Convexity Theories 0 cont. Foundations

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Abstract. The main issue of this paper is an axiomatization of the notion of absolutely convergent series involving a set of summands of fixed (but unrestricted) infinite cardinality N. This notion is used to define the category $N_R\text{pnSmod}^1$ of R-prenormed R-semimodules with N-summation whose homomorphisms are contractive. Based on this we introduce left N-convexity theories Γ and the category ΓC of left Γ-convex modules. We show that the closed unit ball functor $N_R\text{pnSmod}^1 \longrightarrow \text{Set}$, the forgetful functor $\Gamma C \longrightarrow \text{Set}$, and the associated Γ-convex module functor $N_R\text{pnSmod}^1 \longrightarrow \Gamma C$ have left adjoints.

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Introduction

The basic definitions of this paper are contained in §1. They concern an axiomatic approach to the notion of "absolutely convergent series" in prenormed semirings and prenormed semimodules . The type of series we are interested in have a set of summands of a fixed, but arbitrary, infinite cardinality N. Semimodules equipped with a family of such "summable" series are said to have N-summation. The section concludes with few statements directly related to the basic definitions. §2 consists of several results involving semimodules with N-summation. In addition we introduce maps from a fixed, but arbitrary, set to a semimodule with N-summation that are "summable". These maps are used in §3 to define, for an arbitrary set A, the free semimodule $\mathcal{L}^N(A)$ with N-summation on the set A. $\mathcal{L}^N(A)$ is a generalization of the well known functional-analytic concept of ℓ_1-space on the set A. The functor $\text{Set} \ni A \mapsto \mathcal{L}^N(A) \in N_R\text{pnSmod}^1$ is the left adjoint of the forgetful functor $N_R\text{pnSmod}^1 \longrightarrow \text{Set}$; here, $N_R\text{pnSmod}^1$ stands for the category of R-prenormed R-semimodules with N-summation and their contractive homomorphisms. In section 4 we deal with N-convexity theories Γ over prenormed semirings with N-summation (generalizing the corresponding concept in [5], §3 and §7), derive few of the properties of Γ-convex modules, and give an explicit construction of the free Γ-convex modules. This construction differs from and is more perspicuous than the one given in [3], §5. In §5 we introduce the functor $\mathcal{O}_\Gamma : N_R\text{pnSmod}^1 \longrightarrow \Gamma C$, where for a given N-convexity theory Γ the category

ΓC is the category of Γ-convex modules and their homomorphisms. $\mathcal{O}_\Gamma(M)$ is the closed unit ball of M equipped with the obvious operation of Γ on the closed unit ball of M. In addition we exhibit the left adjoint $\mathcal{S}^\Gamma$ of $\mathcal{O}_\Gamma$. The paper ends with a section presenting several examples for the previously introduced concepts.

1. The basic definitions

Let R be a semiring (in the sense of [5], §1). Let furthermore N be a fixed set of cardinality $\geq \aleph_0$. Maps from N to R, that is elements of R^N, will be denoted by lower case greek letters with a lower placeholder symbol, e.g. α_* or $\alpha_\square$; occasionally we will write: $\{\alpha_n : n \in N\}$ or $\{\alpha(n) : n \in N\}$ instead of α_*. If we define $\alpha_* \dot{+} \beta_*$ as the map $N \ni n \longmapsto \alpha_n \dot{+} \beta_n \in R$ then R^N becomes a semiring. If $r \in R$ and if we denote the constant map $N \longrightarrow R$ with value r by r_* then $R \ni r \mapsto r_* \in R^N$ is a homomorphism of semirings.

Let M be a semimodule (in the sense of [5], §1) over the semiring R. Then we can again form M^N. Let $\alpha_* \in R^N$ and $\mu'_*, \mu_* \in M^N$, and define $\alpha_*\mu_*$ and $\mu'_* + \mu_*$ pointwise — as in the case of R — then M^N becomes a R^N-semimodule. As before we let m_*, $m \in M$, be the constant map with value m. Then $M \ni m \mapsto m_* \in M^N$ is a homomorphism of R-semimodules.

If R is a partially ordered semiring and M is a partially ordered R-semimodule (in the sense of [5], §1) then we define $\mu'_* \leq \mu_*$, where $\mu'_*, \mu_* \in M^N$, as $\mu'_n \leq \mu_n$ for all $n \in N$. This makes M^N a partially ordered semimodule over the partially ordered semiring R^N. Obviously, $R \ni r \mapsto r_* \in R^N$ and $M \ni m \mapsto m_* \in M^N$ are order preserving. $\mu_* \in M^N$ is said to be bounded if there are $m', m'' \in M$ with $m'_* \leq \mu_* \leq m''_*$, and a family $\{\mu^i_* : i \in I\}$ is called uniformly bounded if there are $m', m'' \in M$ with $m'_* \leq \mu^i_* \leq m''_*$ for all $i \in I$.

If M is an R-prenormed R-semimodule over the prenormed semiring R and if $\|\ \|$ denotes the prenorm with value cone C (see [5], §2) then, with $\mu_* \in M^N$, we denote by $\|\mu_*\|$ the map $N \ni n \mapsto \|\mu_n\| \in C$; hence $\|\mu_*\|$ is in C^N. This makes M^N a R^N-prenormed R^N-semimodule over the prenormed semiring R^N with value cone C^N.

Finally a construction that will be used later. Let A be some set and $\chi : N \longrightarrow A$ a set map. Let furthermore M be an R-semimodule and $\mu_* \in M^N$. Given $a \in A$ we denote by $\mu_*^{\chi^{-1}(a)}$ the map $N \longrightarrow M$ given by

$$N \ni n \mapsto \begin{cases} \mu_n & \text{, if } \chi(n) = a; \\ 0 & \text{, otherwise.} \end{cases}$$

In other words, $\mu_*^{\chi^{-1}(a)}$ is given by the formulae

$$\mu_*^{\chi^{-1}(a)}|\chi^{-1}(a) = \mu_*|\chi^{-1}(a) \qquad \text{and} \qquad \mu_*^{\chi^{-1}(a)}|N \setminus \chi^{-1}(a) = 0_*|N \setminus \chi^{-1}(a).$$

Note that any subset of N can be obtained as $\chi^{-1}(a)$, provided that A contains at least two elements, and that each partition of N can be written as $\{\chi^{-1}(a) : a \in A'\}$ for some subset A' of A, provided that A is large enough.

In the following definition we refer to the concept of positive semiring. According to [5], §1, a positive semiring is a partially ordered semiring with 0 as its smallest element.

(1.1) Definition. Let C be a positive semiring. By a *left N-summation* for C is meant a pair (S_C, Σ_C) consisting of a C-subsemimodule S_C of C^N and a C-homomorphism $\Sigma_C : S_C \longrightarrow C$ such that

(i) $C^{(N)} := \{\alpha_* \in C^N : \text{supp } \alpha_* \text{ is finite}\}$ is contained in S_C and for all $\alpha_* \in C^{(N)}$, $\Sigma_C(\alpha_*) = \sum'\{\alpha_n : n \in \text{supp } \alpha_*\}$, where $\sum'$ stands for the usual sum of finitely many elements in C;

(ii) for all $\alpha_* \in S_C$ and $\beta_* \in C^N$ with $\beta_* \leq \alpha_*, \beta_*$ is in S_C and $\Sigma_C(\beta_*) \leq \Sigma_C(\alpha_*)$;

(iii) for every $\alpha_* \in S_C$ and every map $\varphi : N \longrightarrow N, \alpha_*^{\varphi^{-1}(n)}$ is in S_C for all $n \in N$, and the map $\alpha_*^{\varphi^{-1}}$ given by $N \ni n \mapsto \Sigma_C(\alpha_*^{\varphi^{-1}(n)}) \in C$ is in S_C and satisfies $\Sigma_C(\alpha_*^{\varphi^{-1}}) = \Sigma_C(\alpha_*)$;

(iv) if α_* is in C^N and there exists a map $\varphi : N \longrightarrow N$ such that $\alpha_*^{\varphi^{-1}(n)}$ is in S_C for all $n \in N$ and that $\alpha_*^{\varphi^{-1}}$ is in S_C then α_* is in S_C.

(1.2) Lemma. *Let C be a positive semiring with left N-summation (S_C, Σ_C). Then the conditions (1.1), (i)-(iii), imply that for every $\alpha_* \in S_C$ the inequalities $\alpha_n \leq \Sigma_C(\alpha_*), n \in N$, hold; in particular, if* $\sup\{\alpha_n : n \in N\}$ *exists,* $\sup\{\alpha_n : n \in N\} \leq \Sigma_C(\alpha_*)$ *is satisfied.* □

The next definition uses the notion of prenormed semiring. Due to [5], (2.1), this is a semiring R together with a map $\| \; \| : R \longrightarrow C$, where C is a positive and complete (with respect to the partial order) semiring, such that

(o) $\|0\| = 0$ and $\|1\| = 1$;

(i) $\|r_1 + r_2\| \leq \|r_1\| + \|r_2\|$, for all $r_1, r_2 \in R$;

(ii) $\|r_1 r_2\| \leq \|r_1\|\|r_2\|$, for all $r_1, r_2 \in R$.

The semiring C is called the value cone of R and $\| \; \|$ is said to be the prenorm of R.

(1.3) Definition. Let R be a prenormed semiring with prenorm $\| \; \| : R \longrightarrow C$. By a *left N-summation* for R is meant a left N-summation (S_C, Σ_C) for C together with a pair (S_R, Σ_R) consisting of an R-subsemimodule S_R of R^N and an R-homomorphism $\Sigma_R : S_R \longrightarrow R$ such that

(o) $\alpha_* \in R^N$ is in S_R if and only if $\|\alpha_*\|$ is in S_C;

(i) $R^{(N)} := \{\alpha_* \in R^N : \text{supp } \alpha_* \text{ is finite}\}$ is contained in S_R and for all $\alpha_* \in R^{(N)}$, $\Sigma_R(\alpha_*) = \sum'\{\alpha_n : n \in \text{supp } \alpha_*\}$, where $\sum'$ stands for the usual sum of finitely many elements in R;

(ii) for all $\alpha_* \in S_R$ and $\beta_* \in R^N$ with $\|\beta_*\| \leq \|\alpha_*\|$, β_* is in S_R and $\|\Sigma_R \beta_*\| \leq \Sigma_C \|\alpha_*\|$;

(iii) for every $\alpha_* \in S_R$ and every $\varphi : N \longrightarrow N, \alpha_*^{\varphi^{-1}(n)}$ is in S_R for all $n \in N$, and the map $\alpha_*^{\varphi^{-1}}$ given by $N \ni n \mapsto \Sigma_R(\alpha_*^{\varphi^{-1}(n)}) \in R$ is in S_R and satisfies $\Sigma_R(\alpha_*^{\varphi^{-1}}) = \Sigma_R(\alpha_*)$.

Definition (1.4) requires the notion of R-prenormed left R-semimodule over the prenormed semiring R with prenorm $\| \ \| : R \longrightarrow C$. By this is meant a left R-semimodule M together with a map $\| \ \| : M \longrightarrow C$ such that

(o) $\|0\| = 0$;
(i) $\|m_1 + m_2\| \leq \|m_1\| + \|m_2\|$, for all $m_1, m_2 \in M$;
(ii) $\|rm\| \leq \|r\|\|m\|$, for all $r \in R, m \in M$.

(1.4) Definition. Let R be a prenormed semiring with prenorm $\| \ \| : R \longrightarrow C$ and N-summation (S_R, Σ_R) for R and (S_C, Σ_C) for C. An R-prenormed left R-semimodule M with *left N-summation* is an R-prenormed left R-semimodule together with a pair (S_M, Σ_M) consisting of an R-subsemimodule S_M of M^N and an R-homomorphism $\Sigma_M : S_M \longrightarrow M$ such that

(o) $\mu_* \in M^N$ is in S_M if and only if $\|\mu_*\|$ is in S_C;
(i) $M^{(N)} := \{\mu_* \in M^N : \text{supp } \mu_* \text{ is finite}\}$ is contained in S_M and for all $\mu_* \in M^{(N)}, \Sigma_M(\mu_*) = \sum'\{\mu_n : n \in \text{supp } \mu_*\}$, where $\sum'$ stands for the usual sum of finitely many elements in M;
(ii) for all $\mu_* \in S_M$ and $\nu_* \in M^N$ with $\|\nu_*\| \leq \|\mu_*\|$, ν_* is in S_M and $\|\Sigma_M \nu_*\| \leq \Sigma_C\|\mu_*\|$;
(iii) for every $\mu_* \in S_M$ and every $\varphi : N \longrightarrow N, \mu_*^{\varphi^{-1}(n)}$ is in S_M for all $n \in N$, and the map $\mu_*^{\varphi^{-1}}$ given by $N \ni n \mapsto \Sigma_M(\mu_*^{\varphi^{-1}(n)}) \in M$ is in S_M and satisfies $\Sigma_M(\mu_*^{\varphi^{-1}}) = \Sigma_M(\mu_*)$.

Note that (1.3) is a special case of (1.4) and that (1.1) can be viewed as a special case of (1.3) by setting $R = C, \| \ \| = \text{id}_C, S_R = S_C$, and $\Sigma_R = \Sigma_C$. Hence the statements following (1.5) concerning semimodules apply to prenormed semirings and positive semirings as well.

(1.5) Definition. Let M and M' be R-prenormed R-semimodule with left N-summation (S_M, Σ_M) resp. $(S_{M'}, \Sigma_{M'})$. Then a *homomorphism* from M to M' is a map $f : M \longrightarrow M'$ such that

(i) f is a homomorphism of left R-semimodules from M to M' satisfying

$$f^N(S_M) \subseteq S_{M'} \quad \text{and} \quad \Sigma_{M'}(f^N(\mu_*)) = f(\Sigma_M(\mu_*)) \quad \text{, for all } \mu_* \in S_M;$$

(ii) there is a $c \in C$ (depending on f) with

$$\|\Sigma_{M'} f^N(\mu_*)\| \leq (\Sigma_C(\|\mu_*\|))c \qquad \text{, for all } \mu_* \in S_M.$$

It is clear from (1.18) that the totality of R-prenormed R-semimodules with left N-summation together with their homomorphisms and the set-theoretical composition of these forms a category $N_R$pnSmod.

A homomorphism of R-prenormed R-semimodules is called *contractive* (or a *contraction*) if $c=1$ can be chosen in (1.5), (ii). Again it is easy to see that the totality of R-prenormed R-semimodules together with their contractive homomorphisms forms a subcategory $N_R\mathrm{pnSmod}^1$ of $N_R\mathrm{pnSmod}$.

We close with three statements directly related to the above definitions.

(1.6) Lemma. *(1.4), (o), implies that, whenever $\mu_* \in S_M$ and $\nu_* \in M^N$ with $\|\nu_*\| \le \|\mu_*\|$ holds then $\nu_* \in S_M$ is satisfied.*

Proof. Due to (1.4), (o), $\|\mu_*\|$ is in S_C. Hence (1.1), (ii), shows that $\|\nu_*\|$ is in S_C and thus, by (1.4), (o), ν_* is in S_M. □

(1.7) Lemma. *(1.4), (o) and (ii), imply that, whenever $\mu_* \in S_M$ and $\varphi : N \longrightarrow N$ is a map, $\mu_*^{\varphi^{-1}(n)}$ is in S_M, for all $n \in N$, and $\mu_*^{\varphi^{-1}}$ is in S_M.*

Proof. Again $\|\mu_*\|$ is in S_C. Since $\|\mu_*^{\varphi^{-1}(n)}\| \le \|\mu_*\|$, $\|\mu_*^{\varphi^{-1}(n)}\|$ is in S_C and thus $\mu_*^{\varphi^{-1}(n)}$ is in S_M. By (1.4), (ii),

$$\|\Sigma_M(\mu_*^{\varphi^{-1}(n)})\| \le \Sigma_C(\|\mu_*^{\varphi^{-1}(n)}\|) = \Sigma_C(\|\mu_*\|^{\varphi^{-1}(n)}).$$

Due to (1.1), (iii), $N \ni n \mapsto \Sigma_C(\mu_*^{\varphi^{-1}(n)}) \in C$ is in S_C as $\|\mu_*\|$ is in S_C. Therefore the map $N \ni n \mapsto \|\Sigma_M(\mu_*^{\varphi^{-1}(n)})\| \in C$ is in S_C and hence the map $\mu_*^{\varphi^{-1}}$, that is $N \ni n \mapsto \Sigma_M(\mu_*^{\varphi^{-1}(n)})$, is in S_M. □

(1.8) Lemma. *Let M be an R-prenormed R-semimodule with left N-summation (S_M, Σ_M) and denote the N-summation of R by (S_R, Σ_R). If $\alpha_* \in S_R$ and $\mu^* \in S_M$ then $\alpha_*\mu^*$, defined as the map $N \ni n \mapsto \alpha_n\mu^n \in M$, is in S_M.*

Proof. Since α_* is in S_R, it follows from (1.3), (o), that $\|\alpha_*\|$ is in S_C, whence $\|\alpha_n\| \le \Sigma_C\|\alpha_*\|$, $n \in N$, on account of (1.2). Therefore $\|\alpha_*\mu^*\| \le (\Sigma_C\|\alpha_*\|)\|\mu^*\|$, and (1.1) and (1.4), (o), show that the right hand side of this inequality is in S_C. Thus $\|\alpha_*\mu^*\|$ is in S_C due to (1.1), (ii), whence $\alpha_*\mu^*$ is in S_M by (1.4), (o).

2. Elementary results

(2.1) Lemma. *Let M be an R-prenormed R-semimodule with left N-summation (S_M, Σ_M). Let furthermore $\mu_* \in S_M$ and $\nu_* \in M^N$ be such that there is a bijection $\overline{\varphi} : \mathrm{supp}\ \mu_* \longrightarrow \mathrm{supp}\ \nu_*$ with $\mu_n = \nu_{\overline{\varphi}(n)}$, for all $n \in \mathrm{supp}\ \mu_*$. Then ν_* is in S_M and $\Sigma_M(\nu_*) = \Sigma_M(\mu_*)$.*

Proof. Extend $\overline{\varphi}$ to some map $\varphi : N \longrightarrow N$. Then one checks quickly that $\nu_* = \mu_*^{\varphi^{-1}}$ holds. Hence (1.1), (iii), shows that ν_* is in S_M and that $\Sigma_M(\nu_*) = \Sigma_M(\mu_*)$ is valid. □

(2.2) Corollary. *Let M be an R-prenormed R-semimodule with left N-summation (S_M, Σ_M). Let furthermore $f : A \longrightarrow M$ be a map with* $\operatorname{card}(\operatorname{supp} f) \leq \operatorname{card} N$. *Then there is a map $\chi : N \longrightarrow A$, a $\mu_* \in M^N$, and sets A' and N' with* $\operatorname{supp} f \subseteq A' \subseteq A$, $\operatorname{supp} \mu_* \subseteq N' \subseteq N$, *and* $\operatorname{card} A' \leq \operatorname{card} N$ *such that*

a) *$A' \subseteq \chi(N)$ and $A'|\chi|N'$ is a bijection*

b) *$\mu_n = f(\chi(n))$, for all $n \in N'$.*

Moreover, if $\overline{\chi}$, $\overline{\mu}_*$, $\overline{A}'$ and $\overline{N}'$ is another set of data satisfying the above conditions, for all $n \in N'$, then μ_* is in S_M if and only if $\overline{\mu}_*$ is in S_M, in which case $\Sigma_M(\mu_*) = \Sigma_M(\overline{\mu}_*)$ holds.

Proof. The existence of χ, μ_*, A', and N' is obvious. The balance of (2.2) is an immediate consequence of (2.1). □

(2.3) Remark. Suppose we are in the situation of (2.2) but are dealing with a family $\{f_n : n \in \tilde{N}\}$ of maps $A \longrightarrow N$, whose index set satisfies $\operatorname{card}\tilde{N} \leq \operatorname{card} N$, instead of looking at a single such map. If $\operatorname{card}(\operatorname{supp} f_n) \leq \operatorname{card} N$, for all $n \in \tilde{N}$, then, as $\operatorname{card} N^2 = \operatorname{card} N$, $\operatorname{card}(\bigcup\{\operatorname{supp} f_n : n \in \tilde{N}\}) \leq \operatorname{card} N$ holds. Therefore, in (2.2), χ, A', and N' can be chosen such that for every f_n, $n \in \tilde{N}$, and the (now uniquely determined) μ_{*n}, $n \in \tilde{N}$, the conditions in (2.2) are satisfied. □

Due to (2.2) we can formulate

(2.4) Definition. Let M be an R-prenormed R-semimodule with left N-summation (S_M, Σ_M). Let furthermore A be any set. Then we define $S_{M,A}$ as the set of maps $f : A \longrightarrow M$ such that $\operatorname{card}(\operatorname{supp} f) \leq \operatorname{card} N$ and for some data χ, μ_*, A', and N' in (2.2), $\mu_* \in S_M$ holds. Moreover, we define $\Sigma_{M,A}(f) := \Sigma_M(\mu_*)$, for all $f \in S_{M,A}$, to obtain a map $\Sigma_{M,A} : S_{M.A} \longrightarrow M$.

Note that $S_{M,N} = S_M$ and $\Sigma_{M,N} = \Sigma_M$ hold. We will write occasionally $\Sigma_{M,A}\{f(a) : a \in A\}$ instead of $\Sigma_{M,A}(f)$.

For $f : A \longrightarrow M$ we denote by $\|f\|$ the map $A \ni a \mapsto \|f(a)\| \in C$.

(2.5) Lemma. *For every set A and every R-prenormed R-semimodule M with left N-summation, $S_{M,A}$ is an R-subsemimodule of M^A and $\Sigma_{M,A} : S_{M,A} \longrightarrow M$ is an R-homomorphism of R-prenormed semimodules; in particular, $\|\Sigma_{M,A}(f)\| \leq \Sigma_{C,A}(\|f\|)$.*

Proof. Let f and g be in $S_{M,A}$. Choose the data in (2.2) to serve both f and g (see (2.3)). If μ_* corresponds to f and ν_* corresponds to g then, obviously, $\mu_* + \nu_*$ corresponds to $f + g$. Hence $f + g$ belongs to $S_{M,A}$. Similarly one shows that rf, $r \in R$, $f \in S_{M,A}$, also belongs to $S_{M,A}$. Moreover,

$$\Sigma_{M,A}(f + g) = \Sigma_M(\mu_* + \nu_*) = \Sigma_M(\mu_*) + \Sigma_M(\nu_*) = \Sigma_{M,A}(f) + \Sigma_{M,A}(g)$$

and

$$\Sigma_{M,A}(rf) = \Sigma_M(r_*\mu_*) = r\Sigma_M(\mu_*) = r\Sigma_{M,A}(f).$$

Finally,

$$\|\Sigma_{M,A}(f)\| = \|\Sigma_M(\mu_*)\| \leq \Sigma_C(\|\mu_*\|) = \Sigma_{C,A}(\|f\|). \qquad \square$$

(2.6) Lemma. *Let M be an R-prenormed R-semimodule with left N-summation (S_M, Σ_M). Furthermore let A be any set. Then $f \in M^A$ belongs to $S_{M,A}$ if and only if $\|f\| \in C^A$ belongs to $S_{C,A}$.*

Proof. (1.4),(o), and (2.3). $\square$

(2.7) Lemma. *Let M be an R-prenormed R-semimodule with left N-summation (S_M, Σ_M). Then, for any set $A, M^{(A)} := \{f \in M^A : \mathrm{supp}\ f$ is finite$\}$ is contained in $S_{M,A}$ and for all $f \in M^{(A)}, \Sigma_{M,A}(f) = \sum'\{f(a) : a \in \mathrm{supp}\ \ f\}$, where $\sum'$ stands for the usual sum of finitely many elements in M.*

Proof. (1.4), (i), and (2.2). $\square$

(2.8) Lemma. *Let M be an R-prenormed R-semimodule with left N-summation (S_M, Σ_M). Furthermore let A be any set. If f is in $S_{M,A}$ and $g \in M^A$ with $\|g\| \leq \|f\|$ then g is in $S_{M,A}$ and $\|\Sigma_{M,A}(g)\| \leq \Sigma_{C,A}(\|f\|)$.*

Proof. (1.4),(ii), and (2.2). $\square$

(2.9) Lemma. *Let M be an R-prenormed R-semimodule with left N-summation (S_M, Σ_M). Furthermore let A be any set. If f is in $S_{M,A}$ and $\psi : A \longrightarrow A$ is any map then, for any $a \in A$, the map $f^{\psi^{-1}(a)}$ given by*

$$A \ni b \mapsto \begin{cases} f(b) & \text{, if } \psi(b) = a; \\ 0 & \text{, otherwise.} \end{cases}$$

is in $S_{M,A}$; moreover, the map $f^{\psi^{-1}}$ given by $A \ni a \mapsto \Sigma_{M,A}(f^{\psi^{-1}(a)}) \in M$ is in $S_{M,A}$ and $\Sigma_{M,A}(f^{\psi^{-1}}) = \Sigma_{M,A}(f)$.

Proof. (1.4),(iii), and (2.2). $\square$

(2.10) Corollary. *Let M be an R-prenormed R-semimodule with left N-summation (S_M, Σ_M). Furthermore let A be any set and $\pi : A \longrightarrow A$ be any bijection.*

Then $f \in M^A$ is in $S_{M,A}$ if and only if $f^\pi := f \circ \pi$ is in $S_{M,A}$, in which case $\Sigma_{M,A}(f^\pi) = \Sigma_{M,A}(f)$.

Proof. Immediate consequence of (2.9) $\square$

(2.11) Lemma. *Let M be an R-prenormed R-semimodule with left N-summation (S_M, Σ_M). Furthermore let A and B be any sets. For $f \in S_{M,A\times B}$ define $f(a,\text{-}) : B \longrightarrow M$ resp. $f(\text{-},b) : A \longrightarrow M$ as the maps*

$$B \ni b \mapsto f(a,b) \in M \qquad \textit{resp.} \qquad A \ni a \mapsto f(a,b) \in M.$$

Then $f(a,\text{-})$ in in $S_{M,B}$, for all $a \in A$, and $f(\text{-},b)$ is in $S_{M,A}$, for all $b \in B$, and the maps

$$A \ni a \mapsto \Sigma_{M,B}(f(a,\text{-})) \in M \qquad \textit{resp.} \qquad B \ni b \mapsto \Sigma_{M,A}(f(\text{-},b)) \in M$$

are in $S_{M,A}$ resp. $S_{M,B}$ and satisfy

$$\begin{aligned}\Sigma_{M,A}\{\Sigma_{M,B}\{f(a,b) : b \in B\} : a \in A\} &= \Sigma_{M,B}\{\Sigma_{M,A}\{f(a,b) : a \in A\} : b \in B\} \\ &= \Sigma_{M,A\times B}(f).\end{aligned}$$

Proof. By (2.2), (2.11) can be reduced to the case $A = B = N$. In this situation choose a bijection $N^2 \longrightarrow N$, use (2.1), and apply (1.4), (iii), twice. $\square$

As a special case of (2.11) we obtain

(2.12) Corollary. *Let M be an R-prenormed R-semimodule with left N-summation (S_M, Σ_M). If $\alpha_* \in S_R$ and $m \in M$ then the map $\alpha_* m$ given by $N \ni n \mapsto \alpha_n m \in M$ is in S_M and $\Sigma_M(\alpha_* m) = (\Sigma_R \alpha_*)m$. Similarly if $r \in R$ and $\mu_* \in S_M$ then the map $r\mu_*$ given by $N \ni n \mapsto r\mu_n \in M$ is in S_M and $\Sigma_M(r\mu_*) = r(\Sigma_M \mu_*)$.*
$\square$

(2.13) Corollary. *Let $f \in M^A$, $\varphi : A \longrightarrow B$ a map. For $b \in B$ let $\|f\|^{\varphi^{-1}(b)}$ be the map*

$$A \ni a \mapsto \begin{cases} \|f(a)\| & \text{, if } \varphi(a) = b; \\ 0 & \text{, otherwise.} \end{cases}$$

Suppose that $\|f\|^{\varphi^{-1}(b)}$ is in $S_{C,A}$ for every $b \in B$ and that the map

$$B \ni b \mapsto \begin{cases} \Sigma_{C,A}(\|f\|^{\varphi^{-1}(b)}) & \text{, if } b \in \varphi(A); \\ 0 & \text{, otherwise;} \end{cases}$$

is in $S_{C,B}$. Then f is in $S_{M,A}$.

Proof. This is an immediate consequence of (1.1),(iii'),(1.4), (o), and (2.2). $\square$

If should be pointed out that, with all index sets assumed to have cardinality $\leq \mathrm{card}N$, there is a correspondence between certain axioms in [7], §6, and some of the results obtained here. This correspondence is as follows:

$$\begin{aligned} \text{Equivalent Families Axiom} &\equiv (2.9), \\ \text{Unary Sum Axiom} &\equiv (1.4), (i), \\ \text{Generalized Partition Axiom} &\equiv (1.4), (iii), \\ \text{Weak Double Sum Axiom} &\equiv (2.11). \end{aligned}$$

3. The closed unit ball functor

As in [5] one defines the closed unit ball functor $\mathcal{B}_N : N_R\mathrm{pnSmod}^1 \longrightarrow \mathrm{Set}$. Its value on the object M of $N_R\mathrm{pnSmod}^1$ is

$$\mathcal{B}_N(M) := \{m \in M : \|m\| \leq 1\}.$$

(3.1) Theorem. $\mathcal{B}_N : N_R\mathrm{pnSmod}^1 \longrightarrow \mathrm{Set}$ *has a left adjoint* $\mathcal{L}^N$.

Proof. We put $\mathcal{L}^N(\emptyset) := \{0\}$. If $A \neq \emptyset$ is any nonempty set, we put – as a set –

$$\mathcal{L}^N(A) := S_{R,A} \subseteq R^A.$$

Due to (2.5), $\mathcal{L}^N(A)$ is an R-subsemimodule of R^A.

Next we define a prenorm $|||\ ||| : \mathcal{L}^N(A) \longrightarrow C$ by putting

$$|||f||| := \Sigma_{C,A}(\|f\|) = \Sigma_{C,A}\{\|f(a)\| : a \in A\} \qquad , f \in \mathcal{L}^N(A). \tag{3.2}$$

Since $\|f+g\| \leq \|f\| + \|g\|$, f and g in $\mathcal{L}^N(A)$, we obtain from (2.5) and (2.8) $|||f+g||| \leq |||f||| + |||g|||$. $|||rf||| \leq \|r\| \cdot |||f|||$, $r \in R$, $f \in \mathcal{L}^N(A)$, follows similarly. This means that $\mathcal{L}^N(A)$ is an R-prenormed R-semimodule.

It remains to define $(S_{\mathcal{L}^N(A)}, \Sigma_{\mathcal{L}^N(A)})$. Let $F_* \in (R^A)^N$. Given $a \in A$ we denote the map $N \ni n \mapsto F_n(a) \in R$ by $F_*(a)$. Due to (1.3), (o), $F_*(a)$ is in S_R if and only if $\|F_*(a)\|$, that is the map $N \ni n \mapsto \|F_n(a)\| \in C$, is in S_C. In this case we can form the map $\Sigma_C\|F_*\| : A \longrightarrow C$ that is given by

$$(\Sigma_C\|F_*\|)(a) := \Sigma_C(\|F_*(a)\|) \qquad , a \in A.$$

With these notations we have

$$\begin{aligned} S :&= S_{\mathcal{L}^N(A)} \\ &= \{F_* \in \mathcal{L}^N(A)^N : \|F_*(a)\| \in S_C, \text{ for all } a \in A, \text{ and } \Sigma_C\|F_*\| \in S_{C,A}\}. \end{aligned}$$

In order to define $\Sigma := \Sigma_{\mathcal{L}^N(A)}$, let F_* be in S. Since $\|F_*(a)\|$ is in S_C we have that $F_*(a)$, that is the map $N \ni n \mapsto F_n(a) \in R$, is in S_R. Due to (1.3), (ii), we obtain $\|\Sigma_R F_*(a)\| \leq \Sigma_C\|F_*(a)\|$. If $\Sigma_R F_*$ denotes the map $A \ni a \mapsto \Sigma_R F_*(a) \in R$ then we have $\|\Sigma_R F_*\| \leq \Sigma_C\|F_*\|$. Since the latter function is in $S_{C,A}$, (2.8) shows that

$\|\Sigma_R F_*\|$ is in $S_{C,A}$, whence $\Sigma_R F_*$ is in $S_{R,A}$ due to (2.6). In other words, $\Sigma_R F_*$ is in $\mathcal{L}^N(A)$, and we put $\Sigma F_* := \Sigma_R F_*$.

At this point we have to show that (S, Σ) is a left N-summation for $\mathcal{L}^N(A)$. So, let F_* and G_* be in S. Then $F_* + G_*$ is the map $N \ni n \mapsto F_n + G_n \in \mathcal{L}^N(A)$ and hence $F_* + G_* \in \mathcal{L}^N(A)^N$. Moreover, $(F_n + G_n)(a) = F_n(a) + G_n(a)$ for all $a \in A$. Thus, $\|(F_* + G_*)(a)\| \leq \|F_*(a)\| + \|G_*(a)\|$. Since both $\|F_*(a)\|$ and $\|G_*(a)\|$ are in S_C, (1.1) shows that $\|F_*(a)\| + \|G_*(a)\|$ is in S_C. Therefore (1.1), (ii), implies that $\|(F_* + G_*)(a)\|$ is also in S_C and that

$$\Sigma_C(\|(F_* + G_*)(a)\|) \leq \Sigma_C(\|F_*(a)\| + \|G_*(a)\|) = \Sigma_C(\|F_*(a)\|) + \Sigma_C(\|G_*(a)\|)$$

holds for all $a \in A$. This means that $\Sigma_C\|F_* + G_*\| \leq \Sigma_C\|F_*\| + \Sigma_C\|G_*\|$ is valid. By assumption both $\Sigma_C\|F_*\|$ and $\Sigma_C\|G_*\|$ are in S_C, whence $\Sigma_C\|F_*\| + \Sigma_C\|G_*\|$ is in S_C. Thus (1.1), (ii), shows that $\Sigma_C\|F_* + G_*\|$ is in S_C. Therefore $F_* + G_*$ is in S. Similarly, but more simply, one shows that $F_* \in S$ and $r \in R$ implies $rF_* \in S$. Thus we have shown that S is an R-subsemimodule of $\mathcal{L}^N(A)^N$.

Next we need to prove that Σ is a homomorphism of R-semimodules. Again let F_* and G_* be in S. Then $\Sigma_R(F_*(a) + G_*(a)) = \Sigma_R F_*(a) + \Sigma_R G_*(a)$ and thus $\Sigma_R(F_* + G_*) = \Sigma_R F_* + \Sigma_R G_*$. Moreover, $(\Sigma_R(F_* + G_*))_* = (\Sigma_R F_*)_* + (\Sigma_R G_*)_*$, whence

$$\begin{aligned}\Sigma(F_* + G_*) &= \Sigma_R((\Sigma_R(F_* + G_*))_*) = \Sigma_R((\Sigma_R F_*)_* + (\Sigma_R G_*)_*)\\ &= \Sigma_R((\Sigma_R F_*)_*) + \Sigma_R((\Sigma_R G_*)_*) = \Sigma F_* + \Sigma G_*.\end{aligned}$$

Similarly one obtains $\Sigma r F_* = r\Sigma F_*$, and Σ is recognized as a homomorphism of R-semimodules.

Now we wish to verify (1.4), (o). For this, let F_* be in $\mathcal{L}^N(A)^N$. We have to show that $F_* \in S$ is equivalent with $|||F_*||| \in S_C$, where $|||F_*|||$ is the map $N \ni n \mapsto |||F_n||| \in C$. Let (F_*) denote the map $A \times N \ni (a, n) \mapsto \|F_n(a)\| \in C$. Suppose that $F_* \in S$ holds. Let $\varphi : A \times N \longrightarrow A$ be the projection onto the first factor. Then, for every $a \in A$,

$$(F_*)^{\varphi^{-1}(a)}(b, n) = \begin{cases} \|F_n(a)\| & \text{, if } b = a, \\ 0 & \text{, otherwise.} \end{cases}$$

Since $\|F_*(a)\|$ is in S_C, $(F_*)^{\varphi^{-1}(a)}$ belongs to $S_{C,A\times N}$ due to (2.1). Moreover, $\Sigma_C\|F_*(a)\| = \Sigma_{C,A\times N}(F_*)^{\varphi^{-1}(a)}$. Since $\Sigma_{C,A}\|F_*\|$ is in $S_{C,A}$, (2.12) shows that (F_*) is in $S_{C,A\times N}$. Thus (2.11) implies that, with $\psi : A \times N \longrightarrow N$ the projection onto the second factor, $(F_*)^{\psi^{-1}(n)}$ is in $S_{C,A\times N}$ for every $n \in N$. But due to (2.1)

$$\Sigma_{C,A\times N}(F_*)^{\psi^{-1}(n)} = \Sigma_{C,A}\{\|F_n(a)\| : a \in A\} = |||F_n||| \qquad , n \in N.$$

Hence (2.11) shows that $|||F_*|||$ is in S_C. The same type of argument shows that $|||F_*||| \in S_C$ implies $F_* \in S$.

On to (1.4), (i). Here we are dealing with $F_* \in \mathcal{L}^N(A)^N$ with finite support. Hence $\|F_*(a)\|$ has finite support and is therefore in S_C, for all $a \in A$. In particular,

$$\Sigma_C\|F_*(a)\| = \sum{}'\{\|F_n(a)\| : n \in \operatorname{supp} F_*\}$$

and hence

$$\Sigma_C \|F_*\| = \sum{}' \{\|F_n\| : n \in \text{supp } F_*\}.$$

Since $S_{C,A}$ is an R-semimodule, the right hand side of the last equation, and thus $\Sigma_C\|F_*\|$, is in $S_{C,A}$. Therefore F_* is in S. Moreover, with F_* having finite support, so does $F_*(a)$, for all $a \in A$, and

$$\Sigma_R F_*(a) = \sum{}' \{F_n(a) : n \in \text{supp } F_*\} \qquad , a \in A.$$

Consequently,

$$\Sigma F_* = \Sigma_R F_* = \sum{}' \{F_n : n \in \text{supp } F_*\},$$

and (1.4), (i), is satisfied.

Next comes (1.4), (ii). Let $F_* \in S$ and $G_* \in \mathcal{L}^N(A)^N$ with $|||G_*||| \leq |||F_*|||$. By (1.4), (o), $|||F_*||| \in S_C$ holds. Due to (1.1), (ii), $|||G_*|||$ is in S_C. By (1.4),(o), G_* is in S. Since $(\Sigma_R G_*)(a) = \Sigma_R(G_*(a)) = \Sigma_R\{G_n(a) : n \in N\}$, it follows from (1.3), (ii), that

$$\|(\Sigma_R G_*)(a)\| \leq \Sigma_C\{\|G_n(a)\| : n \in N\} = \Sigma_C\|G_*(a)\|.$$

Therefore we obtain, as in the proof of (1.4), (o),

$$\begin{aligned} |||\Sigma G_*||| &= \Sigma_{C,A}\{\|(\Sigma_R G_*)(a)\| : a \in A\} \leq \Sigma_{C,A}\{\Sigma_C\{\|G_n(a)\| : n \in N\} : a \in A\} \\ &= \Sigma_{C,A\times N}((G_*)) = \Sigma_C|||G_*||| \end{aligned}$$

and, through (1.1), (iii),

$$|||\Sigma G_*||| \leq \Sigma_C|||G_*||| \leq \Sigma_C|||F_*|||.$$

There remains (1.4), (iii). Let $F_* \in S$ and let $\varphi : N \longrightarrow N$ be a map. By (1.7), $F_*^{\varphi^{-1}(n)}$ is in S, for all $n \in N$, and $F_*^{\varphi^{-1}}$ is in S. Moreover, for every $a \in A$, due to (1.3), (iii),

$$\begin{aligned} (\Sigma F_*)(a) &= \Sigma_R(F_*(a)) = \Sigma_R(F_*(a)^{\varphi^{-1}}) = \Sigma_R\{(\Sigma_R(F_*(a)^{\varphi^{-1}(n)})) : n \in N\} \\ &= \Sigma_R\{(\Sigma_R(F_*^{\varphi^{-1}(n)})(a)) : n \in N\} = \Sigma_R\{(\Sigma F_*^{\varphi^{-1}(n)})(a) : n \in N\} \\ &= (\Sigma\{\Sigma F_*^{\varphi^{-1}(n)} : n \in N\})(a) = (\Sigma F_*^{\varphi^{-1}})(a), \end{aligned}$$

and thus $\Sigma(F_*^{\varphi^{-1}}) = \Sigma(F_*)$, as had to be shown.

At this point we know that $\mathcal{L}^N(A)$ is an R-prenormed R-semimodule with left N-summation $(S_{\mathcal{L}^N(A)}, \Sigma_{\mathcal{L}^N(A)})$.

For $a \in A$ let δ^a be the Dirac function at a on A, that is the map $A \longrightarrow R$ with $\delta^a(a) = 1$ and $\delta^a(b) = 0$, for all $a \neq b \in A$. By (1.3), (i), δ^a is in $\mathcal{L}^N(A)$ and $|||\delta^a||| = 1$ holds. The map $A \ni a \mapsto \delta^a \in \mathcal{B}_N(\mathcal{L}^N(A))$ is denoted by δ and is called the Dirac map on A. We claim that $\delta : A \longrightarrow \mathcal{B}_N(\mathcal{L}^N(A))$ is universal with respect to $N_R$pnSmod^1. For this let M be an R-prenormed R-semimodule with left N-summation (S_M, Σ_M) and let $h : A \longrightarrow \mathcal{B}_N(M)$ be any set map. Let furthermore $f \in \mathcal{L}^N(A)$ and consider the map fh given by $A \ni a \mapsto f(a)h(a) \in M$.

Then $\|fh\|(a) = \|f(a)h(a)\| \leq \|f(a)\|\|h(a)\| \leq \|f(a)\|$, for every $a \in A$. By (1.3), (o), $f \in S_{R,A}$ implies $\|f\| \in S_{C,A}$. The last inequalities together with (1.1), (ii), show $\|fh\| \in S_{C,A}$. Hence (1.4), (o), implies $fh \in S_{M,A}$. Hence we have $\overline{h}(f) := \Sigma_{M,A}(fh) \in M$. Thus we obtain the map $\overline{h} : \mathcal{L}^N(A) \longrightarrow M$. By (1.4), (i), $\overline{h}(\delta^a) = h(a)$, for all $a \in A$. Since $\Sigma_{M,A}$ is a homomorphism of R-semimodules, the same is true for $\overline{h}$. We want to show that $\overline{h}$ is a homomorphism of R-prenormed R-semimodules with left N-summation. Let $F_* \in S$. Then $|||F_*||| \in S_C$ due to (1.4), (o). Since, for any $f \in \mathcal{L}^N(A)$, $|||\overline{h}(f)||| \leq |||f|||$ holds, we have $|||\overline{h}^N(F_*)||| \leq |||F_*|||$ and thus $|||\overline{h}^N(F_*)||| \in S_C$ by (1.1), (ii), and hence $\overline{h}^N(F_*) \in S_C$ by (1.4), (o). That is, $\overline{h}^N(S) \subseteq S_M$. Now consider F_* as a map $A \times N \longrightarrow R$. Since $|||F_*|||$ is in S_C, it follows from (2.12) that F_* is in $S_{R,A\times N}$. Let $p : A \times N \longrightarrow A$ be the first projection and put $\tilde{h} := h \circ p$. Then $F_* \cdot \tilde{h}$ is in $M^{A\times N}$, and it follows from (2.8) that $F_* \cdot \tilde{h}$ is in $S_{M,A\times N}$. Hence (2.11) and (2.12) lead to

$$\begin{aligned}\Sigma_{M,A\times N}(F_* \cdot \tilde{h}) &= \Sigma_{M,A}\{\Sigma_M\{F_n(a) \cdot h(a) : n \in N\} : a \in A\} \\ &= \Sigma_{M,A}\{(\Sigma_R\{F_n(a) : n \in N\}) \cdot h(a) : a \in A\} \\ &= \Sigma_{M,A}\{(\Sigma_R F_*(a)) \cdot h(a) : a \in A\} = \Sigma_{M,A}((\Sigma F_*) \cdot h) = \overline{h}(\Sigma F_*).\end{aligned}$$

On the other hand

$$\begin{aligned}M_{M,A\times N}(F_* \cdot \tilde{h}) &= \Sigma_M\{\Sigma_{M,A}\{F_n(a)h(a) : a \in A\} : n \in N\} \\ &= \Sigma_M\{\overline{h}(F_n) : n \in N\} = \Sigma_M(\overline{h}^N(F_*)).\end{aligned}$$

This means that $\Sigma_M(\overline{h}^N(F_*)) = \overline{h}(\Sigma F_*)$, which is the formula in (1.5), (i). Finally, as $|||\overline{h}^N(F_*)||| \leq |||F_*|||$ has been shown before, (1.4), (ii), implies

$$\|\Sigma_M \overline{h}^N(F_*)\| \leq \Sigma_C |||\overline{h}^N(F_*)||| \leq (\Sigma_C |||F_*|||) \cdot 1,$$

showing (1.5), (ii), as well as proving that $\overline{h}$ is a contraction.

The very final step is now to show that $\overline{h}$ is unique. So, let $h' : \mathcal{L}^N(A) \longrightarrow M$ be a contractive homomorphism of R-prenormed R-semimodules with left N-summation such that $h = h' \circ \delta$. Let $f \in \mathcal{L}^N(A)$. As in (2.2) we have supp $f \subseteq A' \subseteq A$, $N' \subseteq N$, with card$A' \leq$ card$N$, and $\chi : N \longrightarrow A$ satisfying (2.2), a). Let f_* be the map

$$N \ni n \mapsto \begin{cases} f(\chi(n))\delta^{\chi(n)} & \text{, if } n \in N' \\ 0 & \text{, otherwise.} \end{cases}$$

Obviously, f_* is in $(R^A)^N$. Since f is in $S_{R,A}$, f_* is in S and $\Sigma f_* = f$. Since $h'^N(f_*)$ is the map

$$N \ni n \mapsto \begin{cases} h'(f(\chi(n))\delta^{\chi(n)}) = f(\chi(n))h'(\delta^{\chi(n)}) & \text{, if } n \in N'; \\ 0 & \text{, otherwise;} \end{cases}$$

we have

$$\begin{aligned}h'(f) = h'(\Sigma f_*) = \Sigma_M(h'^N(f_*)) &= \Sigma_{M,A}\{f(a)h'(\delta^a) : a \in A\} \\ &= \Sigma_{M,A}\{f(a)h(a) : a \in A\} = \overline{h}(f),\end{aligned}$$

proving the required uniqueness. $\square$

4. N-convexity theories

(4.1) Definition. Let R be a prenormed semiring with left N-summation (S_R, Σ_R). By a left *N-convexity theory over R* is meant a subset Γ of S_R such that

(o) $\|\alpha_*\| \in S_C$ and $\Sigma_C \|\alpha_*\| \leq 1$, for all $\alpha_* \in \Gamma$,
(i) $\delta^n_* \in \Gamma$, for all $n \in N$,
(ii) for all $\alpha_*, \beta^n_*, n \in N$, in Γ the map $\langle \alpha_\square, \beta^\square_* \rangle$ given by $N \ni n \mapsto \Sigma_R \alpha_* \beta^*_n \in R$ is in Γ.

It is a simple consequence of (1.1), (ii) – (iv), that $\langle \alpha_\square, \beta^\square_* \rangle$ satisfies (4.1), (o).

Let X be any set and denote the elements of X^N by lower case letters with an upper placeholder symbol, e. g. x^* or $x^\square$.

(4.2) Definition. Let Γ be a left N-convexity theory over R. By a left Γ-convex module is meant a set $X \neq \emptyset$ together with a map

$$\Gamma \times X^N \ni (\alpha_*, x^*) \mapsto \langle \alpha_*, x^* \rangle \in X$$

such that

(i) $\langle \delta^n_*, x^* \rangle = x^n$, for all $n \in N$, $\alpha_* \in \Gamma$, $x^* \in X^N$,
(ii) $\langle \alpha_\square, \langle \beta^\square_*, x^* \rangle \rangle = \langle \langle \alpha_\square, \beta^\square_* \rangle, x^* \rangle$, for all $\alpha_* \in \Gamma$, $\beta^\square_* \in \Gamma^N$, $x^* \in X^N$.

(4.3) Definition. Let Γ be a left N-convexity theory. By a homomorphism of left Γ-convex modules $X \longrightarrow X'$ is meant a map $f : X \longrightarrow X'$ such that

$$f(\langle \alpha_*, x^* \rangle) = \langle \alpha_*, f^N(x^*) \rangle \qquad , \text{ for all } \alpha_* \in \Gamma, x^* \in X^N.$$

Let Γ be an N-convexity theory. Then the totality of left Γ-convex modules and their homomorphisms, with composition the set-theoretical one, form a category ΓC, the category of left Γ-convex modules. Clearly, ΓC is an algebraic category. Since it has a rank ([2], p.56), it has free objects on any set. However, we want to construct such free objects explicitly. First, three technical statements about Γ-convex modules. They correspond to [4], (2.4), (iii), (iv), and (viii), and the proofs there carry over to the current situation with nominal changes only.

(4.4) Lemma. *Let X be a left Γ-convex module, let $\alpha_* \in \Gamma$ with* supp $\alpha_* \subseteq N_0 \subseteq N$, *and let $y^*, z^* \in X^N$ be such that $y^n = z^n$, for all $n \in N_0$. Then $\langle \alpha_*, y^* \rangle = \langle \alpha_*, z^* \rangle$.* $\square$

(4.5) Lemma. *Let X be a left Γ-convex module, let $\alpha_* \in \Gamma$ and $x^* \in X$. For any bijection $\sigma : N \longrightarrow N$ define ${}^\sigma\alpha_*$ resp. ${}^\sigma x^*$ as the maps*

$$N \ni n \mapsto \alpha_{\sigma(n)} \in R \qquad \textit{resp.} \qquad N \ni n \mapsto x^{\sigma(n)} \in X.$$

Then $\langle\alpha_*, x^*\rangle = \langle{}^\sigma\alpha_*, {}^\sigma x^*\rangle$. □

(4.6) Lemma. *Let* X *be a left* Γ*-convex module,* α_* *and* β_* *in* Γ*, and* x^* *and* y^* *in* X*. Let furthermore* $\varphi : N \longrightarrow N$ *be an injection and assume*

$$\beta_n = \alpha_{\varphi^{-1}(n)} \quad , n \in \varphi(N), \qquad \text{and} \qquad \beta_n = 0 \quad , n \notin \varphi(N);$$
$$x^n = y^{\varphi^{-1}(n)} \quad , n \in \varphi(N).$$

Then $\langle\alpha_*, y^*\rangle = \langle\beta_*, x^*\rangle$. □

Next we have

(4.7) Theorem. (see [3], 5.4) *The forgetful functor* $\mathcal{V}_\Gamma : \Gamma C \longrightarrow$ Set *has a left adjoint* $\mathcal{F}^\Gamma$.

Proof. Let R be the prenormed semiring with left N-summation (S_R, Σ_R) that appears in the definition (4.1) of Γ. Given any set A we define $\mathcal{F}^\Gamma(A)$ as the set of maps $f : A \longrightarrow R$ such that card(supp f) $\leq$ card$N$ and for some data χ, β_*, A', and N' in (2.2), $\beta_* \in \Gamma$ holds. In order to make $\mathcal{F}^\Gamma(A)$ a left Γ-convex module we have to define $\langle\alpha_*, f^*\rangle$, for all $\alpha_* \in \Gamma$ and $f^* \in \mathcal{F}^\Gamma(A)^N$. According to (2.3) we can choose χ, A' and N' such that for every $f^n \in \mathcal{F}^\Gamma(A) \subseteq S_{R,A}$, $n \in N$, the conditions in (2.2) are satisfied. Suppose that β_*^n is associated to f^n via the data χ, A' and N'. Then $\beta_*^n \in \Gamma$, for all $n \in N$. Due to (4.1), (ii), $\langle\alpha_\Box, \beta_*^\Box\rangle$ is in Γ and, as is seen easily, supp $\langle\alpha_\Box, \beta_*^\Box\rangle \subseteq \bigcup\{\text{supp } \beta_*^n : n \in N\} \subseteq N'$. Hence there is a unique $f \in \mathcal{F}^\Gamma(A)$ with supp $f \subseteq A'$ and $f(\chi(n)) = \langle\alpha_\Box, \beta_*^\Box\rangle$, for all $n \in N'$. Denote this f by $\langle\alpha_*, f^*\rangle$. By (2.2), f does not depend on the data chosen. It is clear from the construction that (4.2), (i), is satisfied. In order to verify (4.2), (ii), it suffices to show the existence and equality of the two terms

$$\langle\alpha_\Box, \langle\beta_*^\Box, \gamma_\Delta^*\rangle\rangle \qquad \text{and} \qquad \langle\langle\alpha_\Box, \beta_*^\Box\rangle, \gamma_\Delta^*\rangle$$

for all $\alpha_\Box \in \Gamma$, $\beta_*^\Box \in \Gamma^N$, $\gamma_\Delta^* \in \Gamma^N$. However, the existence of these expressions is an immediate consequence of (4.1), (ii). As for equality, consider the map $\alpha_\Box\beta_*^\Box\gamma_t^*$ given by

$$N \times N \ni (n, p) \mapsto \alpha_n\beta_p^n\gamma_t^p \in R.$$

Since by (1.2), $\|\alpha_n\beta_p^n\gamma_t^p\| \leq \|\alpha_n\|$, it follows from (1.1), (o) and (ii), and (1.4), (o), that the map $N \ni n \mapsto \alpha_n\beta_p^n\gamma_t^p \in R$ is in S_R, whence $\Sigma_R\{\alpha_n\beta_p^n\gamma_t^p : n \in N\}$ is defined. Since, due to (1.2) and (1.3), (ii),

$$\begin{aligned}\|\Sigma_R\{\alpha_n\beta_p^n\gamma_t^p : n \in N\}\| &\leq \Sigma_C\{\|\alpha_n\|\|\beta_p^n\|\|\gamma_t^p\| : n \in N\}\\ &\leq \Sigma_C\{\|\alpha_n\|\|\beta_p^n\| : n \in N\},\end{aligned}$$

and since by (1.1), (iv), the right hand side of this inequality (as a function of p) is in S_C, it follows from (1.3), (o), that

$$N \ni p \mapsto \Sigma_R\{\alpha_n\beta_p^n\gamma_t^p : n \in N\} \in R$$

is in S_R. Hence (1.1), (iii'), and the use of a bijection $N^2 \longrightarrow N$ show that $\alpha_\Box\beta_*^\Box\gamma_t^*$ is in $S_{R,N\times N}$. Therefore (2.11) leads to the desired equality. Which means, that

$\mathcal{F}^\Gamma(A)$ is a left Γ-convex module. The Dirac map $\delta : A \longrightarrow \mathcal{F}^\Gamma(A)$ assigns to each $a \in A$ the Dirac function δ^a at a. We claim that $\delta : A \longrightarrow \mathcal{F}^\Gamma(A)$ is a universal arrow. Let X be a left Γ-convex module and let $h : A \longrightarrow X$ be a set map. We want to define an appropriate map $h' : \mathcal{F}^\Gamma(A) \longrightarrow X$. Let $f \in \mathcal{F}^\Gamma(A)$. Choose for f the data χ, β_*, A', and N' and denote by h^* the map

$$N \ni n \mapsto \begin{cases} h(\chi(n)) & \text{, if } n \in N'; \\ x_0 & \text{, if } n \notin N'; \end{cases}$$

where x_0 is some element of X. It follows from (4.4) – (4.6) that $\langle \beta_*, h^* \rangle$ is independent of x_0 and of the data chosen, and we put $h'(f) := \langle \beta_*, h^* \rangle$. Obviously, $h'(\delta^a) = h(a)$, $a \in A$, whence $h' \circ \delta = h$ holds. Next we check that h' is a homomorphism of left Γ-convex modules. Let $\alpha_* \in \Gamma$ and $f^* \in \mathcal{F}^\Gamma(A)^N$. By (2.3) we can choose data so that they serve for all f^n, $n \in N$. If β_*^n is the element of Γ associated with f^n, $n \in N$, via such data then

$$h'(\langle \alpha_*, f^* \rangle) = \langle \langle \alpha_\Box, \beta_*^\Box \rangle, h^* \rangle,$$

while
$$\langle \alpha_*, h'^N(f^*) \rangle = \langle \alpha_\Box, \langle \beta_*^\Box, h^* \rangle \rangle,$$

whence the equality of the left hand sides of the last two equations follows from (4.2), (ii), showing that h' is a homomorphism of left Γ-convex modules. As for uniqueness of the required factorization, let $\tilde{h} : \mathcal{F}^\Gamma(A) \longrightarrow X$ be a homomorphism of left Γ-convex modules with $\tilde{h} \circ \delta = h$. For $f \in \mathcal{F}^\Gamma(A)$ choose the data as above to obtain $\beta_* \in \Gamma$ associated with f. Denote furthermore by $\delta^{\chi(*)}$ the map $N \ni n \mapsto \delta^{\chi(n)} \in \mathcal{F}^\Gamma(A)$. An easy computation shows that $f = \langle \beta_*, \delta^{\chi(*)} \rangle$. Hence

$$\tilde{h}(f) = \tilde{h}(\langle \beta_*, \delta^{\chi(*)} \rangle) = \langle \beta_*, \tilde{h}^N(\delta^{\chi(*)}) \rangle = \langle \beta_*, h^* \rangle = h'(f). \qquad \Box$$

5. The associated Γ-convex module functor

(5.1) Proposition. *Let Γ be a left N-convexity theory over R. Then there is a functor $\mathcal{O}_\Gamma : N_R\mathrm{pnSmod}^1 \longrightarrow \Gamma C$ whose object function is the following:*

if M is an R-prenormed R-semimodule with N-summation then the set underlying $\mathcal{O}_\Gamma(M)$ is $\mathcal{B}_N(M)$ and the Γ-convex module structure on $\mathcal{O}_\Gamma(M)$ is given by

$$\Gamma \times \mathcal{O}_\Gamma(M)^N \ni (\alpha_*, \mu^*) \mapsto \Sigma_M(\alpha_*\mu^*) =: \langle \alpha_*, \mu^* \rangle \in \mathcal{O}_\Gamma(M).$$

Proof. Since $\|\alpha_*\mu^*\| \le \|\alpha_*\|$ and since $\|\alpha_*\| \in S_C$ by (4.1), (o), it follows from (1.1), (ii), that $\|\alpha_*\mu^*\|$ is in S_C and thus $\alpha_*\mu^*$ is in S_M, due to (1.4), (o). Hence $\Sigma_M(\alpha_*\mu^*)$ is well defined and

$$\|\Sigma_M(\alpha_*\mu^*)\| \le \Sigma_C(\|\alpha_*\mu^*\|) \le \Sigma_C(\|\alpha_*\|\|\mu^*\|) \le \Sigma_C(\|\alpha_*\|) \le 1,$$

whence $\langle\alpha_*, \mu^*\rangle = \Sigma_M(\alpha_*\mu^*)$ is in $\mathcal{O}_\Gamma(M)$. By (1.4), (i),

$$\langle\delta^n_*, \mu^*\rangle = \Sigma_M\{\delta^n_p\mu^p : p \in N\} = \mu^n \qquad \text{, for all } n \in N.$$

This verifies (4.2), (i). Now denote by $\alpha_\Box\beta^\Box_*\mu^*$ the map $N \times N \ni (n,p) \mapsto \alpha_n\beta^n_p\mu^p \in M$. Then $\|\alpha_n\beta^n_p\mu^p\| \le \|\alpha_n\|\|\beta^n_p\|\|\mu^p\| \le \|\alpha_n\|\|\beta^n_p\|$, for all $n, p \in N$. By (1.1), (iv), the right hand side of these inequalities, as a map $N \times N \longrightarrow C$, is in $S_{C,N\times N}$. Therefore $\|\alpha_\Box\beta^\Box_*\mu^*\|$ is in $S_{C,N\times N}$ by (1.1), (ii), and $\alpha_\Box\beta^\Box_*\mu^*$ is in $S_{M,N\times N}$ by (1.4), (o). Hence (2.11) and (2.12) imply

$$\begin{aligned}\Sigma_{M,N\times N}(\alpha_\Box\beta^\Box_*\mu^*) &= \Sigma_M\{\Sigma_M\{\alpha_n\beta^n_p\mu^p : n \in N\} : p \in N\}\\ &= \Sigma_M\{(\Sigma_M\{\alpha_n\beta^n_p : n \in N\})\mu^p : p \in N\}\\ &= \langle\langle\alpha_\Box, \beta^\Box_*\rangle, \mu^*\rangle\end{aligned}$$

and

$$\begin{aligned}\Sigma_{M,N\times N}(\alpha_\Box\beta^\Box_*\mu^*) &= \Sigma_M\{\Sigma_M\{\alpha_n\beta^n_p\mu^p : p \in N\} : n \in N\}\\ &= \Sigma_M\{\alpha_n(\Sigma_M\{\beta^n_p\mu^p : p \in N\}) : n \in N\}\\ &= \langle\alpha_\Box, \langle\beta^\Box_*, \mu^*\rangle\rangle,\end{aligned}$$

which is (4.2), (ii). Thus the object function described in (5.1) is indeed in ΓC. Finally, given $f : M \longrightarrow M'$ in $N_R\text{pnSmod}^1$, we obtain from (1.5), (i),

$$\begin{aligned}f(\langle\alpha_*, \mu^*\rangle) = f(\Sigma_M\alpha_*\mu^*) &= \Sigma_{M'}(f^N(\alpha_*\mu^*))\\ &= \Sigma_{M'}(\alpha_* f^N(\mu^*)) = \langle\alpha_*, f^N(\mu^*)\rangle,\end{aligned}$$

showing that f induces a homomorphism of Γ-convex modules. $\Box$

Since the value cone C of a prenormed semiring is partially ordered, we obtain an induced partial order on C^N (it was described at the beginning of §1). If T is a subset of C^N, then $\inf T$ refers to this partial order on C^N.

(5.2) Theorem. *Let R be a prenormed semiring with left N-summation. Suppose that the value cone C of R satisfies the following conditions*

CO: *C is complete (in the sense of* [5], §1*);*

IS: *for every $T \subseteq S_C$, $\inf\{\Sigma_C(t_*) : t_* \in T\} = \Sigma_C(\inf\{t_* : t_* \in T\})$;*

LD: *for every $\emptyset \ne U \subseteq C$, $u_0 := \inf\{u : u \in U\}$, and every $t \in C$ with $u_0 < u_0 + t$ there is a $u_1 \in U$ with $u_0 \le u_1 \le u_0 + t$;*

LIM: *for every $U \subseteq C$ and every $c \in C$, $\inf\{cu : u \in U\} = c\inf\{u : u \in U\}$;*

OP: *for every $c_* \in S_C$ there is a $d_* \in C^N$ such that*

- (o) *$c_* + d_* \in S_C$;*
- (i) *if c_n is not a maximal element of C (with respect to the partial order of C) then $c_n < c_n + d_n$.*

Then for every left N-convexity theory Γ over R, $\mathcal{O}_\Gamma$ has a left adjoint $\mathcal{S}^\Gamma$.

Proof. Let X be a Γ-convex module and denote the set underlying X again by X. Form, as in the proof of (3.1), the R-semimodule $\mathcal{L}^N(X)$ together with the Dirac map $\delta : X \longrightarrow \mathcal{L}^N(X)$ and consider the set

$$S := \{(\delta^{\langle\alpha_*, x^*\rangle}, \langle\alpha_*, \delta^{x^*}\rangle) : \alpha_* \in \Gamma, x^* \in X^N\} \subseteq \mathcal{L}^N(X) \times \mathcal{L}^N(X),$$

where δ^{x^*} is the map $N \ni n \mapsto \delta^{x^n} \in \mathcal{L}^N(X)$. If $h : \mathcal{L}^N(X) \longrightarrow M$ is a contracting homomorphism of R-prenormed R-semimodules with N-summation we say that h is S-compatible if $S_h := \{(f, f') : h(f) = h(f')\} \subseteq \mathcal{L}^N(X) \times \mathcal{L}^N(X)$ contains S. Clearly, there are such contracting homomorphisms, e.g. the zero homomorphism. Let $\sim$ be the intersection of all these S_h. Then $\sim$ is an equivalence relation, $\mathcal{S}^\Gamma(X) := \mathcal{L}^N(X)/\sim$ is an R-semimodule, and the quotient map $q : \mathcal{L}^N(X) \longrightarrow \mathcal{S}^\Gamma(X)$ is a homomorphism of R-semimodules satisfying

$$q(\delta^{\langle\alpha_*, x^*\rangle}) = q(\langle\alpha_*, \delta^{x^*}\rangle) \qquad \text{, for all } \alpha_* \in \Gamma, x^* \in X^N.$$

Now define $||| \; ||| : \mathcal{S}^\Gamma(X) \longrightarrow C$ by

$$|||s||| := \inf\{|||f||| : q(f) = s\} \qquad , s \in \mathcal{S}^\Gamma(X).$$

Since C is complete (in the sense of [5], §1), the above infimum exists, and it follows easily from [5], (2.10), (IA), – which is a simple consequence of (IS) – and (LIM) that $||| \; ||| : \mathcal{S}^\Gamma(X) \longrightarrow C$ is a prenorm.

Next we put $S_{\mathcal{S}^\Gamma(X)} := q^N(S_{\mathcal{L}^N(X)})$. Suppose now that f_*, f'_* are in $S_{\mathcal{L}^N(X)}$ and satisfy $q^N(f_*) = q^N(f'_*)$, that is $q(f_n) = q(f'_n)$ for all $n \in N$. Then $(f_n, f'_n) \in \sim$ and hence $(f_n, f'_n) \in S_h$ for all contracting homomorphisms $h : \mathcal{L}^N(X) \longrightarrow M$ that are S-compatible, for all $n \in N$. This shows that $h^N(f_*) = h^N(f'_*)$ and therefore

$$h(\Sigma_{\mathcal{L}^N(X)} f_*) = \Sigma_M(h^N(f_*)) = \Sigma_M(h^N(f'_*)) = h(\Sigma_{\mathcal{L}^N(X)} f'_*),$$

that is we obtain the relation $(\Sigma_{\mathcal{L}^N(X)} f_*, \Sigma_{\mathcal{L}^N(X)} f'_*) \in S_h$ for all such h, whence $(\Sigma_{\mathcal{L}^N(X)} f_*, \Sigma_{\mathcal{L}^N(X)} f'_*) \in\sim$ holds. Thus we can define

$$\Sigma_{\mathcal{S}^\Gamma(X)}(s_*) := q(\Sigma_{\mathcal{L}^N(X)} f_*) \qquad , s_* \in S_{\mathcal{S}^\Gamma(X)},$$

where $f_* \in S^N_{\mathcal{L}}(X)$ is chosen such that $g^N(f_*) = s_*$. In particular, $q(\Sigma_{\mathcal{L}^N(X)}(f_*)) = \Sigma_{\mathcal{S}^\Gamma(X)}(q^N(f_*))$ for all $f_* \in S_{\mathcal{L}^N(X)}$. One checks easily that $S^\Gamma_{\mathcal{S}}(X)$ is an R-subsemimodule of $\mathcal{S}^\Gamma(X)^N$ and that $\Sigma_{\mathcal{S}^\Gamma(X)} : S_{\mathcal{S}^\Gamma(X)} \longrightarrow \mathcal{S}^\Gamma(X)$ is a homomorphism of R-semimodules.

Next we verify (1.4), (o), for $\mathcal{S}^\Gamma(X)$. Let $s_* \in S_{\mathcal{S}^\Gamma(X)}$ and choose $f_* \in S_{\mathcal{L}^N(X)}$ with $q^N(f_*) = s_*$. Then $|||s_n||| \leq |||f_n|||$, $n \in N$. Since $|||f_*|||$ is in S_C, so is $|||s_*|||$ due to (1.1), (ii). Conversely, assume $s_* \in \mathcal{S}^\Gamma(X)^N$ and $|||s_*||| \in S_C$. We apply (OP) to $c_* := |||s_*|||$ and obtain d_* with the properties stated there. If $|||s_n|||$ is a maximal element of C, choose $f_* \in \mathcal{L}^N(X)$ such that $q(f_n) = s_n$. Then $|||f_n||| = |||s_n|||$. If $|||s_n|||$ is not a maximal element of C, (LD) implies the existence of an $f_n \in \mathcal{L}^N(X)$ with $q(f_n) = s_n$ and $|||s_n||| \leq |||f_n||| \leq |||s_n||| + d_n$. Hence f_* is in $\mathcal{L}^N(X)$ and $|||f_*||| \leq |||s_*||| + d_* \in S_C$, whence $|||f_*|||$ is in S_C due to (1.1), (ii), and thus f_* is in $S_{\mathcal{L}^N(X)}$ by (1.4), (o). Therefore $s_* = q^N(f_*)$ is in $S_{\mathcal{S}^\Gamma(X)}$.

(1.4), (i), is trivially satisfied in the current situation.

On to (1.4), (ii). It follows from (1.5) that $s_* \in S_{\mathcal{S}^\Gamma(X)}$ and $t_* \in \mathcal{S}^\Gamma(X)^N$ with $|||t_*||| \leq |||s_*|||$ implies $t_* \in S_{\mathcal{S}^\Gamma(X)}$. Hence there are $g_* \in S_{\mathcal{L}^N(X)}$ with $t_* = q^N(g_*)$ and

$$|||\Sigma_{\mathcal{S}^\Gamma(X)} t_*||| = |||q(\Sigma_{\mathcal{L}^N(X)} g_*)||| \leq |||\Sigma_{\mathcal{L}^N(X)} g_*||| \leq \Sigma_C |||g_*|||,$$

due to (1.4), (ii), for $\mathcal{L}^N(X)$. Hence we have

$$\begin{aligned} |||\Sigma_{\mathcal{S}^\Gamma(X)} t_*||| &\leq \inf\{\Sigma_C |||g_*||| : q^N(g_*) = t_*\} \\ &= \Sigma_C(\inf\{|||g_*||| : q^N(g_*) = t_*\}) = \Sigma_C |||t_*|||. \end{aligned}$$

Finally (1.4), (iii). Let $\varphi : N \longrightarrow N$ be a map and let $s_* \in S_{\mathcal{S}^\Gamma(X)}$. Then there is an $f_* \in S_{\mathcal{L}^N(X)}$ with $s_* = q^N(f_*)$. Since $s_*^{\varphi^{-1}(n)} = q^N(f_*^{\varphi^{-1}(n)})$ and since $f_*^{\varphi^{-1}(n)}$ is in $S_{\mathcal{L}^N(X)}$, due to (1.4), (iii), applied to $\mathcal{L}^N(X)$, we have $s_*^{\varphi^{-1}(n)} \in S_{\mathcal{S}^\Gamma(X)}$ for every $n \in N$. Moreover, by definition, $\Sigma_{\mathcal{S}^\Gamma(X)} s_*^{\varphi^{-1}(n)} = q(\Sigma_{\mathcal{L}^N(X)} f_*^{\varphi^{-1}(n)})$. Since $f_*^{\varphi^{-1}}$, that is the map $N \ni n \mapsto \Sigma_{\mathcal{L}^N(X)} f_*^{\varphi^{-1}(n)} \in \mathcal{L}^N(X)$, is in $S_{\mathcal{L}^N(X)}$ it follows that $s_*^{\varphi^{-1}}$ is in $S_{\mathcal{S}^\Gamma(X)}$. Finally

$$\Sigma_{\mathcal{S}^\Gamma(X)} s_*^{\varphi^{-1}} = q(\Sigma_{\mathcal{L}^N(X)} f_*^{\varphi^{-1}}) = q(\Sigma_{\mathcal{L}^N(X)} f_*) = \Sigma_{\mathcal{S}^\Gamma(X)} s_*.$$

Thus we have shown that $\mathcal{S}^\Gamma(X)$ is an R-prenormed R-semimodule with left N-summation $(S_{\mathcal{S}^\Gamma(X)}, \Sigma_{\mathcal{S}^\Gamma(X)})$.

Additionally we claim that $q : \mathcal{L}^N(X) \longrightarrow \mathcal{S}^\Gamma(X)$ is a contracting homomorphism of R-prenormed R-semimodules with left N-summation. (1.5), (i), is obvious from the construction of $\mathcal{S}^\Gamma(X)$, while (1.5), (ii), – with $c = 1$ – was established in the above verification of (1.4), (ii).

What remains to be done is to show that $\mathcal{B}_N(q) \circ \delta : X \longrightarrow \mathcal{O}_\Gamma(\mathcal{S}^\Gamma(X))$ is a universal arrow. Since q is a homomorphism of R-prenormed R-semimodules with N-summation, $\mathcal{B}_N(q) \circ \delta$ is a homomorphism of left Γ-convex modules. Let $h : X \longrightarrow \mathcal{O}_\Gamma(M)$ be such a homomorphism. Due to (3.1) there is a contractive homomorphism $h' : \mathcal{L}^N(X) \longrightarrow M$ of R-prenormed R-semimodules with N-summation with $\mathcal{B}_N(h') \circ \delta = h$. Since h is a homomorphism of left Γ-convex modules, h' is S-compatible and hence gives rise to a factorization $h' = h'' \circ q$, where $h'' : \mathcal{S}^\Gamma(X) \longrightarrow M$ is a contractive homomorphism of R-prenormed R-semimodules with N-summation. Hence $h = \mathcal{B}_N(h'' \circ q) \circ \delta$, which is the required factorization. We claim that h determines $\overline{h}$ uniquely. So, let $\widetilde{h} \circ q \circ \delta = h$ be another factorization. Each $s \in \mathcal{S}^\Gamma(X)$ can be written as $q(f)$, with $f \in \mathcal{L}^N(X)$. Due to (2.4), f equals $\Sigma_{\mathcal{L}^N(X)}(\alpha_* \delta^{\chi(*)})$, where $\chi : N \longrightarrow A$ is a suitable map and $\alpha_* \in S_R$ is chosen appropriately. Hence we have

$$\begin{aligned} \widetilde{h}(s) = \widetilde{h}(q(f)) &= \widetilde{h}(q(\Sigma_{\mathcal{L}^N(X)}(\alpha_* \delta^{\chi(*)}))) \\ &= \Sigma_M(\alpha_* (\widetilde{h} \circ q)^N(\delta^{\chi(*)})) = \Sigma_M(\alpha_* \widetilde{h}^{\chi(*)}), \end{aligned}$$

where $\widetilde{h}^{\chi(*)}$ is the map $N \ni n \mapsto h(\chi(n)) \in \mathcal{O}_\Gamma(M)$. Thus h determines $\overline{h}$ uniquely. $\square$

6. Examples

Clearly, every positive semiring C has $(C^{(N)}, \sum'_C)$, with $\sum'_C$ the usual sum in C, as a left N-summation. Similarly, every prenormed semiring R has $(R^{(N)}, \sum'_R)$ as a left N-summation, just as $(M^{(N)}, \sum'_N)$ is a left N-summation for every R-prenormed R-semimodule. This means that the positive (resp. prenormed) semirings, the R-prenormed R-semimodules, and the finitary convexity theories discussed in [5], §1 - §3, are special cases of the notions investigated in the present paper.

One checks quite easily that the Banach semirings R discussed in [5], §6, are another instance of the concepts treated here. The cone of such a Banach semiring R has to satisfy suitable properties (see [5], 4.14 and 4.15); S_C is then the subset of $C^{\mathbb{N}}$ consisting of all those α_* for which $\Sigma\alpha_*$ has a limit (in the sense of [5], §4), while $\Sigma_C\alpha_* := \sup\{\alpha_n : n \in \mathbb{N}\}$. In addition, S_R is the set of those $\beta_* \in R^{\mathbb{N}}$ for which $\Sigma\beta_*$ is an absolute Cauchy sum (in the sense of [5], §4), while $\Sigma_R\beta_*$ is the limit (in the sense of [5], §4) of the infinite sum $\sum \beta_*$. Analogously one obtains (S_M, Σ_M) for each Banach R-semimodule M; in particular, each Banach R-semimodule is an R-prenormed R-semimodule with $\mathbb{N}$-summation in current terminology, while the converse in general fails to be correct. However, if $C := \mathbb{R}_+ = \{r \in \mathbb{R} : r \geq 0\}$, $R := \mathbb{R}$, $\| \; \| : \mathbb{R} \longrightarrow \mathbb{R}_+$ is the usual absolute value, and S_C is the set of all $\alpha_* \in \mathbb{R}_+^{\mathbb{N}}$ for which $\sum \alpha_*$ converges and $\Sigma_C\alpha_* = \sum \alpha_*$ then the Banach spaces over $\mathbb{R}$ (in the sense of functional analysis) are precisely the $\mathbb{R}$-prenormed $\mathbb{R}$-semimodules with $\ltimes$-summation as follows from a well known characterization of Banach spaces ([6], 3.1.2).

Now we want to characterize explicitly the concepts of the present paper in the case where the semiring involved is the smallest semiring that is not a ring. Define on the two-element set $\{0, 1\}$

addition by $\quad 0 + 0 = 0,\ 0 + 1 = 1 + 0 = 1 + 1 = 1,$
multiplication by $\quad 0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0, 1 \cdot 1 = 1,$
partial order by $\quad 0 < 1.$

Then $\{0, 1\}$ equipped with this structure is a complete, commutative, and unital semiring $\mathbb{D}$. Define an N-summation on $\mathbb{D}$ by putting $S_{\mathbb{D}} := \mathbb{D}^N$ and $\Sigma_{\mathbb{D}}\alpha_* := \max\{\alpha_n : n \in N\}$. One checks easily that these data make $\mathbb{D}$ a positive semiring with N-summation.

Put $R := \mathbb{D}$ and define $\| \; \| : R \longrightarrow \mathbb{D}$ by $\|r\| := r$, $r \in R$. Let furthermore $S_R := \mathbb{D}^N$ and $\Sigma_R := \Sigma_{\mathbb{D}}$. Then $R = \mathbb{D}$ is a prenormed (even normed) semiring with N-summation. A $\mathbb{D}$-prenormed $\mathbb{D}$-semimodule M is commutative, idempotent (i.e. $m + m = m$ for all $m \in M$) monoid together with a submonoid M_0 (corresponding to $\{m \in M : \|m\| = 0\}$); one check easily that M_0 can be an arbitrary submonoid of M. The monoid M has the additional property that 0 is the only element of M that possesses an (additive) inverse. Next we define "$m_1 \leq m_2$" as "there is an $m \in M$ with $m_1 + m = m_2$", for all $m_1, m_2 \in M$. One checks easily that this is a partial order relation on M (in particular, $m_1 \leq m_2$ and $m_2 \leq m_1$ imply $m_1 = m_2$) that is compatible with the additive monoid structure on M. With respect to this order relation, M has finite suprema and $\sup\{m_1, \ldots, m_n\} = m_1 + \ldots + m_n$. Hence

M_0 is closed under finite suprema. If the $\mathbb{D}$-prenormed $\mathbb{D}$-semimodule M has N-summation (S_M, Σ_M) then (1.4), (o), implies $S_M = M^N$. If $\mu_* \in M^N$ and $T \subseteq N$ we denote by μ_*^T the map given by $\mu_*^T|T = \mu_*|T$ and $\mu_*^T|N \setminus T = 0_*|N \setminus T$. Then (1.4), (iii), shows that $\Sigma_M \mu_*^T \leq \Sigma_M \mu_*$ holds for all $\mu_* \in M^N$ and $T \subseteq N$. It follows from (3.1) that for every $m \in M$, $m = \Sigma_M m_*$ is valid. An immediate consequence of this is $\Sigma_M \mu_* = \sup\{\mu_n : n \in N\}$. This means that M has N-suprema, that is suprema of all subsets of M of cardinality $\leq \mathrm{card} N$.

Conversely one checks easily that any commutative, idempotent, partially ordered monoid M with a distinguished submonoid M_0 such that the partial order is compatible with the monoid structure and has N-suprema is in fact a $\mathbb{D}$-prenormed $\mathbb{D}$-semimodule with N-summation (M^N, Σ_M), where $\Sigma_M \mu_* = \sup\{\mu_n : n \in N\}$.

Let $\Gamma_{\mathbb{D}} := \mathbb{D}^N$. It is easy to see that $\Gamma_{\mathbb{D}}$ is an N-convexity theory over $\mathbb{D}$. Let X be a Γ-convex module. Then we say that for $x_1, x_2 \in X$ the relation $x_1 \leq x_2$ is valid precisely when there are $\alpha_* \in \Gamma$ and $x^* \in X^N$ such that

(i) there is an $i \in \mathrm{supp}\ \alpha_*$ with $x^i = x_1$,

(ii) $\langle \alpha_*, x^* \rangle = x_2$.

One checks easily that this defines a partial order on X and that (with respect to this partial order) X has N-suprema. In fact, $\langle \alpha_*, x^* \rangle = \sup\{x^n : n \in \mathrm{supp}\ \alpha_*\}$. Moreover, if $Y \subseteq X$ is a subset of cardinality $\leq \mathrm{card} N$, let $\varphi : Y \longrightarrow N$ be an injective map, and define α_* resp. x^* as the maps (with $y_0 \in X$ chosen arbitrarily)

$$N \ni n \mapsto \begin{cases} 1 & \text{, if } n \in \mathrm{im}\varphi; \\ 0 & \text{, otherwise;} \end{cases} \qquad \text{resp.} \qquad N \ni n \mapsto \begin{cases} y & \text{, if } n = \varphi(y); \\ y_0 & \text{, otherwise.} \end{cases}$$

Then $\sup(Y) = \langle \alpha_*, x^* \rangle$.

Conversely, if X is a partially ordered set that has N-suprema, define

$$\langle \alpha_*, x^* \rangle := \sup\{x^n : n \in \mathrm{supp}\ \alpha_*\} \qquad , \alpha_* \in \Gamma \text{ and } x^* \in X^N.$$

A simple computation shows that this makes X a $\Gamma_{\mathbb{D}}$-convex module. Finally one concludes from (4.3) that a map $f : X \longrightarrow X'$ between $\Gamma_{\mathbb{D}}$-convex modules is a homomorphism of $\Gamma_{\mathbb{D}}$-convex modules precisely when for each subset $Y \subseteq X$ of cardinality $\leq \mathrm{card} N$, $\sup(f(Y)) = f(\sup(Y))$ gilt. Hence the category $\Gamma_{\mathbb{D}} C$ is isomorphic to the category of partially ordered sets with N-suprema and N-suprema preserving maps.

Instead of $\Gamma_{\mathbb{D}}$ one could take the set $\Gamma_{\mathbb{D},}$ of all $\alpha_* \in \Gamma_{\mathbb{D}}$ with $\mathrm{card}(\mathrm{supp}\ \alpha_*) < \mathrm{card} N$. A simple computation shows that $\Gamma_{\mathbb{D},}$ is an N-convexity theory over $\mathbb{D}$. Then one obtains the same results as in the case of $\Gamma_{\mathbb{D}}$ except that the requirement "existence of the supremum of every subset Y with $\mathrm{card} Y \leq \mathrm{card} N$" has to be replaced by the requirement "existence of the supremum of every subset Y with $\mathrm{card} Y < \mathrm{card} N$".

Instead of $\Gamma_{\mathbb{D}}$ one could take the set $\Gamma_{\mathbb{D}} \setminus \{0_*\}$. Again it is easy to see that $\Gamma_{\mathbb{D},sc} := \Gamma_{\mathbb{D}} \setminus \{0_*\}$ is an N-convexity theory over $\mathbb{D}$. Again, as before, the same results remain in force, except that the subsets Y in question now have to be nonempty. It should be pointed out, that $\Gamma_{\mathbb{D},sc}$ is the $\mathbb{D}$-analog to the superconvexity theory $\Omega_{sc} := \{\alpha_* \in \mathbb{R}^{\mathbb{N}} : \alpha_n \geq 0$, for all $n \in \mathbb{N}$, and $\Sigma_{\mathbb{N}} \alpha_n = 1\}$. The $\mathbb{D}$-analog to the classical convexity theory $\Omega_c := \{\alpha_* \in \Omega_{sc} : \mathrm{supp}\ \alpha_*$ is finite$\}$ is then $\Gamma_{\mathbb{D},c} := \{\alpha_* \in \Gamma_{\mathbb{D},sc} : \text{supp } \alpha_* \text{ is finite}\}$; results similar to the above hold for $\Gamma_{\mathbb{D},c}$.

References

[1] Birkhoff, G.: *Lattice Theory.* AMS Coll. Publ. XXV (1967).

[2] Manes, E. G.: *Algebraic Theories.* Springer Verlag (1976).

[3] Pareigis, B., and Röhrl, H.: *Left linear theories.* Sem.ber. Fernuniv. Hagen 44, 276-302 (1992).

[4] Pumplün, D., and Röhrl, H.: *Banach spaces and totally convex spaces I.* Comm. in Alg. 12, 953–1019 (1984).

[5] Röhrl, H.: *Convexity theories 0. Foundations.* Sem.ber. Fernuniv. Hagen 44, 333–376 (1992).

[6] Semadeni, Z.: *Banach spaces I.* Polish Sci. Publ. (1971).

[7] Weinert, H. J.: *Generalized semialgebras over semirings.* Springer Lecture Notes in Mathematics 1320, 380–416 (1986).

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