DOUBLE QUANTUM GROUPS

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Abstract. The construction of the Drinfeld double $D(H)$ of a finite dimensional Hopf algebra H was one of the first examples of a quasitriangular Hopf algebra whose category of modules $\mathcal{M}_{D(H)}$ is braided. The braided category of Yetter-Drinfeld modules $\mathcal{DY}_H^H$ is the analogue for infinite dimensional Hopf algebras. It uses a strong dependence between the H-module and the H-comodule structures.

We generalize this construction to the category $\mathcal{M}_A^C(\psi)$ of entwined modules, that is A-modules and C-comodules over Hopf algebras A and C where the structures are only related by an entwining map $\psi : C \otimes A \longrightarrow A \otimes C$. We show that $\mathcal{M}_A^C(\psi)$ is braided iff there is an r-map $r : C \otimes C \longrightarrow A \otimes A$ satisfying suitable axioms that generalize the axioms of an R-matrix. For finite dimensional C there is a quasitriangular Hopf algebra structure on $\mathrm{Hom}(C, A)$, the *quantum group double*, generalizing the construction of the Drinfeld double.

Contents

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Introduction

Let A be an algebra and C be a coalgebra. If one considers modules that are simultaneously right A-modules and right C-comodules, then it is reasonable to ask for compatibility conditions of these two structures. In [3] and [4] Brzeziński introduced a compatibility which he called an entwining structure and which consists of a certain homomorphism $\psi : C \otimes A \longrightarrow A \otimes C$. The category of such (entwined) modules will be denoted by $\mathcal{M}_A^C(\psi)$. Special cases will be Yetter-Drinfeld modules and Doi-Hopf modules.

The conditions for the entwining map as introduced in [3] have for long been known among category theorists under the name of *distributive laws*. In our situation they can be described as follows. An algebra A defines a monad over the category $\mathcal{M}$ and the category of A-modules is the associated Eilenberg-Moore category. Analogously a coalgebra C is a comonad with Eilenberg-Moore category C-Comod. The compatibility condition for the entwining structure $\psi : C \otimes A \longrightarrow A \otimes C$ turns out to be a mixed distributive law between the monad A and the comonad C. In [2] the distributive laws are given for two monads, in [1] they are given for two comonads. In our situation we find mixed distributive laws between a monad and a comonad. Thus part of the theory described in this paper can be extended to more general pairs of monads and comonads with mixed distributive laws. In particular conditions for mixed distributive laws with a monoidal structure have recently been studied in [11, 12].

In this paper we study necessary and sufficient conditions for $\mathcal{M}_A^C(\psi)$ to become a monoidal and a braided monoidal category. For that purpose we are going to assume that C and A are bialgebras. The monoidal structure of $\mathcal{M}_A^C(\psi)$ will depend on additional compatibility conditions for the homomorphism ψ.

The braiding of $\mathcal{M}_A^C(\psi)$ will be given by an r-map $r : C \otimes C \longrightarrow A \otimes A$ that generalizes the well-known R-matrix for quasitriangular Hopf algebras and the r-map $r : C \otimes C \longrightarrow k$ for coquasitriangular Hopf algebras.

In the special case of Doi-Hopf modules this has already been done in [5]. However, we will show more, even in this special case, namely that *every* braiding for $\mathcal{M}_A^C(\psi)$ is given by such an r-homomorphism. Furthermore we give conditions for an inverse of r. An inverse has already been considered in [5] but the somewhat different conditions there could not be verified with our calculations.

As mentioned above, special cases of our construction are the category of Yetter-Drinfeld modules and the category of Doi-Hopf modules [5], [6]. Since in the case of Yetter-Drinfeld modules $\mathcal{YD}_H^H$ there is an equivalence of categories $\mathcal{YD}_H^H \cong \mathcal{M}_{D(H)}$, where $D(H)$ is the Drinfeld double of H (if H is finite dimensional), we give a similar crossed product construction for a Hopf algebra structure on $\mathrm{Hom}(C, A)$ so that $\mathcal{M}_A^C(\psi) \cong \mathcal{M}_{\mathcal{M}(C,A)}$.

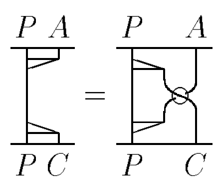
1. The Category of Entwined Modules

Let $\mathcal{M}$ be a symmetric monoidal category with unit object I. Our main interest lies in $\mathcal{M} = k$-Vec, the category of vector spaces over a field k. We can assume that $\mathcal{M}$ is strict.

Let A be an algebra and C be a coalgebra in $\mathcal{M}$. We will study the category of objects that are simultaneously A-modules and C-comodules. If P is a right A-module and a right C-comodule, then $P \otimes C$ becomes an A-module by the A-module structure of P and $P \otimes A$ becomes a C-comodule by the C-comodule structure of P.

In both cases factors from A and C have to be interchanged in a certain way for which we are not using the usual twist map $\sigma : C \otimes A \longrightarrow A \otimes C$ but a new map $\psi : C \otimes A \longrightarrow A \otimes C$ satisfying the axioms given below. We will use the same morphism $\psi : C \otimes A \longrightarrow A \otimes C$ for all objects.

The compatibility of the A-module and C-comodule structures shall be given by the diagram

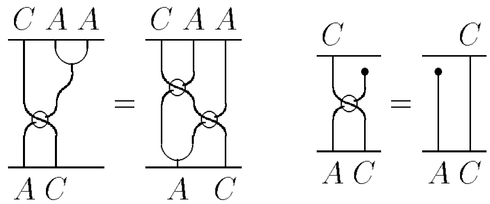


or

$$\delta_P \mu_P = (\mu_P \otimes C)(P \otimes \psi)(\delta_P \otimes A)$$

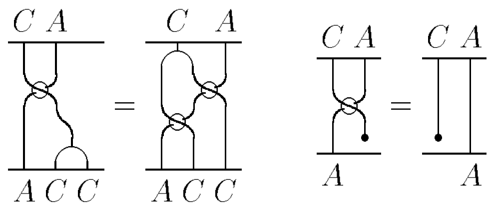
Throughout we will use extensively a graphic notation that was originally suggested by Penrose [18]. By [8] and [17] we may perform our computations with this notation.

We want to impose conditions on ψ such that $\delta_P : P \longrightarrow P \otimes C$ becomes an A-module homomorphism for every A-module P. $P \otimes C$ becomes an A-module (functorially in P) and δ_P an A-module homomorphism if and only if



(1)

We also want that $\mu_P : P \otimes A \longrightarrow P$ becomes a C-comodule homomorphism for every C-comodule P. $P \otimes A$ becomes a C-comodule and μ_P a C-comodule homomorphism if and only if



(2)

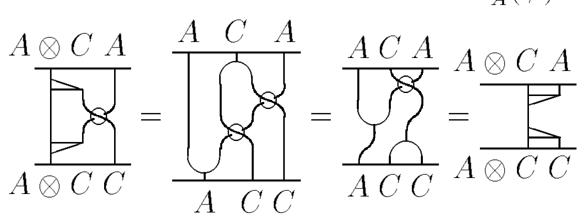
Definition 1.1. The triple (A, C, ψ) satisfying (1) and (2) is called an *entwining structure* with *entwining map* ψ ([3] and [4]).

Let $\mathcal{M}_A^C(\psi)$ be the category of objects that are simultaneously C-comodules and A-modules $(P, \delta_P : P \longrightarrow P \otimes C, \mu_P : P \otimes A \longrightarrow P)$ such that with respect to an entwining structure (A, C, ψ)

$$\delta_P \mu_P = (\mu_P \otimes C)(P \otimes \psi)(\delta_P \otimes A)$$

holds. These objects will be called *entwined modules*. Morphisms shall be A-module and C-comodule morphisms.

Remark 1.2. Observe that both $A \otimes C$ and $C \otimes A$ are entwined modules in $\mathcal{M}_A^C(\psi)$ due to the axioms for ψ. The A-module structure on $A \otimes C$ is given by $(\nabla_A \otimes C)(A \otimes \psi)$ and the C-comodule structure is $A \otimes \Delta_C$. The axiom for entwined modules in $\mathcal{M}_A^C(\psi)$ is satisfied since



i.e. the axiom (2) for ψ holds. The proof for $C \otimes A$ with A-module structure $C \otimes \nabla_A$ and C-comodule structure $(C \otimes \psi)(\Delta_C \otimes A)$ is dual.

More generally, we get objects $M \otimes A$ in $\mathcal{M}_A^C(\psi)$ for each $M \in \mathcal{M}^C$. This defines a left adjoint functor for the underlying functor $U : \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}^C$ to the category of right C-comodules. Furthermore $M \otimes C$ is an object in $\mathcal{M}_A^C(\psi)$ for each $M \in \mathcal{M}_A$. This defines a right adjoint functor for the underlying functor $U : \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}_A$ to the category of right A-modules.

Using (1) and (2), it is easy to see that the entwining map $\psi : C \otimes A \longrightarrow A \otimes C$ itself is a morphism in $\mathcal{M}_A^C(\psi)$. Furthermore, $\mu \otimes C : M \otimes A \otimes C \longrightarrow M \otimes C$ is in $\mathcal{M}_A^C(\psi)$ as well as $\delta : M \longrightarrow M \otimes C$.

2. Endomorphisms of the Underlying Functor ω

Let $\omega : \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}$ be the underlying functor. Observe that $\mathcal{M}_A^C(\psi)$ is a (left) $\mathcal{M}$-category (or $\mathcal{M}$-actegory, a terminology due to R. Street), i.e. there is a functor $\otimes : \mathcal{M} \times \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}_A^C(\psi)$ with coherent associativity and unit conditions, and that ω is an $\mathcal{M}$-functor, i.e. there is a coherent isomorphism $X \otimes \omega(P) \cong \omega(X \otimes P)$ with $X \in \mathcal{M}$ and $P \in \mathcal{M}_A^C(\psi)$ in the sense of [14]. Let $\mathrm{Nat}_{\mathcal{M}}(\omega, \omega)$ denote the set of natural $\mathcal{M}$-transformations. A natural transformation $\varphi(P) : \omega(P) \longrightarrow \omega(P)$ is an $\mathcal{M}$-transformation if $X \otimes \varphi(P) = \varphi(X \otimes P)$ where we identified along the coherent isomorphism $X \otimes \omega(P) \cong \omega(X \otimes P)$.

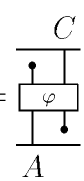
If $\mathcal{M} = k$-Vec, then every natural transformation $\varphi : \omega \longrightarrow \omega$ is an $\mathcal{M}$-transformation by [16] Theorem 6.4. So in that case we have $\mathrm{Nat}(\omega, \omega) = \mathrm{Nat}_{\mathcal{M}}(\omega, \omega)$.

Theorem 2.1. *There is an isomorphism*

$$\mathrm{Nat}_{\mathcal{M}}(\omega, \omega) \cong \mathcal{M}(C, A).$$

Proof. We define $\Pi : \mathrm{Nat}_{\mathcal{M}}(\omega, \omega) \cong \mathcal{M}(C, A)$ by

$$\Pi(\varphi) = (A \otimes \varepsilon)\varphi(A \otimes C)(\eta \otimes C) \qquad \text{or}$$

$\Pi(\varphi) =$ 

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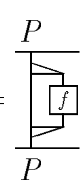
The inverse map $\Sigma : \mathcal{M}(C, A) \longrightarrow \mathrm{Nat}_{\mathcal{M}}(\omega, \omega)$ is defined by

$$\Sigma(f)(P, \delta_P, \mu_P) = \mu_P(P \otimes f)\delta_P \qquad \text{or}$$

$\Sigma(f) =$ 

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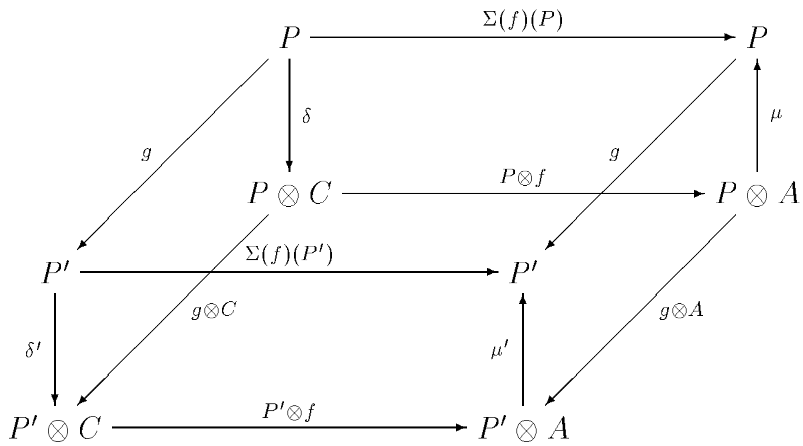
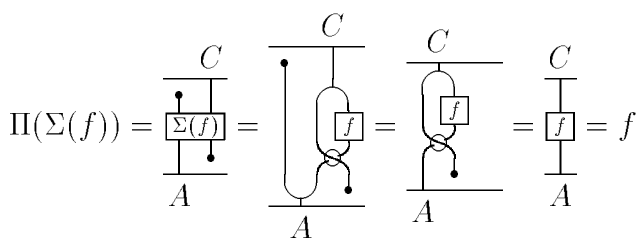
We first check that $\Sigma(f)$ is a natural $\mathcal{M}$-transformation. Let $g : P \longrightarrow P'$ be a morphism in $\mathcal{M}_A^C(\psi)$. Then the commutativity of the diagram

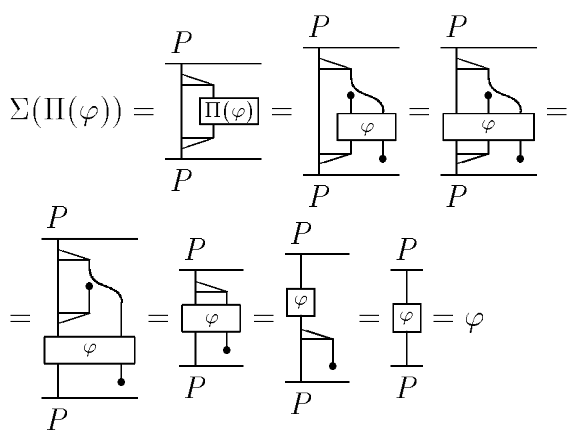


proves that $\Sigma(f)$ is a natural transformation. The (graphical) definition of $\Sigma(f)$ shows immediately $\Sigma(f)(X \otimes P) = X \otimes \Sigma(f)(P)$.

Now we have



and



where we used in the following order that φ is an $\mathcal{M}$-transformation, that $\mu \otimes C$ is a morphism in $\mathcal{M}_A^C(\psi)$, and that δ is a morphism in $\mathcal{M}_A^C(\psi)$ (see Remark 1.2). □

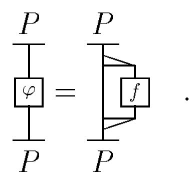
The isomorphism of the theorem can be easily expanded to an isomorphism

$$\mathrm{Nat}_{\mathcal{M}}(\omega \otimes L, \omega \otimes M) \cong \mathcal{M}(C \otimes L, A \otimes M)$$

that is natural in $L, M \in \mathcal{M}$. Also the rest of our considerations can be expanded to this situation. Since we do not have an application for this generalized situation we do not pursue it further. Furthermore there is a close relationship between natural $\mathcal{M}$-transformations as used here and natural transformations in enriched category theory where a similar theorem is known.

The following Corollary and its generalizations is a central tool of this paper. It will be used to describe natural $\mathcal{M}$-transformations by morphisms $f : C \longrightarrow A$ and we will compare morphisms from C to A by comparing their associated natural transformations.

Corollary 2.2. *For every natural (left) $\mathcal{M}$-transformation $\varphi : \omega \longrightarrow \omega$ there is a unique $f : C \longrightarrow A$ such that*



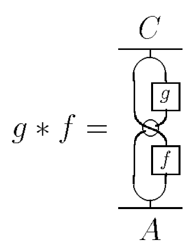
We write the composition-multiplication of natural transformations in the natural order

$$\varphi \circ \psi := (\omega \xrightarrow{\varphi} \omega \xrightarrow{\psi} \omega).$$

Remark 2.3. $\mathrm{Nat}_{\mathcal{M}}(\omega, \omega)$ is an algebra by the composition of natural $\mathcal{M}$-transformations.

Actually $\mathrm{Nat}_{\mathcal{M}}(\omega, \omega)$ is only a monoid. But we prefer to view it as an algebra in the monoidal category of sets. If $\mathcal{M}$ is the category of vector spaces, then $\mathrm{Nat}_{\mathcal{M}}(\omega, \omega)$ also has a vector space structure and is an algebra in the ordinary sense of the word. The same holds for the set $\mathcal{M}(C, A)$.

The set $\mathcal{M}(C, A)$ is an algebra with the following multiplication, called the *entwined convolution*



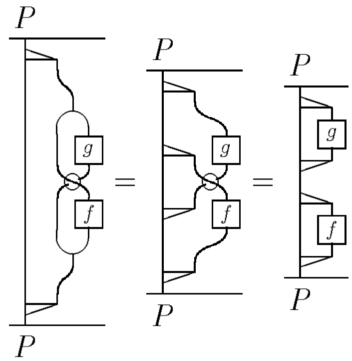
and unit $\eta\varepsilon$. This entwined convolution corresponds to the convolution multiplication on $\mathrm{Hom}(C^{cop}, A)$ if $\psi = \sigma$ the symmetry of $\mathcal{M}$.

Proposition 2.4.

$$\Sigma : \mathcal{M}(C, A) \cong \mathrm{Nat}_{\mathcal{M}}(\omega, \omega)$$

is an isomorphism of algebras.

Proof. We have

$\Sigma(g * f) =$  $= \Sigma(g) \circ \Sigma(f)$.

Obviously we also have $\Sigma(\eta\varepsilon) = \mathrm{id}_\omega$. □

Corollary 2.5. *A natural $\mathcal{M}$-transformation $\varphi : \omega \longrightarrow \omega$ is an isomorphism iff $r = \Pi(\varphi)$ is a $*$-invertible element in $\mathcal{M}(C, A)$.*

3. Tensor Products of ω and Natural Transformations

Let $\omega : \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}$ be the underlying functor. Let $\omega^2 = \omega \otimes \omega := \otimes(\omega, \omega) : \mathcal{M}_A^C(\psi) \times \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}$ denote the tensor product of ω with itself. The category $\mathcal{M}_A^C(\psi)^2 := \mathcal{M}_A^C(\psi) \times \mathcal{M}_A^C(\psi)$ is an $\mathcal{M}^2 = \mathcal{M} \times \mathcal{M}$-category by $(X, Y) \otimes (P, Q) := (X \otimes P, Y \otimes Q)$. The category $\mathcal{M}_A^C(\psi)$ is also an $\mathcal{M}^2$-category by $(X, Y) \otimes P = X \otimes Y \otimes P$. Furthermore the functor ω^2 is an $\mathcal{M}^2$-functor since $(X, Y) \otimes \omega(P) \otimes \omega(Q) \cong \omega(X \otimes P) \otimes \omega(Y \otimes Q)$.

Let τ denote the switch functor $\tau : \mathcal{M}_A^C(\psi) \times \mathcal{M}_A^C(\psi) \ni (P, Q) \mapsto (Q, P) \in \mathcal{M}_A^C(\psi) \times \mathcal{M}_A^C(\psi)$. We will also consider the functor $(\omega \otimes \omega)\tau : \mathcal{M}_A^C(\psi) \times \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}$ defined by $(\omega \otimes \omega)\tau(P, Q) := Q \otimes P$. This functor is an $\mathcal{M}^2$-functor since $\mathcal{M}$ is symmetric. In a similar way we have $\mathcal{M}^n$-functors $\omega^n : \mathcal{M}_A^C(\psi)^n \longrightarrow \mathcal{M}$ and $\omega^n \sigma : \mathcal{M}_A^C(\psi)^n \longrightarrow \mathcal{M}$ where σ is any permutation.

We are going to consider natural $\mathcal{M}^2$-transformations between functors of the type ω^2 and/or $\omega^2\tau$. First we turn to natural $\mathcal{M}^2$-endomorphisms of ω^2.

Again if $\mathcal{M} = k$-Vec, then every natural transformation $\varphi : \omega^2 \longrightarrow \omega^2$ is an $\mathcal{M}^2$-transformation by [16].

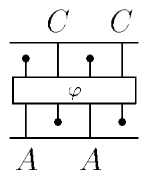
Theorem 3.1. *There is an isomorphism*

$$\mathrm{Nat}_{\mathcal{M}^2}(\omega^2, \omega^2) \cong \mathcal{M}(C \otimes C, A \otimes A).$$

Proof. As before $A \otimes C$ is an object in $\mathcal{M}_A^C(\psi)$. We define an isomorphism Π as follows

$$\Pi : \mathrm{Nat}_{\mathcal{M}^2}(\omega^2, \omega^2) \cong \mathcal{M}(C \otimes C, A \otimes A)$$

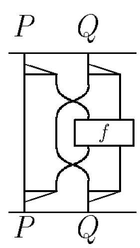
$$\Pi(\varphi) := (A \otimes \varepsilon \otimes A \otimes \varepsilon)\varphi(A \otimes C, A \otimes C)(\eta \otimes C \otimes \eta \otimes C)$$

$\Pi(\varphi) :=$  .

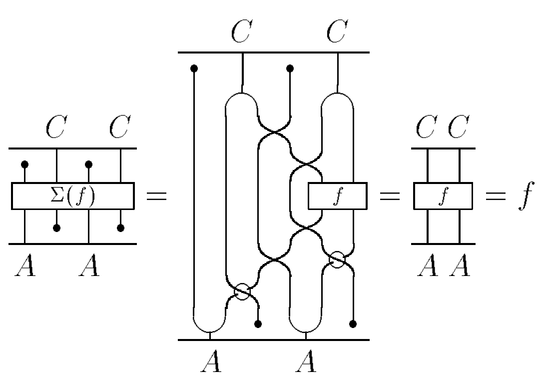
The inverse map is defined by

$$\Sigma : \mathcal{M}(C \otimes C, A \otimes A) \longrightarrow \mathrm{Nat}_{\mathcal{M}^2}(\omega^2, \omega^2)$$

$$\begin{aligned}&\Sigma(f)((P, \delta_P, \mu_P), (Q, \delta_Q, \mu_Q)) := \\ &\qquad (\mu_P \otimes \mu_Q)(P \otimes \sigma \otimes C)(P \otimes Q \otimes f)(P \otimes \sigma \otimes Q)(\delta_P \otimes \delta_Q)\end{aligned}$$

$\Sigma(f) :=$  .

A similar proof as in section 1 shows that $\Sigma(f)$ is a natural $\mathcal{M}^2$-transformation. Now we have

$\Pi(\Sigma(f)) =$ 

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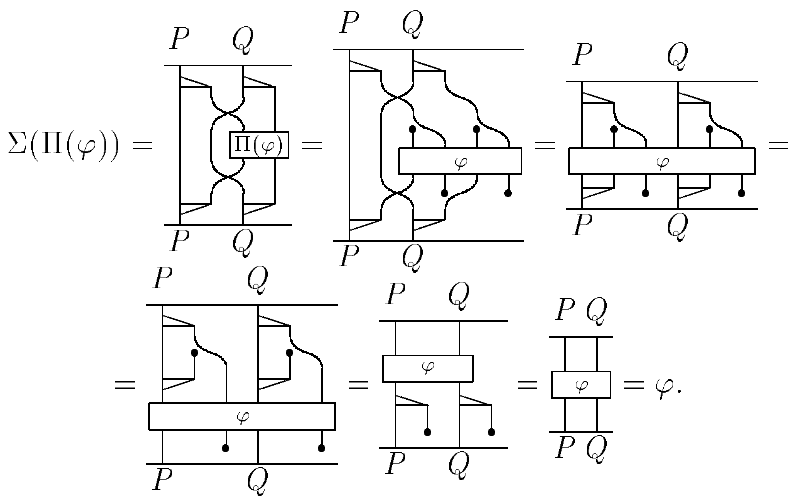
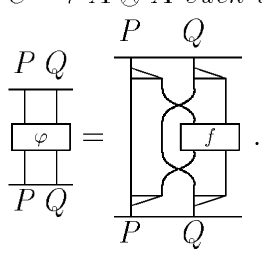


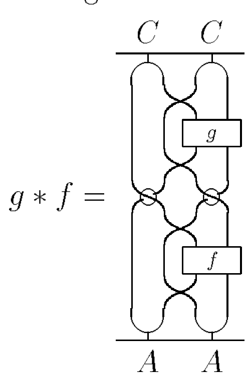
□

Corollary 3.2. *For every natural $\mathcal{M}^2$-transformation $\varphi : \omega^2 \longrightarrow \omega^2$ there is a unique $f : C \otimes C \longrightarrow A \otimes A$ such that*



Remark 3.3. As in section 1, $\mathrm{Nat}_{\mathcal{M}^2}(\omega \otimes \omega, \omega \otimes \omega)$ is an algebra by the composition of natural $\mathcal{M}^2$-transformations (written as composition in the natural order).

The set $\mathcal{M}(C \otimes C, A \otimes A)$ is an algebra with the following multiplication



and unit $\eta\varepsilon$.

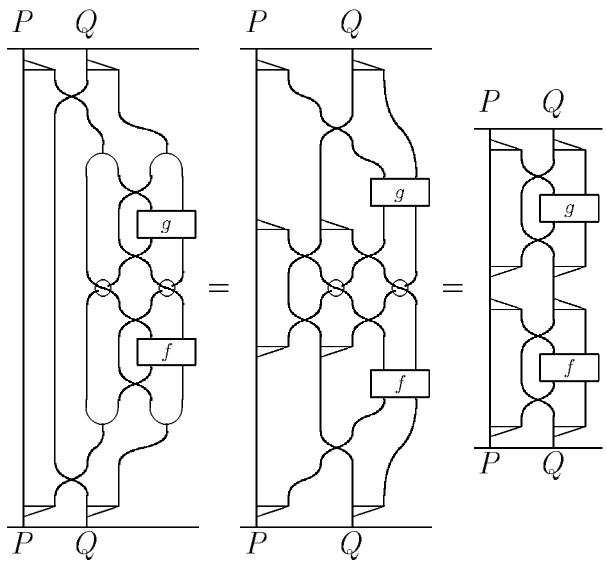
Proposition 3.4.

$$\Sigma : \mathcal{M}(C \otimes C, A \otimes A) \cong \mathrm{Nat}_{\mathcal{M}}(\omega \otimes \omega, \omega \otimes \omega)$$

is an isomorphism of algebras.

Proof.

$\Sigma(g * f) =$



$= \Sigma(g) \circ \Sigma(f).$

Obviously we have $\Sigma(\eta\varepsilon \otimes \eta\varepsilon) = \mathrm{id}$. ☐

Corollary 3.5. *A natural $\mathcal{M}^2$-transformation $\varphi : \omega \otimes \omega \longrightarrow \omega \otimes \omega$ is an isomorphism iff $r = \Pi(\varphi)$ is a $*$-invertible element in $\mathcal{M}(C \otimes C, A \otimes A)$.*

We now turn to natural $\mathcal{M}^2$-transformations from ω^2 to $\omega^2\tau$ resp. from $\omega^2\tau$ to ω^2. We are in the following situation. With the composition of natural $\mathcal{M}^2$-transformations we have algebras, bimodules, and bilinear operations

$$\begin{array}{cc} M_o := \mathrm{Nat}_{\mathcal{M}^2}(\omega^2, \omega^2), & M_t := \mathrm{Nat}_{\mathcal{M}^2}(\omega^2\tau, \omega^2\tau), \\ M_{ot} := {}_{M_o}\mathrm{Nat}_{\mathcal{M}^2}(\omega^2, \omega^2\tau)_{M_t}, & M_{to} := {}_{M_t}\mathrm{Nat}_{\mathcal{M}^2}(\omega^2\tau, \omega^2)_{M_o}, \\ M_{ot} \times M_{to} \longrightarrow M_o, & M_{to} \times M_{ot} \longrightarrow M_t. \end{array}$$

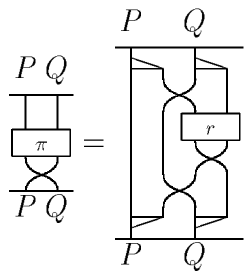
Let $\sigma : X \otimes Y \longrightarrow Y \otimes X$ denote the symmetry of $\mathcal{M}$. Then we have an isomorphism $s : M_o \ni \varphi \mapsto \varphi\sigma \in M_{ot}$ compatible with the left module structure over M_o. We also have an isomorphism $s' : M_o \ni \varphi \mapsto \sigma\varphi \in M_{to}$ compatible with the right module structure over M_o. Furthermore the isomorphism $\bar{s} : M_o \ni \varphi \mapsto \sigma\varphi\sigma \in M_t$ is compatible with the algebra structure.

These isomorphisms induce an isomorphism

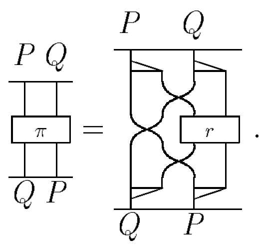
$$\Pi : \mathrm{Nat}_{\mathcal{M}^2}(\omega^2, \omega^2\tau) \cong \mathrm{Nat}_{\mathcal{M}^2}(\omega^2, \omega^2) \cong \mathcal{M}(C \otimes C, A \otimes A)$$

with inverse Σ. Since σ is an isomorphism we get

Theorem 3.6. *For every natural $\mathcal{M}^2$-transformation $\pi : \omega(P) \otimes \omega(Q) \longrightarrow \omega(Q) \otimes \omega(P)$ there is a unique $r : C \otimes C \longrightarrow A \otimes A$ such that*



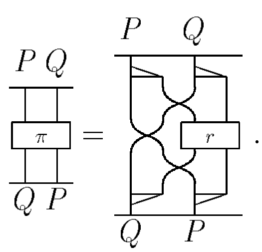
or equivalently



We also get an isomorphism

$$\Pi : \mathrm{Nat}_{\mathcal{M}^2}(\omega^2\tau, \omega^2) \cong \mathrm{Nat}_{\mathcal{M}^2}(\omega^2, \omega^2) \cong \mathcal{M}(C \otimes C, A \otimes A)$$

with inverse Σ that leads to the same condition



The induced isomorphism

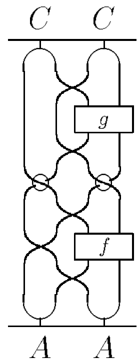
$$\Pi : \mathrm{Nat}_{\mathcal{M}^2}(\omega^2\tau, \omega^2\tau) \cong \mathrm{Nat}_{\mathcal{M}^2}(\omega^2, \omega^2) \cong \mathcal{M}(C \otimes C, A \otimes A)$$

is the same isomorphism as that of Corollary 3.2 and preserves the multiplication.

Corollary 3.7. *The action $M_o \times M_{ot} \longrightarrow M_{ot}$ induces a left module action $\mathcal{M}(C \otimes C, A \otimes A) \times \mathcal{M}(C \otimes C, A \otimes A) \ni (g, f) \mapsto g * f \in$*

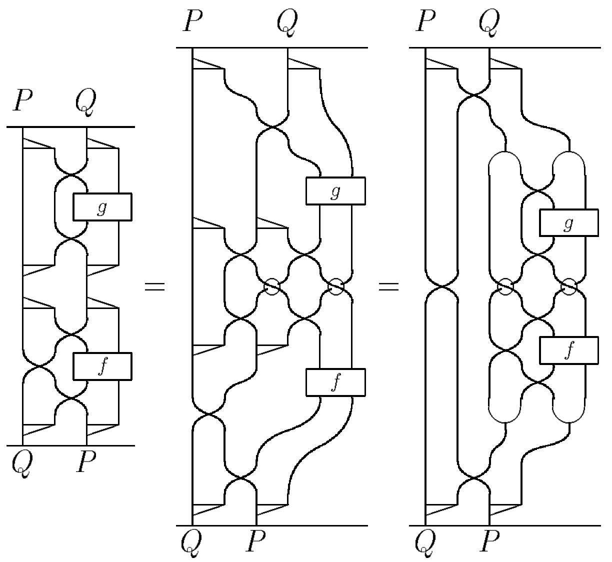
$\mathcal{M}(C \otimes C, A \otimes A)$ by

$$g * f =$$



Proof. Let $\varphi = \Sigma(g) \in M_o$ and $\psi = \Sigma(f) \in M_{ot}$ be given. Then

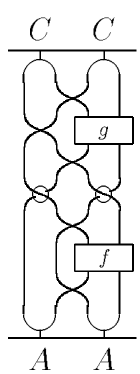
$$\varphi \circ \psi =$$



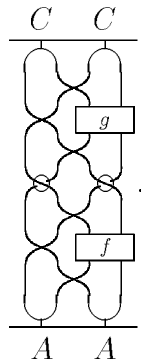
□

In an analogous way M_{to} is a right M_o-module by

$$g * f =$$



and the operation $M_{ot} \times M_{to} \longrightarrow M_o$ (as well as $M_{to} \times M_{ot} \longrightarrow M_t$) is given by

$g * f =$  . (3)

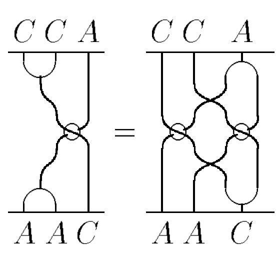
Similar equalities hold for higher tensor powers of ω together with permutations of the arguments.

4. Tensor Products of Entwined Modules

If A is a bialgebra, then the tensor product of two A-modules is again an A-module by the diagonal multiplication. Similarly, if C is a bialgebra, then the tensor product of two C-comodules is a C-comodule by the codiagonal comultiplication. Furthermore I is a unit object for the tensor product if endowed with the trivial A-structure resp. the trivial C-structure. We want to study conditions under which $\mathcal{M}_A^C(\psi)$ becomes a monoidal category with the given multiplication and comultiplication on the tensor product of two modules. The underlying functor will then preserve the tensor product, i.e. it will be a monoidal functor.

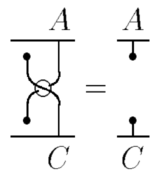
Theorem 4.1. *Let A and C be bialgebras. The category $\mathcal{M}_A^C(\psi)$ is monoidal iff the following additional compatibility conditions for the entwining map $\psi : C \otimes A \longrightarrow A \otimes C$ hold:*

$$(\Delta_A \otimes C)\psi(\nabla_C \otimes A) =$$
$$(A \otimes A \otimes \nabla_C)(A \otimes \sigma \otimes C)(\psi \otimes \psi)(C \otimes \sigma \otimes A)(C \otimes C \otimes \Delta_A)$$

 (4)

and

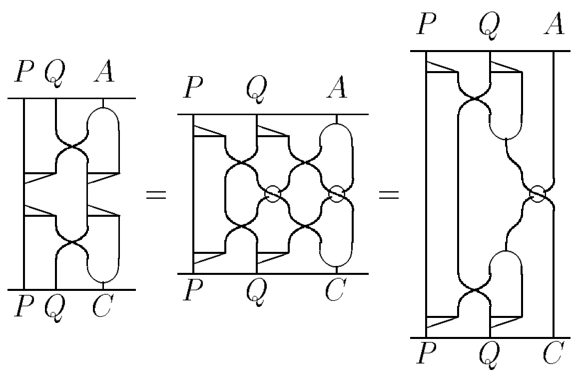
$$(\varepsilon_A \otimes C)\psi(\eta_C \otimes A) = \varepsilon_A \otimes \eta_C$$



(5)

If these conditions are satisfied we call (A, C, ψ) *a* monoidal entwining structure *and* ψ *a* monoidal entwining map. *The tensor product* $P \otimes Q$ *of modules* $P, Q \in \mathcal{M}_A^C(\psi)$ *becomes an object in* $\mathcal{M}_A^C(\psi)$ *with the diagonal module and the codiagonal comodule structure.*

Proof. Let $\psi : C \otimes A \longrightarrow A \otimes C$ be as above. Then we get for $P, Q \in \mathcal{M}_A^C(\psi)$:

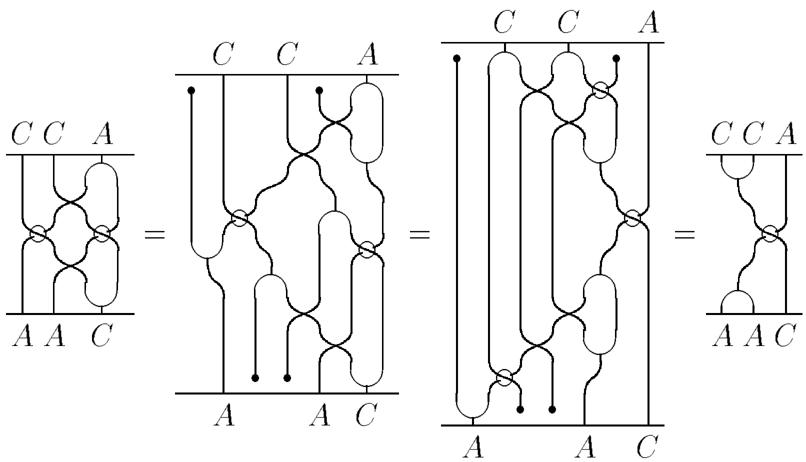


so that $P \otimes Q$ is an entwined module. The unit object I is an entwined module with structure maps $\mu_I := I \otimes \varepsilon_A$ and $\delta_I := I \otimes \eta_C$. Here the second condition for ψ is used.

Now assume that $\mathcal{M}_A^C(\psi)$ is a monoidal category with the tensor product as in $\mathcal{M}$ and the diagonal action of A and codiagonal coaction of C.

I is the unit object iff the condition (5) holds. Condition (4) follows by considering the tensor product of $A \otimes C$ and $C \otimes A$ with the structures defined in Remark 1.2. We use $A \otimes C \otimes C \otimes A$ as an object in $\mathcal{M}_A^C(\psi)$

and get by applying ε_C and η_A



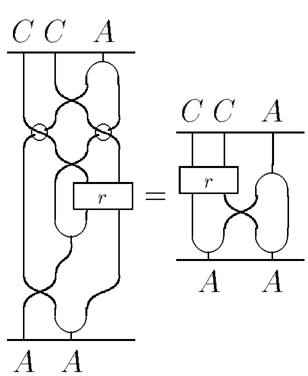
i.e. condition (4). □

5. Double Quantum Groups and Braidings

Now we want to introduce a braiding $\pi(P,Q) : \omega(P)\otimes\omega(Q) \longrightarrow \omega(Q)\otimes\omega(P)$ in $\mathcal{M}_A^C(\psi)$. We consider only braidings that are natural $\mathcal{M}^2$-transformations. If $\mathcal{M} = k$-Vec this is no restriction by [16] since $\mathrm{Nat}(\omega^2, \omega^2\tau) = \mathrm{Nat}_{\mathcal{M}^2}(\omega^2, \omega^2\tau)$. By Theorem 3.6 such a braiding is uniquely determined by a morphism $r : C \otimes C \longrightarrow A \otimes A$.

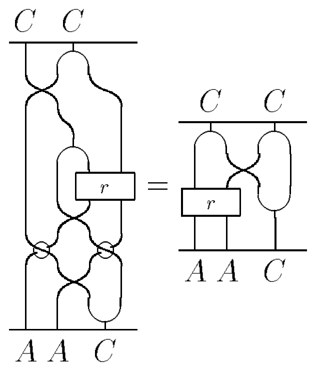
Proposition 5.1. *Let* $\pi(P,Q) : \omega(P) \otimes \omega(Q) \longrightarrow \omega(Q) \otimes \omega(P)$ *be a natural* $\mathcal{M}^2$*-transformation with associated morphism* $r = \Pi(\pi) : C\otimes C \longrightarrow A \otimes A$.

a) π *is a morphism of* A*-modules iff*



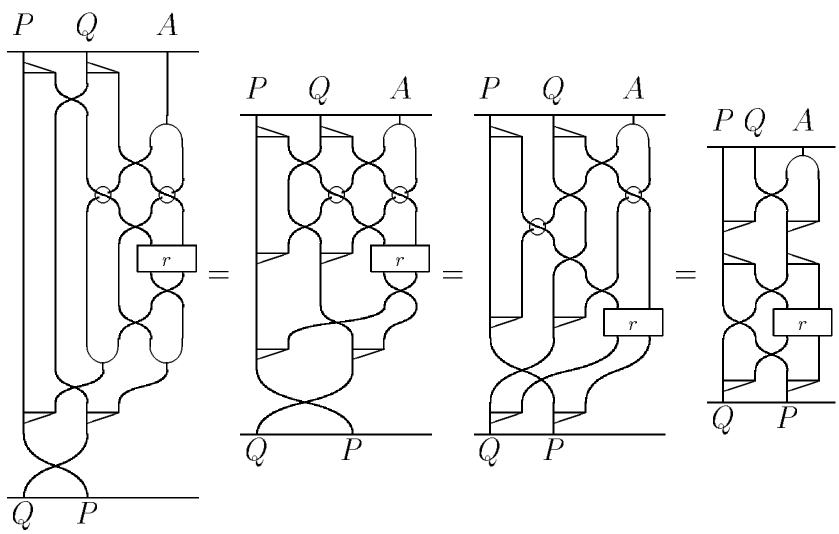
. (6)

b) π *is a morphism of* C*-comodules iff*

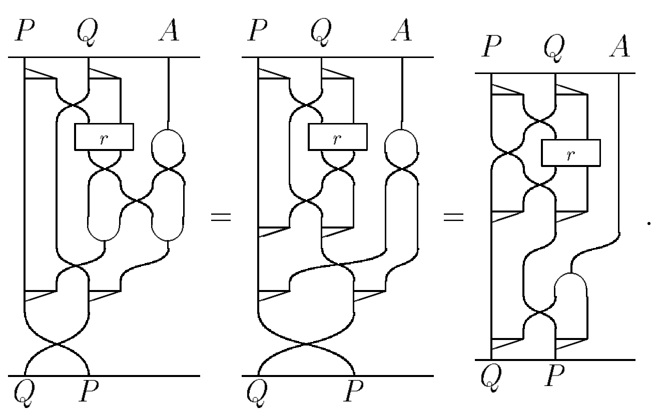


(7)

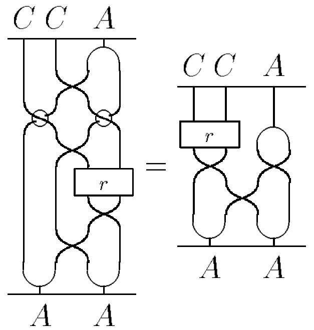
Proof. a) We use the diagonal action of A on the domain or the range of the natural transformation $\pi : \omega(P) \otimes \omega(Q) \longrightarrow \omega(Q) \otimes \omega(P)$ as in the preceding Theorem. We consider the following two equations



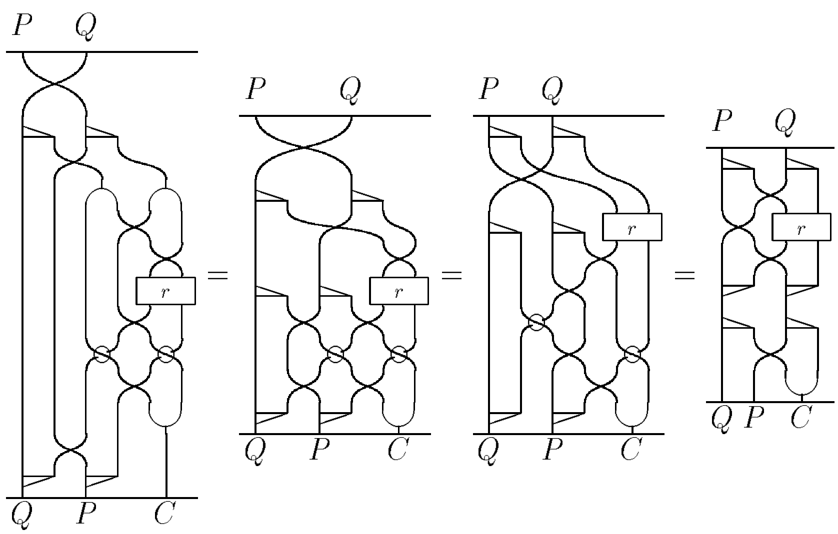
and



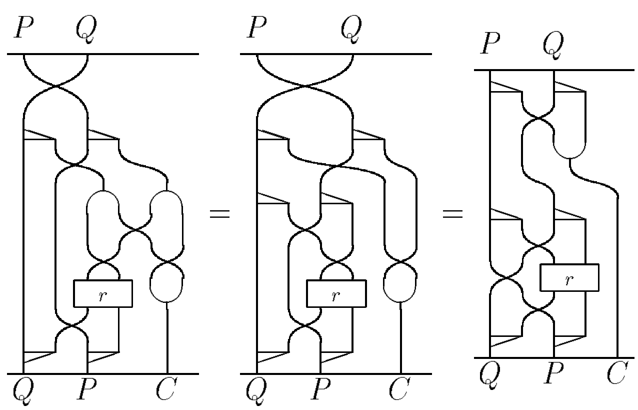
The right hand sides are equal if and only if $\pi(P, Q)$ is an A-module homomorphism. By canceling the last symmetry of $\omega \otimes \omega$ and by applying the uniqueness of Corollary 3.2 we get that the left hand sides are equal if and only if



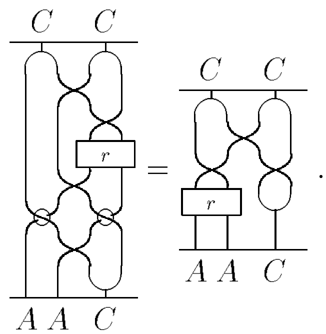
This is equivalent to (6). b) follows from the (top-bottom) dual calculation. The two equations



and



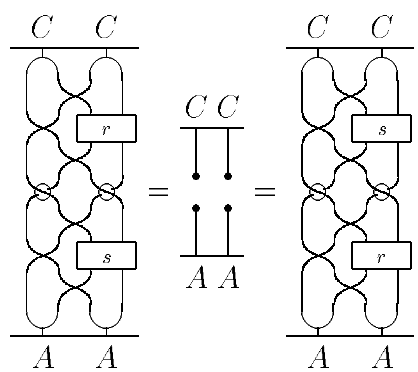
show that $\pi(P,Q)$ is a C-comodule homomorphism if and only if



This is equivalent to (7). ☐

Observe that if (6) and (7) are satisfied for $r : C \otimes C \longrightarrow A \otimes A$, then $\pi = \Sigma(r) : P \otimes Q \longrightarrow Q \otimes P$ is a natural transformation in $\mathcal{M}_A^C(\psi)$.

Proposition 5.2. *For $P, Q \in \mathcal{M}_A^C(\psi)$, a natural $\mathcal{M}^2$-transformation $\pi(P,Q) = \Sigma(r) : \omega(P) \otimes \omega(Q) \longrightarrow \omega(Q) \otimes \omega(P)$ for $r : C \otimes C \longrightarrow A \otimes A$ is an isomorphism if and only if there exists a map $s : C \otimes C \longrightarrow A \otimes A$ such that*



(8)

Proof. This is a consequence of the multiplication on $\mathcal{M}(C \otimes C, A \otimes A)$ described in (3). This multiplication corresponds to the map

$$\mathrm{Nat}_{\mathcal{M}^2}(\omega^2, \omega^2\tau) \times \mathrm{Nat}_{\mathcal{M}^2}(\omega^2\tau, \omega^2) \longrightarrow \mathrm{Nat}_{\mathcal{M}^2}(\omega^2, \omega^2).$$

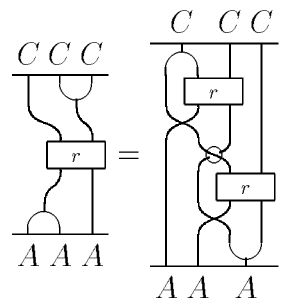
The inverse of $\pi : \omega^2 \longrightarrow \omega^2\tau$ lies in $\mathrm{Nat}_{\mathcal{M}^2}(\omega^2\tau, \omega^2)$. ☐

Proposition 5.3. *A natural $\mathcal{M}^2$-transformation $\pi(P,Q) = \Sigma(r)$: $\omega(P) \otimes \omega(Q) \longrightarrow \omega(Q) \otimes \omega(P)$ for $r : C \otimes C \longrightarrow A \otimes A$, is compatible with tensor products*

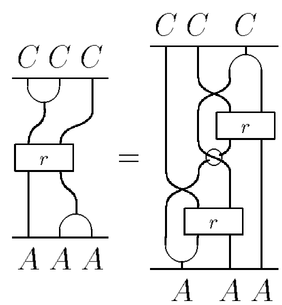
$$(\pi(P,Q) \otimes R) \circ (Q \otimes \pi(P,R)) = \pi(P, Q \otimes R)$$

$$(P \otimes \pi(Q,R)) \circ (\pi(P,R) \otimes Q) = \pi(P \otimes Q, R)$$

if and only if the following hold for r :



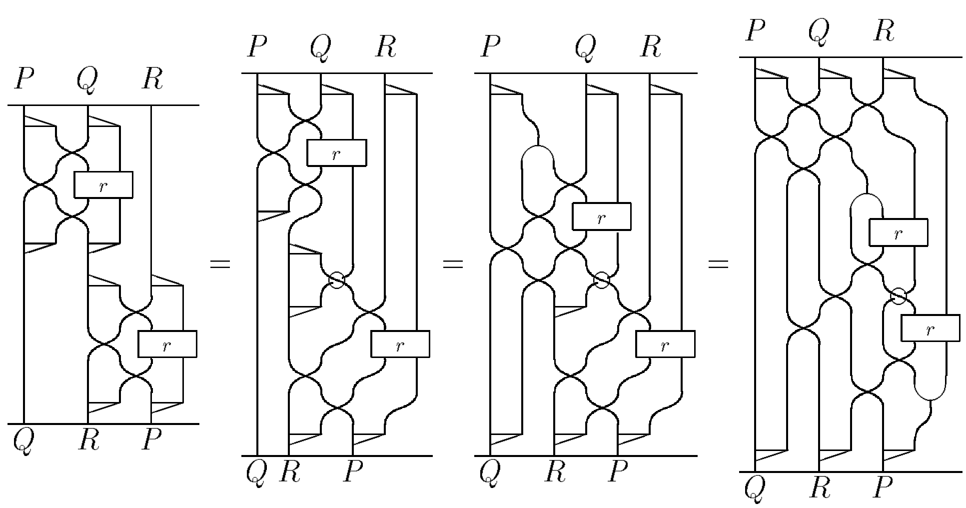
(9)

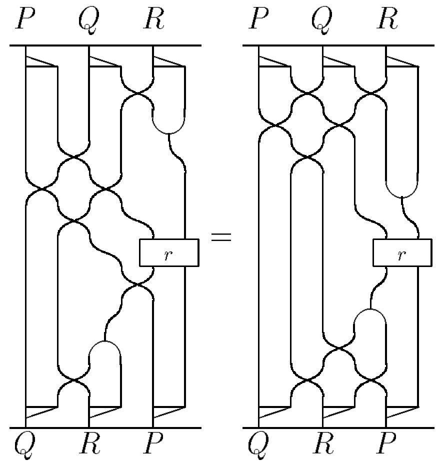


(10)

Proof. The first condition follows from the fact that the natural transformation $\omega(P) \otimes \omega(Q) \otimes \omega(R) \longrightarrow \omega(Q) \otimes \omega(R) \otimes \omega(P)$ is given by a unique morphism $t : C \otimes C \otimes C \longrightarrow A \otimes A \otimes A$ and

$$(\pi(P,Q) \otimes R) \circ (Q \otimes \pi(P,R)) =$$



$\pi(P, Q \otimes R) =$ 

The proof for the second condition is similar. □

Now we use the notation $\psi(c \otimes a) := a_\alpha \otimes c^\alpha$ introduced in [3] for an entwining map in $\mathcal{M} = k$-Vec and write $r(c \otimes d) := r^1(c \otimes d) \otimes r^2(c \otimes d) := R^1(c \otimes d) \otimes R^2(c \otimes d)$ for a morphism $r : C \otimes C \longrightarrow A \otimes A$. For the coalgebra structure and cooperation of C we use Sweedler notation $\Delta(c) := c_{(1)} \otimes c_{(2)}$ for $c \in C$ and $\delta(m) := m_{(0)} \otimes m_{(1)}$ for $m \in M$ respectively.

Definition 5.4. A pair of Hopf algebras C and A together with a monoidal entwining map $\psi : C \otimes A \longrightarrow A \otimes C$ (such that $\mathcal{M}_A^C(\psi)$ is a monoidal category) and together with a morphism $r : C \otimes C \longrightarrow A \otimes A$ is called a *double quantum group* if the following identities hold

1) $r(c \otimes d)\Delta(a) = (a_{(2)\alpha} \otimes a_{(1)\beta})r(c^\beta \otimes d^\alpha)$,
2) $r(c_{(1)} \otimes d_{(1)}) \otimes c_{(2)}d_{(2)} =$
   $r^1(c_{(2)} \otimes d_{(2)})_\alpha \otimes r^2(c_{(2)} \otimes d_{(2)})_\beta \otimes {d_{(1)}}^\alpha {c_{(1)}}^\beta$,
3) $(\Delta \otimes A)r(C \otimes \nabla)(c \otimes d \otimes e) =$
   $R^1(c_{(2)} \otimes d) \otimes r^1({c_{(1)}}^\alpha \otimes e) \otimes R^2(c_{(2)} \otimes d)_\alpha r^2({c_{(1)}}^\alpha \otimes e)$,
4) $(A \otimes \Delta)r(\nabla \otimes C)(c \otimes d \otimes e) =$
   $R^1(d \otimes e_{(2)})_\alpha r^1(c \otimes e^\alpha_{(1)}) \otimes r^2(c \otimes e^\alpha_{(1)}) \otimes R^2(d \otimes e_{(2)})$,
5) there exists a map $s : C \otimes C \longrightarrow A \otimes A$ such that
   $r^2(c_{(2)} \otimes d_{(2)})_\alpha s^1({d_{(1)}}^\beta \otimes {c_{(1)}}^\alpha) \otimes r^1(c_{(2)} \otimes d_{(2)})_\beta s^2({d_{(1)}}^\beta \otimes {c_{(1)}}^\alpha)$
   $= \eta\varepsilon(c) \otimes \eta\varepsilon(d) =$
   $s^2(c_{(2)} \otimes d_{(2)})_\alpha r^1({d_{(1)}}^\beta \otimes {c_{(1)}}^\alpha) \otimes s^1(c_{(2)} \otimes d_{(2)})_\beta r^2({d_{(1)}}^\beta \otimes {c_{(1)}}^\alpha)$.

So in this section we have shown the following

Theorem 5.5. *Let C and A be bialgebras and $\psi : C \otimes A \longrightarrow A \otimes C$ be a monoidal entwining map such that $\mathcal{M}_A^C(\psi)$ is a monoidal category. Then $\mathcal{M}_A^C(\psi)$ is braided with the braid map*

$$\pi(P, Q) : P \otimes Q \longrightarrow Q \otimes P,\ p \otimes q \mapsto (q_{(0)} \otimes p_{(0)})r(p_{(1)} \otimes q_{(1)})$$

iff the associated morphism $r : C \otimes C \longrightarrow A \otimes A$ *satisfies the axioms* 1) *to* 5) *in* Definition 5.4*.*

Proof. Indeed, conditions 1) to 5) of Definition 5.4 are the same as the conditions derived in Propositions 5.1, 5.2, and 5.3. □

6. Quantum Group Doubles

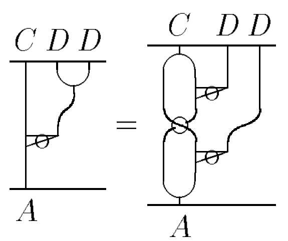
For finite-dimensional Hopf algebras H one knows that the category of Yetter-Drinfeld modules is equivalent as a braided monoidal category to the category of modules over the Drinfeld double $\mathcal{YD}_H^H \cong \mathcal{M}_{D(H)}$. In this section we want to generalize this result to the braided monoidal category $\mathcal{M}_A^C(\psi)$.

Let (C, A, ψ) be an entwining structure. Assume furthermore that the functor $\mathcal{M}(C \otimes V, A) : \mathcal{M} \longrightarrow \text{Set}$ (as a functor in V) is representable by an object $D = \mathcal{M}[C, A] \in \mathcal{M}$:

$$\mathcal{M}(C \otimes V, A) \cong \mathcal{M}(V, \mathcal{M}[C, A]).$$

This is obviously true for $\mathcal{M} = k$-Vec, take $D = \text{Hom}_k(C, A)$. Then there is a universal action $C \otimes D \longrightarrow A$, the counterimage of the identity $\text{id}_D \in \mathcal{M}(D, D)$.

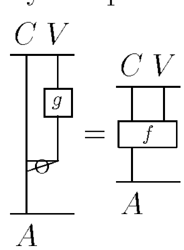
Theorem 6.1. $D := \mathcal{M}[C, A]$ *is an algebra with a* (smash product) *multiplication uniquely defined by*



(11)

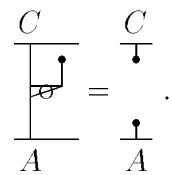
Furthermore there is a functor $\mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}_D$ *compatible with the underlying functors to* $\mathcal{M}$*.*

Proof. Since D represents the functor $\mathcal{M}(C \otimes V, A)$ we have a unique morphism $g : V \longrightarrow D$ for every morphism $f : C \otimes V \longrightarrow A$ such that

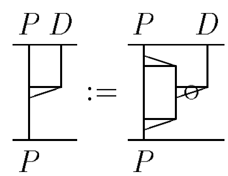


holds.

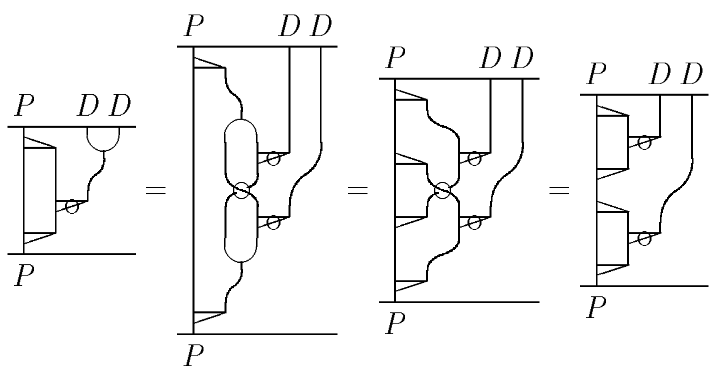
Thus diagram (11) of the theorem uniquely defines a multiplication on D. Since the right hand side of the diagram represents the multiplication on $\mathcal{M}(C,A)$ (Remark 2.3) the associativity of $\mathcal{M}(C,A)$ holds also for $D=\mathcal{M}[C,A]$. The unit of D is defined by the diagram



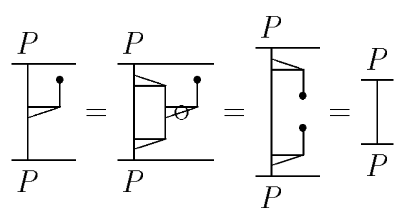
If P is a module in $\mathcal{M}_A^C(\psi)$, then it is also a D-module by



since

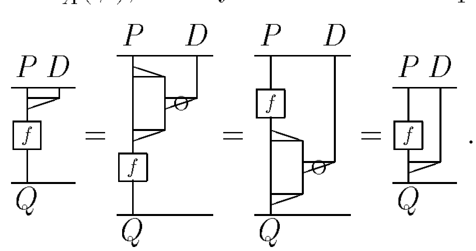


and



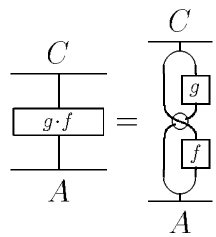
hold.

If $f: P \longrightarrow Q$ is in $\mathcal{M}_A^C(\psi)$, then f is also a D-morphism since



□

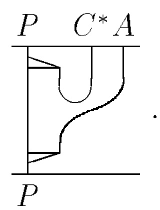
If $\mathcal{M} = k$-Vec, then $\mathcal{M}(C, A) = \mathcal{M}[C, A] = \mathrm{Hom}(C, A)$ as algebras. In fact we have two multiplications on $\mathrm{Hom}(C, A)$, the entwined convolution on $\mathcal{M}(C, A)$ used in Proposition 2.4 and the induced (smash product) multiplication on $\mathcal{M}[C, A]$ in Theorem 6.1. The universal operation of $\mathrm{Hom}(C, A)$ on C is given as evaluation map $C \otimes \mathrm{Hom}(C, A) \longrightarrow A$. Thus the multiplication defined in (11) on $\mathrm{Hom}(C, A)$ is described by



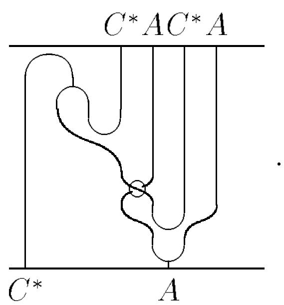
and turns out to be the entwined convolution.

Assume now that C has a (right) dual in $\mathcal{M}$, i.e. there is an object C^* and there are morphisms $\mathrm{ev} : C \otimes C^* \longrightarrow I$ and $\mathrm{db} : I \longrightarrow C^* \otimes C$ such that $(\mathrm{ev} \otimes C)(C \otimes \mathrm{db}) = \mathrm{id}_C$ and $(C^* \otimes \mathrm{ev})(\mathrm{db} \otimes C^*) = \mathrm{id}_{C^*}$. For $\mathcal{M} = k$-Vec this means that C is finite dimensional.

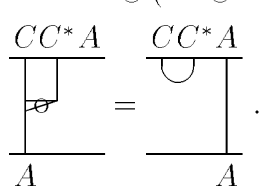
Lemma 6.2. *$C^* \otimes A \cong \mathcal{M}[C, A]$ represents the functor $\mathcal{M}(C \otimes V, A)$. The induced operation of $C^* \otimes A$ on a module $P \in \mathcal{M}_A^C(\psi)$ is*



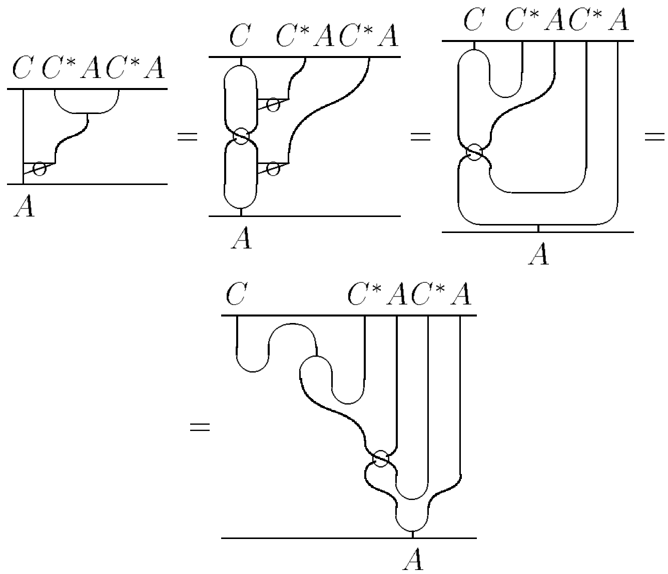
Furthermore the induced multiplication on $C^* \otimes A$ is



Proof. The universal operation $C \otimes (C^* \otimes A) \longrightarrow A$ is given by



Then we have for the definition of the multiplication on $C^* \otimes A$



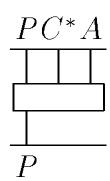
(12)

which proves the assertion by the uniqueness of the multiplication as shown in Theorem 6.1. □

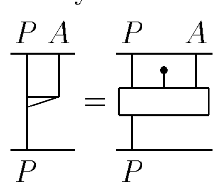
So we have seen that $C^* \otimes A$ and $\mathcal{M}[C, A]$ are canonically isomorphic as algebras with the multiplication from Theorem 6.1 resp. from Lemma 6.2.

Corollary 6.3. *The categories $\mathcal{M}_A^C(\psi)$ and $\mathcal{M}_{C^* \# A}$ are equivalent.*

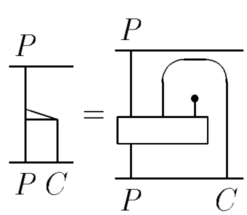
Proof. Let P be a $C^* \otimes A$-module with the operation



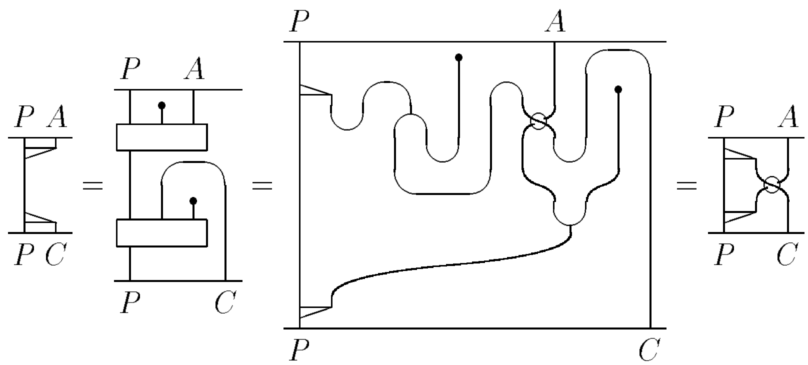
Then it becomes an A-module by



and a C-comodule by



since $A \longrightarrow C^* \otimes A$ and $C^* \longrightarrow C^* \otimes A$ are homomorphisms of algebras. The compatibility condition for objects in $\mathcal{M}_A^C(\psi)$ follows from

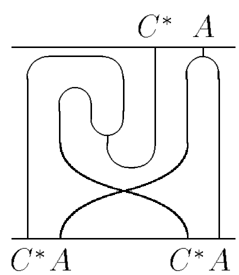


where we applied the smash product multiplication from Lemma 6.2 in the second step. Now it is easy to check that the two structures from $\mathcal{M}_A^C(\psi)$ resp. $\mathcal{M}_{C^*\otimes A}$ are in 1-1-correspondence and that the same holds for morphisms. □

Theorem 6.4. *Assume that (C, A, ψ, r) is a double quantum group and that C has a dual C^*. Then $\mathcal{M}[C, A]$ is a quasitriangular Hopf algebra, the* quantum group double of C and A, *and*

$$\mathcal{M}_A^C(\psi) \cong \mathcal{M}_{\mathcal{M}[C,A]}$$

as braided monoidal categories. The multiplication of $\mathcal{M}[C, A]$ is given in Theorem 6.1 resp. Lemma 6.2. The comultiplication for $C^* \otimes A \cong \mathcal{M}[C, A]$ is



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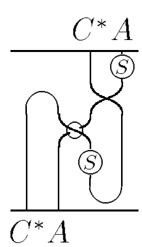
The antipode $S_{\mathcal{M}[C,A]}$ is given by



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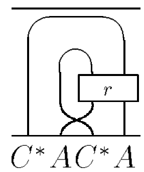
r defines a braiding on $\mathcal{M}_A^C(\psi)$ and therefore gives the R-matrix



Proof. The algebra structure of $\mathcal{M}[C, A] \cong C^* \otimes A$ was discussed above. The equivalence of the two categories is compatible with the underlying functors to $\mathcal{M}$. The tensor product of $\mathcal{M}_A^C(\psi)$ is carried over by the equivalence to a tensor product in $\mathcal{M}_{\mathcal{M}[C,A]}$ that is compatible with the underlying functor. Thus we know that $\mathcal{M}[C, A]$ is a bialgebra [15]. Furthermore $\mathcal{M}[C, A]$ is quasitriangular since $\mathcal{M}_{\mathcal{M}[C,A]}$ is braided. It is a straightforward computation to check the properties of the antipode and the R-matrix. □

We should mention here that the requirements that the A-module structure is diagonal on the tensor factors and that the C-comodule structure is codiagonal amount to saying that the underlying functors $\mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}_A$ and $\mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}^C$ are monoidal functors.

7. Examples

Example 7.1. Let (H, A, C) be a right-right Doi-Hopf datum, i.e. H is a Hopf algebra, A a right H-comodule algebra and C a right H-module coalgebra. As observed in [3] the category of Doi-Hopf modules $\mathcal{M}(H)_A^C$ [7] whose objects are right A-modules and right C-comodules with the compatibility condition

$$(ma)_{(0)} \otimes (ma)_{(1)} = m_{(0)}a_{(0)} \otimes m_{(1)}a_{(1)}$$

is equal to $\mathcal{M}_A^C(\psi)$ with the entwining map $\psi : c \otimes a \mapsto a_{(0)} \otimes ca_{(1)}$. Our results on the unique form of the braiding extend the computations done in [5] for the left-right case. We proved that every braiding in $\mathcal{M}(H)_A^C$ is given in the explicit form

$$\pi(P, Q) : P \otimes Q \longrightarrow Q \otimes P,\ p \otimes q \mapsto (q_{(0)} \otimes p_{(0)})r(p_{(1)} \otimes q_{(1)})$$

where r satisfies the conditions from Definition 5.4.

Example 7.2. Let A be a bialgebra, $C := k$, $\psi := id_A$, and consider the monoidal category $\mathcal{M}_A = \mathcal{M}_A^C(\psi)$ of right A-modules. The above equations show that $\mathcal{M}_A$ is braided if and only if there exists an invertible element $R = R^1 \otimes R^2 = r = r^1 \otimes r^2 \in A \otimes A$ with the properties

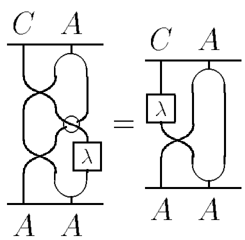
$$R(\Delta(a)) = (\tau\Delta(a))R$$

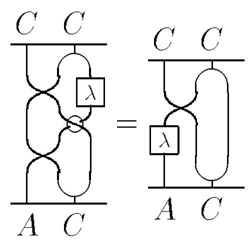
$$(\Delta \otimes A)(R) = R^1 \otimes r^1 \otimes R^2 r^2$$
$$(A \otimes \Delta)(R) = R^1 r^1 \otimes r^2 \otimes R^2$$

and that the braiding is given by applying the twist map and multiplying with R. This means that (k, A, id_A) is a double quantum group iff A is a quasitriangular bialgebra with R-matrix R.

Example 7.3. In a similar way by setting $A := k$ and $\psi := id_C$ for a bialgebra C we see that the category $\mathcal{M}^C$ of right C-comodules is braided if and only if C is coquasitriangular.

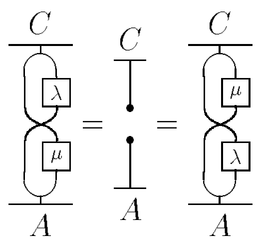
Remark 7.4. We want to compute the conditions for the existence of a braiding on $\mathcal{M}_A^C(\psi)$ given by a map $r := \eta_A \varepsilon_C \otimes \lambda$ for $\lambda \in \mathrm{Hom}(C, A)$. In this special case the conditions for A-linearity and C-colinearity from Proposition 5.1 can be simplified to



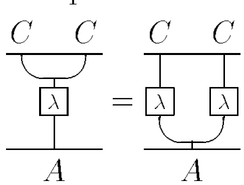
(6') and


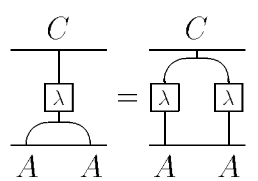
(7')

If λ is twisted convolution invertible, that is there exists a map $\mu \in \mathrm{Hom}(C, A)$ such that



then the condition (8) in Proposition 5.2 is satisfied by taking $s := \mu \otimes \eta_A \varepsilon_C$ for the inverse braiding. The compatibility conditions in Proposition 5.3 turn out to be equivalent to λ being multiplicative and comultiplicative :



(9') and


. (10')

Example 7.5. An example for the above considered simpler situation is the category of right-right Yetter-Drinfeld modules $\mathcal{YD}_H^H$ over a Hopf algebra H with bijective antipode S. As we can derive from [9] and [6] $\mathcal{YD}_H^H$ is equal to the category of Doi-Hopf modules $\mathcal{M}_H^H(H^{op} \otimes H)$ where the right $H^{op} \otimes H$-module structure on H is given by $h \cdot (f \otimes g) := fhg$

and the right $H^{op} \otimes H$-comodule structure is $\delta(h) := h_{(2)} \otimes S(h_{(1)}) \otimes h_{(3)}$. So with the entwining map $\psi : H \otimes H \longrightarrow H \otimes H, g \otimes h \mapsto h_{(2)} \otimes S(h_{(1)}) g h_{(3)}$ we have $\mathcal{YD}_H^H = \mathcal{M}_H^H(\psi)$ (see also [3]). It is well-known ([20]) that $\mathcal{YD}_H^H$ is braided by $\pi(M,N) : M \otimes N \longrightarrow N \otimes M, m \otimes n \mapsto n_{(0)} \otimes m n_{(1)}$ which corresponds to the map $r := g \otimes h \mapsto \eta\varepsilon(g) \otimes id(h)$ [5]. Since the inverse of the antipode S^{-1} is the twisted convolution inverse of the identity map and the above conditions are satisfied by ψ we see that the inverse of the braiding is given by $S^{-1} \otimes \eta\varepsilon$, that is $\pi^{-1}(M,N)(m \otimes n) = n S^{-1}(m_{(1)}) \otimes m_{(0)}$.

Example 7.6. Besides Doi-Hopf modules there is another category of modules which can be considered as entwined modules. For an algebra A and a coalgebra C in $\mathcal{M}$ simply take the symmetry σ as trivial entwining map. A right A-module and right C-comodule M becomes an object in $\mathcal{M}_A^C(\sigma)$ if

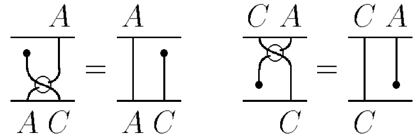
$$\delta(ma) = m_{(0)} \cdot a \otimes m_{(1)}$$

Modules with this property are called *dimodules* and were studied by Fred Long [10]. An example can be obtained by considering a group algebra $kG =: A =: C$ and the category of G-graded modules $M = \bigoplus_{g \in G} M_g$ such that each M_g is a G-module. This is equivalent to a kG-comodule structure on M with $\delta(m) = m \otimes g$ for all $m \in M_g$, $g \in G$, and we get $\delta(m \cdot h) = m \cdot h \otimes g$ for all $m \in M_g$ since $m \cdot h \in M_g$.

Recently P. Schauenburg [19] showed that for infinite dimensional A there exist entwining structures which do not come from a Doi-Hopf datum. So our results on monoidal structures in $\mathcal{M}_A^C(\psi)$ apply to a larger variety of categories. We should point out, however, that in this example the connecting Hopf algebra is just the base field k, that is A-C-dimodules are Doi-Hopf modules over (k, A, C).

Now let A be a quasitriangular bialgebra with R-matrix $R = R^1 \otimes R^2$ and inverse $S = S^1 \otimes S^2$ and C be a coquasitriangular bialgebra by $r : C \otimes C \longrightarrow k$ with convolution inverse s. Then it is easy to check that the map $c \otimes d \mapsto r(c \otimes d) R^1 \otimes R^2$ satisfies the above equations and therefore defines a braiding on the category $\mathcal{M}_A^C(\sigma)$ with inverse given by $c \otimes d \mapsto s(d \otimes c) S^2 \otimes S^1$.

Example 7.7. If (C, A, ψ, r) is a double quantum group and the equations



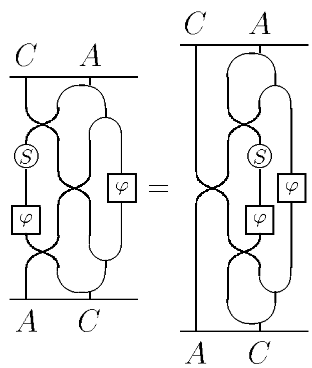
hold for ψ, then $(A, r(\eta_C \otimes \eta_C))$ is a quasitriangular Hopf algebra and $(C, (\varepsilon_A \otimes \varepsilon_A) r)$ is a coquasitriangular Hopf algebra. This is easily seen by directly applying the conditions for a double quantum group. In particular the category of A-modules as well as the category of C-comodules is a full, braided subcategory of $\mathcal{M}_A^C(\psi)$ by using the trivial coaction with C resp. the trivial action with A. The additional condition on ψ is essential and the category of Yetter-Drinfeld modules is an immediate counterexample. The trivial entwining structure σ fulfills this condition and so we find that the category of dimodules from the previous example is monoidal and braided if and only if A is quasitriangular and C is coquasitriangular.

The following proposition can be used to obtain many examples of monoidal entwining structures.

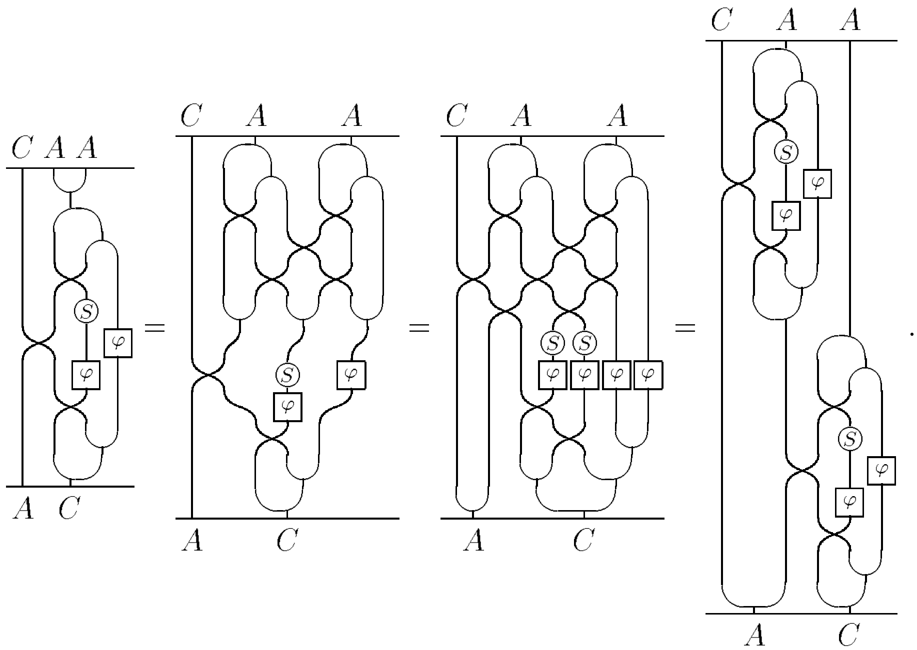
Proposition 7.8. *Let A, C be Hopf algebras and let $\varphi : A \longrightarrow C$ be a bialgebra homomorphism. Then $\psi : C \otimes A \longrightarrow A \otimes C$ with $\psi(c \otimes a) := \sum a_{(2)} \otimes S(\varphi(a_{(1)}))c\varphi(a_{(3)})$ is a monoidal entwining map.*

Proof. We check the axioms

(1a) The entwining map is



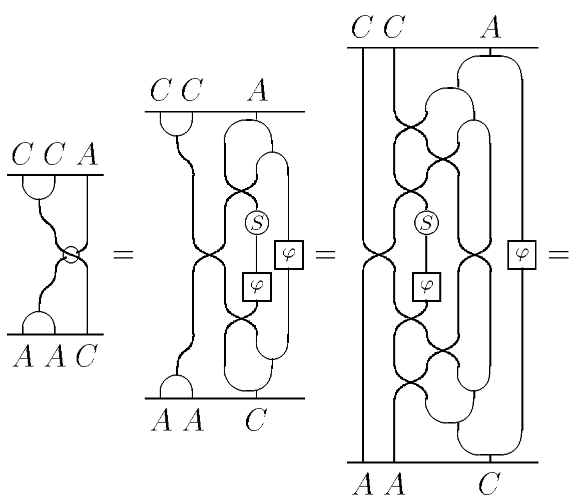
Observe that S commutes with φ, since φ is a bialgebra homomorphism. Thus we have

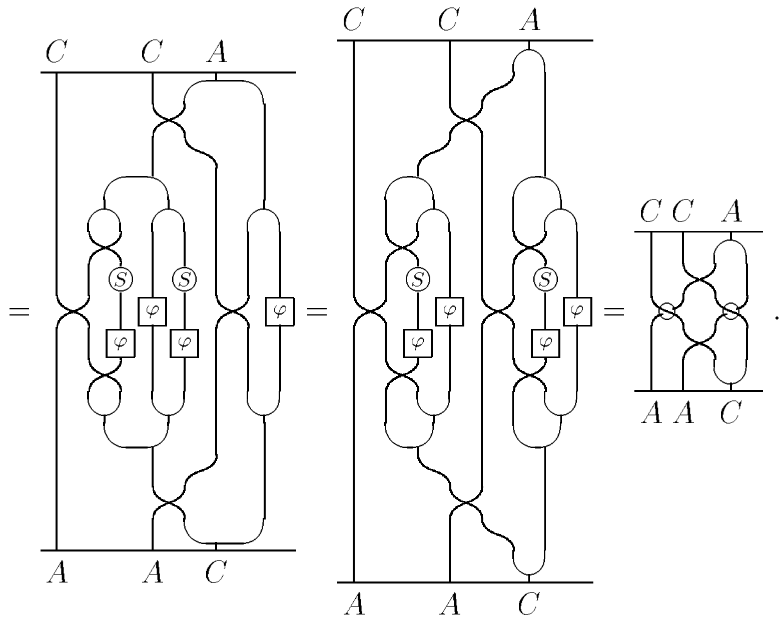


(1b) $\psi(c \otimes 1) = 1 \otimes S\varphi(1)c \cdot 1 = 1 \otimes a$.

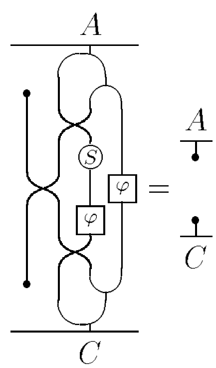
(2a) and (2b) are dual to (1a) and (1b).

(4)





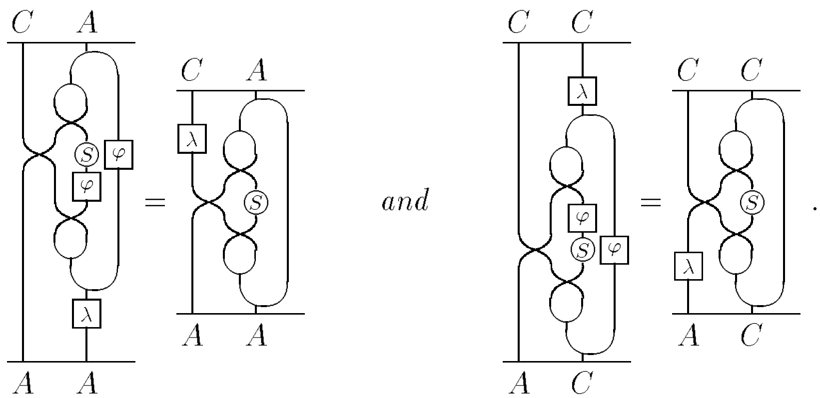
(5)



□

If $\varphi : A \longrightarrow C$ is a bialgebra homomorphism then C is a right A-module by the adjoint action and A is a right C-comodule by the coadjoint action.

Proposition 7.9. *Let $\lambda : C \longrightarrow A$ be a twisted convolution invertible bialgebra homomorphism, such that*



Then $\eta\varepsilon \otimes \lambda : C \otimes C \longrightarrow A \otimes A$ is an r-map.

Proof. The conditions of the proposition say something similar to the fact that $\lambda : C \longrightarrow A$ is an A-right-module homomorphism with respect to the adjoint actions and a C-right-comodule homomorphism with respect to the coadjoint coactions, both up to the extra middle factor of A resp. C that is preserved.

By Remark 7.4 we only have to show that that conditions (6') and (7') are satisfied. This can be easily checked by using the entwining map of Proposition 7.8. □

Example 7.10. We apply these propositions to a group homomorphism $\varphi : H \longrightarrow G$ with G a finite group and the group rings $C = kG$ and $A = kH$. This defines the quantum group double with multiplication as in Proposition 6.2

$$(e^x \otimes a)(e^y \otimes b) = \delta_{\varphi(a^{-1})x\varphi(a),y} e^x \otimes ab,$$

where $x, y \in G, a, b \in H$ and e^x are in the dual basis for kG^*. The comultiplication is

$$\Delta(e^x \otimes a) = \sum_{y \in G} (e^y \otimes a) \otimes (e^{xy^{-1}} \otimes a).$$

Observe that H is a right H-group by the adjoint action and that G is a right H-group by $x \cdot a := \varphi(a^{-1})x\varphi(a)$. Assume that $\lambda : G \longrightarrow H$ is a homomorphism of H-groups, also satisfying $y^{-1}xy = \varphi\lambda(y^{-1})x\varphi\lambda(y)$ (i.e. $\varphi\lambda(y)y^{-1}$ is central). Then λ induces a (twisted) convolution invertible homomorphism of bialgebras with inverse $\mu(x) := \lambda(x^{-1})$ and the induced homomorphism satisfies the conditions in Proposition 7.9. Hence the quantum double $C^*\#A = kG^*\#kH$ is quasitriangular

with R-matrix
$$R = \sum_{x \in G} e^x \otimes 1_A \otimes \varepsilon_C \otimes \lambda(x).$$

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