If $\mathcal{M} = \mathbb{K}$-Vek then $\mathcal{M}^H$ has kernels. So for a finite coquasitriangular Hopf algebra H over a field $\mathbb{K}$ the category of H-comodules $\mathcal{M}^H$ satisfies the properties for $\mathcal{C}$ as required at the beginning of this section.

We return now to the general case. If $\mathcal{C}$ is right closed and has difference kernels then ${}_A\mathcal{C}_A$ is also right closed with $[M, N]_A$ the difference kernel in

$$[M, N]_A \longrightarrow [M, N] \rightrightarrows [M \otimes A, N].$$

Techniques as in [9] can be used to prove this.

We consider the pair of adjoint functors -$\otimes A$ and $[A, \text{-}]$. Let $\eta : M \longrightarrow [A, M \otimes A]$ be the unit and $\varepsilon : [A, M] \otimes A \longrightarrow M$ be the counit of the adjoint pair. Then the isomorphism $\mathrm{Mor}_{\mathcal{C}}(M \otimes A, N) \cong \mathrm{Mor}_{\mathcal{C}}(M, [A, M])$ is given by $f \mapsto [A, f]\eta$ and $g \mapsto \varepsilon[g \otimes 1_A]$, in particular we get $\varepsilon([A, f]\eta \otimes 1_A) = f$.

Let M be a right A-module with structure morphism $\rho_M : M \otimes A \longrightarrow M$. Let

$$K \xrightarrow{\iota_K} M \underset{[A,\rho_M\sigma^2_{M,A}]\eta}{\overset{[A,\rho_M]\eta}{\rightrightarrows}} [A, M]$$

be a difference kernel (equalizer). (We abbreviate $\sigma^2_{M,A} := \sigma_{A,M}\sigma_{M,A}$.)

Lemma 3.3. *K is an A-submodule of M.*

Proof. We have

$$
\begin{aligned}
&\epsilon([A,\rho_M]\eta\rho_M(\iota_K\otimes 1_A)\otimes 1_A)\\
&=\epsilon([A,\rho_M]\eta\otimes 1_A)(\rho_M\otimes 1_A)(\iota_K\otimes 1_A\otimes 1_A)\\
&=\rho_M(\rho_M\otimes 1_A)(\iota_K\otimes 1_A\otimes 1_A)\\
&=\rho_M(1_A\otimes\nabla)(\iota_K\otimes 1_A\otimes 1_A)\\
&=\rho_M(1_A\otimes\nabla)(1_M\otimes\sigma_{A,A})(\iota_K\otimes 1_A\otimes 1_A) \quad \text{since } A \text{ is commutative}\\
&=\rho_M(\rho_M\otimes 1_A)(\iota_K\otimes 1_A\otimes 1_A)(1_M\otimes\sigma_{A,A})\\
&=\rho_M(\rho_M\otimes 1_A)(\sigma^2_{M,A}\otimes 1_A)(\iota_K\otimes 1_A\otimes 1_A)(1_M\otimes\sigma_{A,A}) \quad \text{(by definition of } \iota_K)\\
&=\rho_M(1_A\otimes\nabla)(1_M\otimes\sigma_{A,A})(\sigma^2_{M,A}\otimes 1_A)(1_M\otimes\sigma_{A,A})(\iota_K\otimes 1_A\otimes 1_A)\\
&=\rho_M(\rho_M\otimes 1_A)\sigma^2_{M\otimes A,A}(\iota_K\otimes 1_A\otimes 1_A) \quad \text{(property of } \sigma)\\
&=\rho_M\sigma^2_{M,A}(\rho_M\otimes 1_A)(\iota_K\otimes 1_A\otimes 1_A)\\
&=\epsilon([A,\rho_M\sigma^2_{M,A}]\eta\otimes 1_A)(\rho_M\otimes 1_A)(\iota_K\otimes 1_A\otimes 1_A)\\
&=\epsilon([A,\rho_M\sigma^2_{M,A}]\eta\rho_M(\iota_K\otimes 1_A)\otimes 1_A)
\end{aligned}
$$

which implies $[A,\rho_M]\eta\rho_M(\iota_K\otimes 1_A)=[A,\rho_M\sigma^2_{M,A}]\eta\rho_M(\iota_K\otimes 1_A)$. So there is a unique morphism $\rho_K : K\otimes A\longrightarrow K$ such that $\iota_K\rho_K=\rho_M(\iota_K\otimes 1_A)$, since ι_K is the difference kernel of $[A,\rho_M]\eta$ and $[A,\rho_M\sigma^2_{M,A}]\eta$. □

Lemma 3.4. *K is dyslectic.*

Proof. We have $\iota_K\rho_K\sigma^2_{K,A}=\rho_M(\iota_K\otimes 1_A)\sigma^2_{K,A}=\rho_M\sigma^2_{M,A}(\iota_K\otimes 1_A)=\epsilon([A,\rho_M\sigma^2_{M,A}]\eta\otimes 1_A)(\iota_K\otimes 1_A)=\epsilon([A,\rho_M\sigma^2_{M,A}]\eta\iota_K\otimes 1_A)=\epsilon([A,\rho_M]\eta\iota_K\otimes 1_A)=\rho_M(\iota_K\otimes 1_A)=\iota_K\rho_K$. □

Lemma 3.5. *If $P\in\mathrm{dys}\,\mathcal{C}_A$ is dyslectic, $M\in\mathcal{C}_A$ and $f : P\longrightarrow M$ an A-homomorphism, then f factors uniquely through $\iota_K : K\longrightarrow M$.*

Proof. We have to show that $[A,\rho_M]\eta f=[A,\rho_M\sigma^2_{M,A}]\eta f$. This follows from

$$
\begin{aligned}
&\epsilon([A,\rho_M]\eta f\otimes 1_A)=\epsilon([A,\rho_M]\eta\otimes 1_A)(f\otimes 1_A)=\rho_M(f\otimes 1_A)=f\rho_P=f\rho_P\sigma^2_{P,A}\\
&=\rho_M(f\otimes 1_A)\sigma^2_{P,A}=\rho_M\sigma^2_{M,A}(f\otimes 1_A)=\epsilon([A,\rho_M\sigma^2_{M,A}]\eta\otimes 1_A)(f\otimes 1_A)\\
&=\epsilon([A,\rho_M\sigma^2_{M,A}]\eta f\otimes 1_A). \quad \square
\end{aligned}
$$

Theorem 3.6. *Let $\mathcal{C}$ be as in Theorem 2.5, be right closed and have difference kernels. Let A be a commutative algebra in $\mathcal{C}$. Then the category of dyslectic A-modules $\mathrm{dys}\,\mathcal{C}_A$ is a coreflexive subcategory of $\mathcal{C}_A$.*

Proof. We have to show that the construction of K as in the previous Lemmas defines a right adjoint functor to the embedding of $\mathrm{dys}\,\mathcal{C}_A$ into $\mathcal{C}_A$. But this is demonstrated by the universal property given in 3.5. □

We remark, that we only needed a right adjoint functor $[A,\text{-}]$ for $\text{-}\otimes A$ in the proof.

4. Examples of suitable braided base categories

We close with some special examples of braided monoidal categories $\mathcal{C}$ (cocomplete such that the tensor product preserves arbitrary colimits), that may be used as base categories. One special example is the category of Yetter-Drinfel'd modules $\mathcal{YD}_H^H$ over a Hopf algebra H. By [14] it can be viewed as a category of comodules over a coquasitriangular Hopf algebra.

For an arbitrary cocommutative Hopf algebra H such a category can also be obtained in the following way. Consider the category of right H-modules which is a cocomplete symmetric monoidal category such that the tensor product preserves arbitrary colimits.

Observe that H acts on itself by the adjoint action

$$\alpha : H \otimes H \ni h \otimes k \mapsto \sum S(k_{(1)}) h k_{(2)} \in H.$$

H is a right H-module Hopf algebra by the adjoint action as can be easily checked. Thus H is a Hopf algebra in the category $\mathcal{M}_H$ and the category $\mathcal{M}_H^{H\bullet}$ of H-comodules in $\mathcal{M}_H$ is a monoidal category. A $\mathbb{K}$-module M is in $\mathcal{M}_H^{H\bullet}$ iff

a) it is a right H-module $\rho : M \otimes H \longrightarrow M$,

b) it is a right H-comodule $\delta : M \longrightarrow M \otimes H$, and

c) $\delta(mh) = \delta(m)h$ or $\delta(mh) = \sum m_{(0)} h_{(1)} \otimes m_{(1)} \cdot h_{(2)} = \sum m_{(0)} h_{(1)} \otimes S(h_{(2)}) m_{(1)} h_{(3)}$.

In view of the cocommutativity of H the last condition is equivalent to

$$\sum (mh_1)_0 \otimes h_2 (mh_1)_1 = \sum m_0 h_1 \otimes m_1 h_2$$

which is the Yetter-Drinfel'd condition [13]. Thus $\mathcal{M}_H^{H\bullet} = \mathcal{YD}_H^H$ is a braided monoidal category [2, 15], the braiding morphism being defined by

$$\sigma : M \otimes N \ni m \otimes n \mapsto \sum n_{(0)} \otimes m n_{(1)} \in N \otimes M.$$

Obviously $\mathcal{M}_H^{H\bullet}$ is not symmetric since $\sum m_{(0)} n_{(1)} \otimes n_{(0)} m_{(1)} n_{(2)} = m \otimes n$ does not hold in general.

A special case for this structure is obtained by choosing $H = \mathbb{K}G$, a group algebra with a finite group G. In this situation the name of dyslectic algebra was first introduced by Haran [4].

Now we can give an example of a module which is not dyslectic. Let $G = C_2 \times C_2$ be the Klein four group. Let $\mathrm{char}(\mathbb{K}) \neq 2$. Then the Hopf algebra $H := \mathbb{K}G = \mathbb{K}[s,t]/(s^2-1, t^2-1)$ is coquasitriangular with $r : H \otimes H \longrightarrow \mathbb{K}$ defined by $r(s \otimes s) = r(t \otimes s) = r(t \otimes t) = 1$, $r(s \otimes t) = -1$. Observe that $r(st \otimes t) = r(s \otimes t) r(t \otimes t) = -1$. Let $A = \mathbb{K}1 \oplus \mathbb{K}x$ with $\delta(1) = 1 \otimes 1$, $\delta(x) = x \otimes t$. Then A becomes a commutative algebra in $\mathcal{M}^H$ by $x^2 = 1$. Let $M = \mathbb{K}y \oplus \mathbb{K}z$ with $\delta(y) = y \otimes s$ and $\delta(z) = z \otimes st$. Then M becomes an A-module in $\mathcal{M}^H$ by $yx = z$ and $zx = y$. In particular we get $\sigma(y \otimes x) = -x \otimes y$, $\sigma^2(y \otimes x) = -y \otimes x$ and $\sigma(z \otimes x) = -x \otimes z$, $\sigma^2(z \otimes x) = -z \otimes x$. The maximal dyslectic submodule K of M turns out to be zero, since $\rho_M((\alpha y + \beta z) \otimes x) =$

$\alpha z + \beta y$ and $\rho_M \sigma^2((\alpha y + \beta z) \otimes x) = -\alpha z - \beta y$. In particular M is not dyslectic. It is easy to check that the diagram in Prop. 2.2 with $N = A$ has a non-commutative upper left hand square so it induces no braiding for $\mathcal{M}_A^H$. If there was a map $\tilde{\sigma}$ induced by σ, then $\tilde{\sigma}(y \otimes_A x) = -x \otimes_A y = x \otimes_A y = 0$. So there is no induced braiding on $\mathcal{M}_A^H$.

We close with another example of a suitable monoidal category. Let G be a group and X be a (right) G-set. G itself can be considered as a G-set by the (right) adjoint action. The Freyd-Yetter category $\mathrm{Cr}(G)$ of crossed G-sets consists of pairs $(X, |.|)$ with a G-set X and a G-morphism $|.| : X \longrightarrow G$ as objects and G-morphisms $f : X \longrightarrow Y$ such that $|.|_Y f = |.|_X$. By [3] Thm. 4.2.2 this is a braided monoidal category with $(X, |.|) \otimes (Y, |.|) = (X \times Y, |.|)$, such that $|(x, y)| = |x||y|$, and the braiding $\sigma_{X,Y}(x, y) = (y, x|y|)$. The unit object I of this category is the one point set being mapped into the unit of G.

An algebra in this category is a set A with maps $|.| : A \longrightarrow G$, $A \times G \longrightarrow A$, $A \times A \longrightarrow A$ and $\{1\} \longrightarrow A$ such that

$$\begin{array}{cc} a(gg') = (ag)g', & ae = a, \\ |ab| = |a||b|, & |1| = e, \\ (ab)g = (ag)(bg), & 1g = 1, \\ (ab)c = a(bc), & 1a = a = a1, \\ |ag| = g^{-1}ag. & \end{array}$$

The algebra is commutative iff

$$ab = b(a|b|).$$

So A is a commutative algebra in $\mathrm{Cr}(G)$ iff A is a crossed semi-module [6]. (We thank the referee for drawing our attention to this fact.)

An A-module is a set X with maps $|.| : X \longrightarrow G$, $X \times G \longrightarrow X$ and $X \times A \longrightarrow X$ such that

$$\begin{array}{cc} x(gg') = (xg)g', & xe = x, \\ |xa| = |x||a|, & (xa)g = (xg)(ag), \\ x(ab) = (xa)b, & x1 = x, \\ |xg| = g^{-1}|x|g. & \end{array}$$

A module X over a commutative algebra A is dyslectic iff

$$xa = (x(a|a|^{-1}|x|))|a|.$$

In particular the dyslectic part of an A-module X is

$$K = \{x \in X | \forall a \in A : xa = (x(a|a|^{-1}|x|))|a|\}.$$

References

1. V. G. Drinfel'd: Quantum groups, "Proc. Int. Congr. Math.", Berkeley, CA 1986, Vol. 1 (1987), 798-820.
2. D. Fischman, S. Montgomery, A Schur double centralizer theorem for cotriangular Hopf algebras and generalized Lie algebras. preprint (1992), 20 pp.
3. P. J. Freyd, D. N. Yetter, Braided compact closed categories with applications to low dimansional topology. *Advances in Math.* **77** (1989) 156-182.
4. S. Haran, An invitation to dyslectic geometry. to appear in *J. Alg.* – preprint (1992), 36 pp.
5. A. Joyal, R. Street, Tortile Yang-Baxter operators in tensor categories. *J. Pure Appl. Algebra* **71** (1991) 43-51.
6. A. Joyal, R. Street, Braided tensor categories. *Advances in Math.* (to appear).
7. C. Kassel, V. Turaev, Double construction for monoidal categories. *Publications de l'Institut de Rech. Math. Avancée.* **507/P294** (Strasbourg, 1992) ISSN 0755-3390.
8. R. G. Larson, J. Towber, Two dual classes of bialgebras related to 'quantum groups' and 'quantum Lie algebras'. *Comm. alg.* **19** (1991), 3295-3345.
9. B. Pareigis, Non-additive ring and module theory I. General theory of monoids. *Publicationes Mathematicae* **24** Debrecen (1977), 190-204.
10. —, Non-additive ring and module theory V. Projective and coflat objects. *Algebra Berichte* **40**, Verlag Reinhard Fischer, München (1980), 33 pp.
11. —, Endomorphism bialgebras of diagrams and of non-commutative algebras and spaces. preprint (1992), 34 pp.
12. —, Reconstruction of Hopf algebras of hidden symmetries, preprint (1993) 10 pp.
13. D. Radford, J. Towber, Yetter-Drinfel'd categories associated to an arbitrary bialgebra. preprint (1992) 23 pp.
14. P. Schauenburg, On coquasitriangular Hopf algebras and the quantum Yang-Baxter equation. *Algebra Berichte* **67**, Verlag Reinhard Fischer, München, (1992), 76 pp.
15. P. Schauenburg, Hopf modules and Yetter-Drinfel'd modules. to appear *J. Alg.* – preprint (1993) 16 pp.

Mathematisches Institut der Universität München, Germany
E-mail address: Bodo.Pareigis@Mathematik.Uni-Muenchen.d400.de