INTEGRATION BY PARTS

FOR ABSOLUTELY CONTINUOUS FUNCTIONS

Let $f, g \in AC[0,1]$.

This means that

$$f \in C[0,1], \quad f(x) = f(0) + \int_0^x \tilde{f}(y)\,dy \quad \text{for some } \tilde{f} \in L^1[0,1]$$

$$g \in C[0,1], \quad g(x) = g(0) + \int_0^x \tilde{g}(y)\,dy \quad \text{for some } \tilde{g} \in L^1[0,1].$$

$\tilde{f}, \tilde{g}$ are uniquely determined by f, g. Although a priori f, g are not differentiable in the classical sense, they admit a <u>weak</u> notion of derivative defined almost everywhere:

$$f'(x) := \tilde{f}(x) \quad \text{a.e.}$$

$$g'(x) := \tilde{g}(x) \quad \text{a.e.}$$

<u>Claim</u> : this weak derivative too satisfies the rule of integration by parts

$$\int_0^1 f(x) g'(x)\,dx = f(1)g(1) - f(0)g(0) - \int_0^1 f'(x) g(x)\,dx \qquad (\bullet)$$

Note : this result is not true under the assumption that $f, g \in C[0,1]$ and f, g differentiable pointwise a.e (counterexample: the Cantor function). Absolute continuity is needed.

Proof of (•)

$$\int_0^1 f(x)\, g'(x)\, dx = \int_0^1 \left[f(0) + \int_0^x \tilde{f}(y)\, dy \right] \tilde{g}(x)\, dx$$

$$= f(0) \int_0^1 \tilde{g}(x)\, dx + \int_0^1 dx \left(\int_0^x \tilde{f}(y)\, dy \right) \tilde{g}(x)$$

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$$= f(0)g(1) - f(0)g(0) + \int_0^1 dy \left(\int_y^1 dx\, \tilde{g}(x) \right) \tilde{f}(y)$$

$$= f(0)g(1) - f(0)g(0) + \int_0^1 dy \left(g(1) - g(y) \right) \tilde{f}(y)$$

$$= f(0)g(1) - f(0)g(0) + g(1) \int_0^1 dy\, \tilde{f}(y) - \int_0^1 dy\, \tilde{f}(y)\, g(y)$$

$$= \cancel{f(0)g(1)} - f(0)g(0) + g(1)f(1) - \cancel{g(1)f(0)} - \int_0^1 dy\, f'(y)\, g(y)$$

$$= f(1)g(1) - f(0)g(0) - \int_0^1 f'(x)\, g(x)\, dx. \qquad \blacksquare$$

Remark. As a consequence of (•) one sees that fg is absolutely continuous too with

$$(\bullet\bullet) \qquad f(x)g(x) = f(0)g(0) + \int_0^x \left[f'(y)g(y) + f(y)g'(y) \right] dy.$$

Indeed (••) is nothing but (•) in the form $\int_0^x (f'g + fg')\,dy = f(x)g(x) - f(0)g(0)$ ($\int_0^x$ replaces $\int_0^1$, same proof) and $f'g + fg' \in L^1[0,1]$ (because f, g are bounded, and $f' = \tilde{f} \in L^1[0,1]$, $g' = \tilde{g} \in L^1[0,1]$).