Quantum Groups – The Functorial Side

Bodo Pareigis

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Abstract

Quantum groups can be introduced in various ways. We use their functorial construction as automorphism groups of noncommutative spaces. This construction shows in particular why quantum groups should not be considered as groups in the categorical sense.

The linear representations of an ordinary group form a symmetric monoidal category. Because of their noncommutativity quantum groups do not have this symmetry of the tensor product of representations. In many cases, however, it can be replaced by a braiding, for example in the category of Yetter-Drinfeld modules or of modules over a quasitriangular Hopf algebra.

A generalization of the definition of Yetter-Drinfeld modules leads to a categorical riddle: an example of a universal-couniversal problem, that is defined by a simultaneous unit and counit.

Introduction

In this survey paper we want to give an introduction to quantum groups using a functorial construction. It is well known to the specialists that quantum groups are not groups in a category. We discuss how close they are to categorical groups. This slight generalization leads to an unusual behavior of their representations. To get a similar representation theory as for groups one imposes a braid structure on the category of representations. We discuss what kind of elements of a quantum group are responsible for such braid structures and where such braid structure occur naturally.

1 Quantum Automorphism Groups of Noncommutative Spaces

Quantum groups arose from the deformation of function algebras of (Lie-)groups. Function algebras of (Lie-)groups and more generally of manifolds are automatically commutative, because the functions have values in the field of complex numbers or an arbitrary field). But quantum physics requires noncommutative function algebras. Deformation techniques have been successfully applied to classical Lie-groups and to universal enveloping algebras of Lie-algebras to construct quantum groups. In this survey paper we want to pursue a different approach. We will construct quantum groups as *automorphism groups of noncommutative spaces.*

It is a classic technique to describe geometric spaces $\mathcal{X}$ through their function algebra $\mathcal{O}(\mathcal{X}) = \mathrm{Fun}(\mathcal{X}, \mathbf{C})$. For example *state spaces* in physics are described by their *algebra of observables.* Under suitable assumptions this leads to a duality between the category of geometric spaces and of commutative algebras. Famous examples are the duality of the Gelfand-Naimark Theorem and the duality between affine algebraic manifolds and finitely generated commutative algebras.

Quantum theory forces us to consider noncommutative algebras as "function algebras" of *noncommutative geometric spaces* or *quantum spaces*. Thus we define

Definition 1.1 (Manin [6]) The *category of quantum spaces* or of *noncommutative spaces* is the dual of the category of noncommutative (not necessarily commutative) algebras.

This definition should be sufficient for a categorical minded reader. But somehow one is missing the corresponding geometric structures. There have been several attempts to make geometric structures visible. One of the simplest is the following.

The category of quantum spaces can be identified with the category of covariant representable functors on the category Alg of not necessarily commutative algebras. Thus a *quantum space* $\mathcal{X}$ with given function algebra A can be viewed as the representable functor $\mathcal{X}(B) := \mathrm{Alg}(A, B)$. The elements of $\mathcal{X}(B)$ are called the B-*points* of $\mathcal{X}$.

These sets $\mathcal{X}(B)$ can be considered as a replacement for a geometric space associated with the function algebra A. Indeed if A is represented as

$$A = k\langle x_1, x_2, \ldots\rangle/(p_1(x_i), p_2(x_i), \ldots),$$

a residue class algebra of the noncommutative polynomial ring (free algebra) on the variables $x_1, x_2, \ldots$, then each B-point $f : A \longrightarrow B$ is described by the values $b_1, b_2, \ldots$ (with $b_i := f(x_i)$) that are zeros of the polynomials $p_1(x_i), p_2(x_i), \ldots$. So $\mathcal{X}(B)$ can be considered as the set of zeros of certain noncommutative polynomials with coordinates in B.

We want to construct groups in the category of quantum spaces. For this purpose we need to know the "product" of two quantum spaces. There are various reasons based in physics as well as in algebra not to use the categorical product. We introduce a product that is much "smaller" than the categorical product. *Quantum groups* will be defined with respect to this smaller product.

Definition 1.2 Let $\mathcal{X}$ and $\mathcal{Y}$ be quantum spaces. Then their *orthogonal product* is defined by

$$(\mathcal{X} \perp \mathcal{Y})(B) :=$$
$$\{(\xi, \zeta) \in \mathcal{X}(B) \times \mathcal{Y}(B) \,|\forall a \in \mathcal{O}(\mathcal{X}), a' \in \mathcal{O}(\mathcal{Y}) :\ \xi(a)\zeta(a') = \zeta(a')\xi(a)\}\,.$$

The B-points ξ and ζ are called *commuting points*. (The images of the functions ξ and ζ in B commute.)

The functor $\mathcal{X} \perp \mathcal{Y}$ is a representable functor and thus a quantum space with representing algebra $\mathcal{O}(\mathcal{X}) \otimes \mathcal{O}(\mathcal{Y})$. The orthogonal product defines the structure of a monoidal category on the category of quantum spaces.

We want to define an automorphism group for a quantum space $\mathcal{X}$, in the category of quantum spaces. That means the group to be constructed should live in the category of quantum spaces and should act in a suitable way on an other quantum space. For all considerations we use the orthogonal product instead of the categorical product. We begin with the notion of an action of one quantum space $\mathcal{M}$ on another quantum space $\mathcal{X}$.

Definition 1.3 Let $\mathcal{X}$ and $\mathcal{M}$ be quantum spaces. A (left) *action* of $\mathcal{M}$ on $\mathcal{X}$ is a morphism $\lambda : \mathcal{M} \perp \mathcal{X} \longrightarrow \mathcal{X}$.

An action $\lambda : \mathcal{M} \perp \mathcal{X} \longrightarrow \mathcal{X}$ is called a *universal action* if for every action $\nu : \mathcal{Z} \perp \mathcal{X} \longrightarrow \mathcal{X}$ there is a unique factorization $f : \mathcal{Z} \longrightarrow \mathcal{M}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{Z} \perp \mathcal{X} \arrow[d, "f\perp 1"'] \arrow[dr, "\nu"] & \\
\mathcal{M} \perp \mathcal{X} \arrow[r, "\lambda"] & \mathcal{X}
\end{tikzcd} .$$

It is an open question under which general conditions on $\mathcal{X}$ such universal actions exist. But we have the following

Theorem 1.4 (Tambara [13]) *Let $\mathcal{X}$ be a quantum space with finite dimensional representing algebra A. Then there exists a universal action of a quantum space on $\mathcal{X}$.*

A universal action performs a surprising little magic. It turns out that a universal action automatically carries a natural structure of a monoid and that this structure comes in such a way that the action on the quantum space $\mathcal{X}$ is a monoid action i. e. it is associative with unit. Furthermore the monoid structure and the monoid action are defined with respect to the orthogonal product, not with respect to the categorical product.

Proposition 1.5 *Let $\lambda : \mathcal{M} \perp \mathcal{X} \longrightarrow \mathcal{X}$ be a universal action. Then $\mathcal{M}$ is in a unique way a monoid in the monoidal category of quantum spaces with tensor product $\perp$ such that it acts on $\mathcal{X}$ by a monoid action (associative with unit).*

Sketch of proof: The proof is actually quite simple. We give only the definition of the multiplication of the monoid. The commutative diagram

$$\begin{tikzcd}
\mathcal{M} \perp \mathcal{M} \perp \mathcal{X} \arrow[r, "\mathrm{id}\perp\lambda"] \arrow[d, "\mu\perp\mathrm{id}"'] & \mathcal{M} \perp \mathcal{X} \arrow[d, "\lambda"] \\
\mathcal{M} \perp \mathcal{X} \arrow[r, "\lambda"] & \mathcal{X}
\end{tikzcd}$$

defines by the universal property of the action a unique morphism $\mu : \mathcal{M} \perp \mathcal{M} \longrightarrow \mathcal{M}$. For this multiplication of $\mathcal{M}$ the action becomes associative by the same diagram. We leave the rest of the proof to the reader. A complete proof can be found in [8]. □

Thus $\mathcal{M}$ can be considered as the endomorphism monoid of $\mathcal{X}$. It is called the *quantum endomorphism monoid* of $\mathcal{X}$. It turns out to satisfy an additional universal property, it is the universal monoid acting on $\mathcal{X}$ by a monoid action.

Remark 1.6 The universal action and the monoid structure of $\mathcal{M}$ translates back into the representing algebras as a universal algebra homomorphism $A \longrightarrow B \otimes A$, the structure of a *bialgebra* on B (algebra plus coalgebra plus compatibility) and the structure of a *B-comodule algebra* on A through the algebra homomorphism $A \longrightarrow B \otimes A$. In particular every finite dimensional algebra A has a universal bialgebra B that makes A into a B-comodule algebra.

Thus we have achieved part of our task to find a quantum automorphism group of a quantum space. In the simplest set theoretic situation we would now have to collect the invertible endomorphisms of $\mathcal{X}$ and we would then obtain the automorphism group with all its universal properties. But picking elements in $\mathcal{M}$ and checking for invertibility is not quite the right way in the situation of quantum spaces. So we will be looking for submonoids $\mathcal{A}$ of $\mathcal{M}$ that have something like an "invert" function $S : \mathcal{A} \longrightarrow \mathcal{A}$ and hopefully pick the best such submonoid.

Since a reasonable inverse function $S : \mathcal{A} \longrightarrow \mathcal{A}$ does not exist for most quantum monoids (it cannot exist if the quantum monoid is a proper noncommutative space, i. e. the function algebra is noncommutative and if the multiplication of the quantum monoid is also noncommutative) we have to go to the corresponding function algebras. And there, indeed, we may find an inverse function $S : H \longrightarrow H$ that behaves very much like forming inverses in a group. The problem is that this inverse function, usually called the *antipode*, is only a linear map and not an algebra homomorphism. We define the inverse or antipode by

Definition 1.7 A bialgebra H is called a *Hopf algebra* if there is a (unique) (linear) map $S : H \longrightarrow H$, the *antipode*, such that the diagram

$$\begin{tikzcd} H \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & \mathbf{C} \arrow[r, "\eta"] & H \\ H \otimes H \arrow[rr, shift left, "S\otimes 1"] \arrow[rr, shift right, "1\otimes S"'] & & H \otimes H \arrow[u, "\nabla"'] \end{tikzcd}$$

commutes.

The antipode of a Hopf algebra is unique and it is an algebra antihomomorphism and a coalgebra antihomomorphism.

This definition was used by Drinfeld

Definition 1.8 (Drinfeld [3]) A *quantum group* is (the dual of) a (noncommutative and noncocommutative) Hopf algebra.

The next problem we face is to find a universal quantum subgroup of a given quantum monoid. This would then represent the set of invertible elements of the monoid. This problem has already been solved many years ago by M. Takeuchi on the function algebra side.

Theorem 1.9 (Takeuchi [14]) *Let B be a bialgebra. Then there exists a universal Hopf algebra $H(B)$ together with a bialgebra homomorphism $B \longrightarrow H(B)$.*

This guarantees that we may adjoin an antipode to any given bialgebra. Translated back into the language of quantum spaces the theorem says that for every quantum monoid $\mathcal{M}$ there is a universal quantum group $\mathcal{G}(\mathcal{M})$ together with a homomorphism of quantum monoids $\mathcal{G}(\mathcal{M}) \longrightarrow \mathcal{M}$.

In the language of set theoretic monoids and groups this amounts to the construction of the group of invertible elements $G(M)$ of a monoid M. We have obtained the correct functor for the "quantum automorphism group" of a quantum space $\mathcal{X}$. Indeed we have

Corollary 1.10 *Let $\mathcal{X}$ be a quantum space that possesses a universal action. Then there is a universal quantum group $\mathcal{G}$ acting on $\mathcal{X}$ by a monoid action (or as an automorphism group).*

2 Braidings and Representations of Quantum Groups

Let G be a group (in the category of sets). A (complex vector space) representation of G is a complex vector space P together with a module action $\mathbf{C}[G] \otimes P \longrightarrow P$. The representations of a group have two properties that we would like to see for representations of quantum groups as well.

Given two representations P and Q of G then $P \otimes Q$ is also a representation of G with the action $g \cdot (p \otimes q) := gp \otimes gq$. With this tensor product the category $\mathbf{C}[G]$-Mod becomes a monoidal category.

This monoidal category is symmetric with the symmetry map $\tau : P \otimes Q \ni p \otimes q \mapsto q \otimes p \in Q \otimes P$. One easily checks that this is a natural isomorphism of $\mathbf{C}[G]$-modules.

The function algebra of G is $\mathbf{C}^G$, the commutative algebra of maps from G to $\mathbf{C}$. If G is a finite group then the isomorphism

$$\mathrm{Hom}(K[G] \otimes P, P) \cong \mathrm{Hom}(P, P \otimes \mathbf{C}^G)$$

transforms every module action of $\mathbf{C}[G]$ on P to a comodule action of $\mathbf{C}^G = \mathbf{C}[G]^*$ on P. Thus the category of $\mathbf{C}^G$-comodules is equivalent to the category of $\mathbf{C}[G]$-modules or representations of G.

Since the bialgebra B from the previous section is a function algebra and the Hopf algebra $H(B)$ is to be considered as a function algebra as well, the representations of these bialgebras should be comodules over B resp. over $H(B)$. Comodules over B also form a monoidal category with $\delta(p \otimes q) = \sum p_{(0)} \otimes q_{(0)} \otimes p_{(1)} q_{(1)} \in P \otimes Q \otimes B$ (Sweedler notation).

Since the axioms for bialgebras B or Hopf algebras H are self dual, representations could be defined to be modules or comodules over B resp. H. Both definitions give a monoidal category.

The principal question is now if the monoidal category of H-modules or of H-comodules is symmetric or has at least a good interchange map $\tau : P \otimes Q \longrightarrow Q \otimes P$. If H is cocommutative then the category H-Mod is symmetric by the usual interchange map $\tau(p \otimes q) = q \otimes p$. If H is commutative then the category Comod-H is also symmetric by the usual interchange map $\tau(p \otimes q) = q \otimes p$. In general, however, this is not the case.

To introduce the appropriate terminology we define

Definition 2.1 Let $\mathcal{C}$ be a monoidal category. A natural isomorphism $\tau : P \otimes Q \longrightarrow Q \otimes P$ in $\mathcal{C}$ is called a *braiding* if the following diagrams commute:

$$\begin{tikzcd}
P \otimes Q \otimes R \arrow[rr, "\pi(P \otimes Q, R)"] \arrow[dr, "1 \otimes \pi(Q,R)"'] & & R \otimes P \otimes Q \\
& P \otimes R \otimes Q \arrow[ur, "\pi(P,R) \otimes 1"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
P \otimes Q \otimes R \arrow[rr, "\pi(P, Q \otimes R)"] \arrow[dr, "\pi(P,Q) \otimes 1"'] & & Q \otimes R \otimes P \\
& Q \otimes P \otimes R \arrow[ur, "1 \otimes \pi(P,R)"'] &
\end{tikzcd}$$

commute. (We assume that $\mathcal{C}$ is strict, so that we may omit the associativity morphisms.)

We now restrict our discussion to right H-modules over a Hopf algebra H with bijective antipode S and develop necessary and sufficient conditions for Mod-H to have a braiding. Similar considerations hold for comodules.

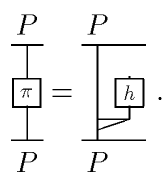
Before we introduce a suitable interchange map $\tau : P \otimes Q \longrightarrow Q \otimes P$, let us study a simpler case of a natural transformation.

We consider the following universal problem. Given an element $h \in H$. Then this element induces a natural transformation $\pi_h : \omega \longrightarrow \omega$ for the underlying functor $\omega : \text{Mod-}H \longrightarrow \text{Vec}$ by the commutative diagram

$$\begin{tikzcd}
\omega(P) \arrow[d, "1\otimes h"'] \arrow[dr, "\pi_h"] & \\
\omega(P)\otimes H \arrow[r, "\mu"] & \omega(P)
\end{tikzcd}$$

Instead of $\pi(P) : \omega(P) \longrightarrow \omega(P)$ we simply write $\pi : P \longrightarrow P$. This leads to the universal problem:

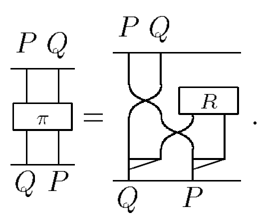
Theorem 2.2 (Pareigis [9]) *For every natural transformation $\pi : P \longrightarrow P$ (in* Vec*) there is a unique $h \in H$ such that*



Actually this theorem says that the algebra H can be reconstructed from the natural endomorphisms of the underlying functor $\omega : \text{Mod-}H \longrightarrow \text{Vec}$.

Now we study a similar universal problem for the tensor product $\omega \otimes \omega$.

Proposition 2.3 *For every natural transformation $\pi : P \otimes Q \longrightarrow Q \otimes P$ there is a unique $R \in H \otimes H$ such that*



It turns out that the natural transformation $\pi : P \otimes Q \longrightarrow Q \otimes P$ is a braiding for the category of right H-modules iff R satisfies the axioms of a *universal R-matrix*, i. e.

1. R is invertible in $H \otimes H$,
2. $\tau(\Delta(h)) = R\Delta(h)R^{-1}$,
3. $(\Delta \otimes \mathrm{id})R = R^{13}R^{23}$,
4. $(\mathrm{id} \otimes \Delta)R = R^{13}R^{12}$,

where $R^{12} = R \otimes 1$, $R^{23} = 1 \otimes R$, and $R^{13} = \sum R_1 \otimes 1 \otimes R_2$ (with $R = \sum R_1 \otimes R_2$). A Hopf algebra together with a universal R-matrix is called a *quasitriangular Hopf algebra*. This is actually what one would like to call a quantum group. But the definition given in the first section has now been generally accepted, especially in view of the fact that there are also coquasitriangular Hopf algebras, i. e. the category of comodules is braided.

In statistical physics these universal R-matrices provide solutions for the quantum Yang-Baxter equation.

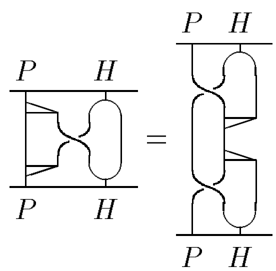
So it is of great interest to know if quasitriangular Hopf algebras exist and how to construct them. We will come back to this question later on.

3 Braidings and Yetter-Drinfeld Modules

There is another setup of Hopf algebras that provides braidings for free. Let us consider the category of vector spaces P that are right H-modules and right H-comodules at the same time together with the following compatibility condition:

$$\sum (p_{(0)} \cdot h_{(1)}) \otimes p_{(1)} h_{(2)} = \sum (p \cdot h_{(2)})_{(0)} \otimes h_{(1)} (p \cdot h_{(2)})_{(1)}$$

or

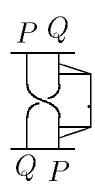


These modules form the category $\mathcal{YD}_H^H$ of *Yetter- Drinfeld modules* over H. The compatibility condition has been known for long in the case of crossed G-sets over a group G. The magic of this compatibility condition is

Proposition 3.1 (Yetter [15]) *The category of Yetter-Drinfeld modules $\mathcal{YD}_H^H$ is a braided monoidal category with the braiding*

$$\tau(P, Q) : P \otimes Q \longrightarrow Q \otimes P, p \otimes q \mapsto \sum q_{(0)} \otimes p \cdot q_{(1)}$$

or



So every quantum group gives rise to a braided monoidal category, the category $\mathcal{YD}_H^H$.

Actually one has an almost universal property.

Theorem 3.2 (Yetter [15]) *Let $\mathcal{C}$ be a small strict braided monoidal category together with a monoidal functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Vec}_f$ to the category of finite dimensional* *vector spaces. Then there is a Hopf algebra H and a factorization*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\widetilde{\mathcal{F}}"] \arrow[d, "\mathcal{F}"'] & \mathcal{YD}_H^H \arrow[d, "\omega"] \\ \mathrm{Vec}_f \arrow[r, tail] & \mathrm{Vec} \end{tikzcd}$$

where $\widetilde{\mathcal{F}}$ preserves the braiding.

So any study of braidings on finite dimensional vector spaces reduces to the study of Yetter-Drinfeld modules. This construction explains also the famous construction of the *Drinfeld double*:

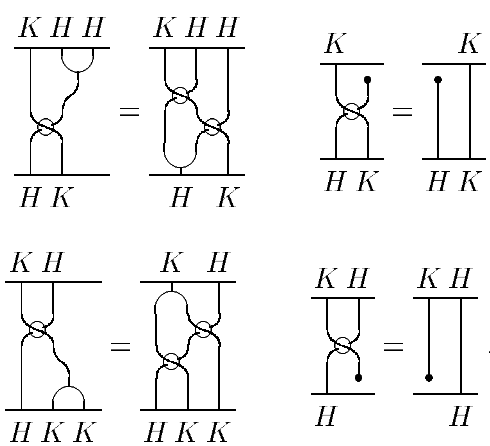
Theorem 3.3 *Let H be a finite dimensional Hopf algebra. Then the category of Yetter-Drinfeld modules $\mathcal{YD}_H^H$ is equivalent as a monoidal category over* Vec *to the category of right modules* Mod-$D(H)$ *over the Drinfeld double $D(H) = H \otimes H^*$ (with appropriate Hopf algebra structure). In particular* Mod-$D(H)$ *is braided and thus $D(H)$ is quasitriangular.*

The famous construction of the Drinfeld double for finite dimensional Hopf algebras provides a wealth of quasitriangular Hopf algebras and thus of solutions of the quantum Yang-Baxter equation.

4 Braidings and Double Quantum Groups

In the category of Yetter-Drinfeld modules (and over the quasitriangular Drinfeld double) we see H-modules and H-comodules simultaneously. We now study the question if a similar setup with two different Hopf algebras is possible.

Definition 4.1 (Brzezinski [1, 2]) Let H be an algebra and K be a coalgebra and let $\psi : K \otimes H \longrightarrow H \otimes K$ be a linear map such that the following hold

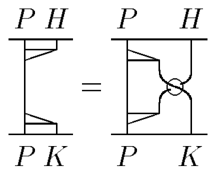


Then (H, K, ψ) is called an *entwining structure*. The map ψ is called an *entwining map*.

Definition 4.2 Let $\mathcal{M}_H^K(\psi)$ be the category of objects that are simultaneously K-comodules and H-modules $(P, \delta_P : P \longrightarrow P \otimes K, \mu_P : P \otimes H \longrightarrow P)$ such that with respect to an entwining structure (H, K, ψ)

$$\delta_P \mu_P = (\mu_P \otimes K)(P \otimes \psi)(\delta_P \otimes H)$$

or



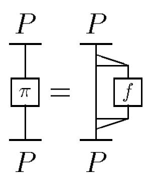
holds. These objects will be called *entwined modules.* Morphisms shall be H-module and K-comodule morphisms.

We consider the following biuniversal problem. Given a homomorphism $f : K \longrightarrow H$ in Vec. Then this homomorphism induces a natural transformation $\pi_f : \omega \longrightarrow \omega$ for the underlying functor $\omega : \mathcal{M}_H^K(\psi) \longrightarrow \mathrm{Vec}$ by the commutative diagram

$$\begin{tikzcd}
\omega(P) \arrow[r, "\delta"] \arrow[drr] & \omega(P) \otimes K \arrow[d, "1\otimes f"] & \\
& \omega(P) \otimes H \arrow[r, "\mu"'] & \omega(P)
\end{tikzcd}$$

Instead of $\pi_f(P) : \omega(P) \longrightarrow \omega(P)$ we write again $\pi : P \longrightarrow P$. This leads to the "biuniversal" problem:

Theorem 4.3 (Hobst-Pareigis [4]) *For every natural transformation $\pi : P \longrightarrow P$ (in* Vec*) there is a unique $f : K \longrightarrow H$ such that*



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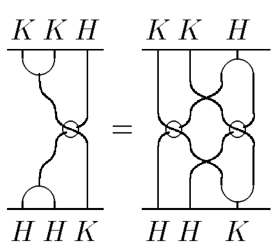
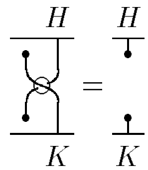
If H is a bialgebra, then the tensor product of two H-modules is again an H-module by the diagonal multiplication. Similarly, if K is a bialgebra, then the tensor product of two K-comodules is a K-comodule by the codiagonal comultiplication. Furthermore $I(= \mathbf{C})$ is a unit object for the tensor product if endowed with the trivial H-structure resp. the trivial K-structure. We want to study conditions under which $\mathcal{M}_H^K(\psi)$ becomes a monoidal category with the given multiplication and comultiplication on the tensor product of two modules. The underlying functor will then preserve the tensor product, i.e. it will be a monoidal functor.

Theorem 4.4 [4] *Let H and K be bialgebras. The category $\mathcal{M}_H^K(\psi)$ is monoidal iff the following additional compatibility conditions for the entwining map $\psi : K \otimes H \longrightarrow H \otimes K$ hold:*



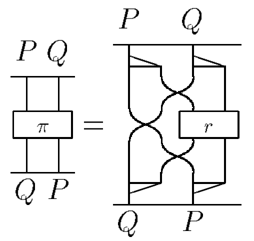
and



If these conditions are satisfied we call (H, K, ψ) *a* monoidal entwining structure *and* ψ *a* monoidal entwining map. *The tensor product* $P \otimes Q$ *of modules* $P, Q \in \mathcal{M}_H^K(\psi)$ *becomes an object in* $\mathcal{M}_H^K(\psi)$ *with the diagonal module and the codiagonal comodule structure.*

An extension of Theorem 4.3 and Proposition 2.3 is

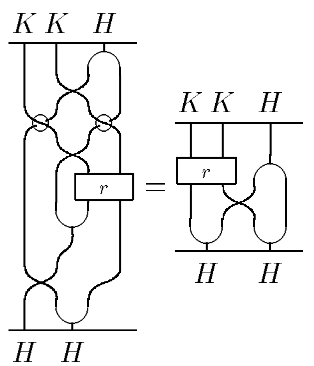
Theorem 4.5 (Hobst-Pareigis [4]) *For every natural transformation* $\tau : P \otimes Q \to Q \otimes P$ *there is a unique* $r : K \otimes K \to H \otimes H$ *such that*



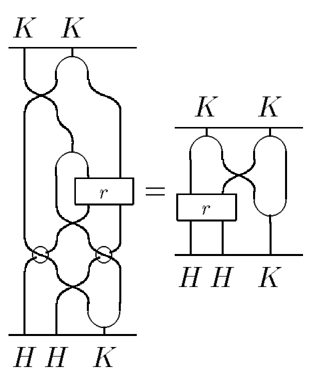
Now we have all the basic tools to determine when $\mathcal{M}_H^K(\psi)$ becomes a braided monoidal category.

Theorem 4.6 [4] *The natural transformation* $\tau : P \otimes Q \to Q \otimes P$ *is a braiding for* $\mathcal{M}_H^K(\psi)$ *if and only if the following conditions are satisfied:*

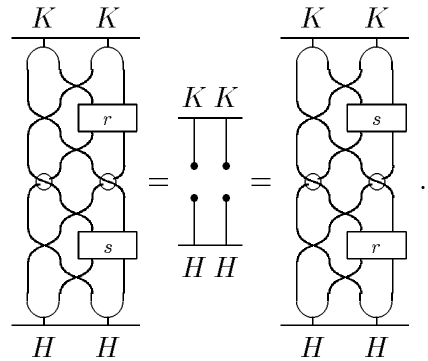
1. τ *is a morphism of* H*-modules or equivalently*



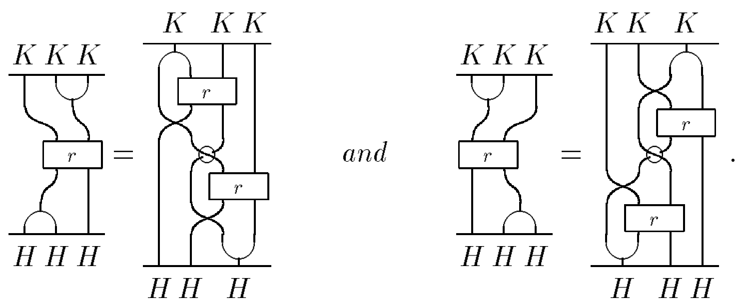
2. τ *is a morphism of* K*-comodules or equivalently*



3. *τ is an isomorphism or equivalently there exists a map $s : K \otimes K \longrightarrow H \otimes H$ such that*



4. *τ is compatible with tensor products or equivalently*



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