Mathematisches Institut der Universität München
Prof. Laszlo Erdos, PhD 09.02.2008

Final Exam in Functional Analysis

Surname: ....

Given name: ....

Date of birth: ....

Problem	1	2	3	4	5	6	7	8	Σ
Max. points	15	15	15	15	15	15	15	15	120

Time: 150 minutes
Materials allowed for use: Pen (blue or black), pencil. If you need extra paper, please raise your hand. You cannot use your own paper. This booklet has 9 sheets (8 problems). Please put your name on ALL SHEETS.
100 points are counted as 100%.
Solutions should contain all necessary intermediate steps. Results without sufficient proofs will not be accepted.
Solutions can be written in German or in English.

NAME: ....

Problem 1.
Let $f \in L^3(1, +\infty)$, and let $g(x) = f(e^x)$ for $x \in (0, +\infty)$. Prove that $g \in L^2(0, +\infty)$.

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Problem 2.
Let $\mathcal{H}$ be a Hilbert space. Suppose that $f_n \overset{n\to\infty}{\longrightarrow} f$ weakly in $\mathcal{H}$, and $\|f_n\| \overset{n\to\infty}{\longrightarrow} \|f\|$. Prove that $f_n \overset{n\to\infty}{\longrightarrow} f$ in norm.

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Problem 3.
Prove that the space $X = \{(x_1, x_2, \ldots) \in l^1 \cap l^2 | \sum_{n=1}^{\infty} x_n = 0\}$ is dense in l^2.

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Problem 4.
Using Fourier series, solve the partial differential equation

$$u_{xx} + u_{xy} + u_{yy} = \cos x \cos y,$$

where $u = u(x, y)$ is a twice continuously differentiable periodic function on $[0, 2\pi] \times [0, 2\pi]$.

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Problem 5.
Let $A = \{(x, y) \in \mathbb{R}^2 : x^2 \leq y\}$ and $B = \{(x, y) \in \mathbb{R}^2 : x > 0, y \leq \ln x\}$. Find a linear functional on $\mathbb{R}^2$ which separates A and B.

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Problem 6.
Prove that there exists $f \in (l^\infty)^*, f \neq 0$, such that $f(x) = 0$ for any periodic $x \in l^\infty$. (We call a sequence $x = (x_1, x_2, \dots)$ periodic if there exists $N \in \mathbb{N}$ such that $x_{n+N} = x_n$ for any n.)

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Problem 7.
Let X, Y be normed spaces, and $T : X \to Y$ a linear operator. Suppose that if a sequence x_n converges weakly to 0 in X, then Tx_n converges weakly to 0 in Y. Prove that T is bounded.

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Problem 8.
Let X, Y, Z be Banach spaces. Let $T : X \to Y$ and $J : Y \to Z$ be linear operators. Suppose that J is bounded and injective, and JT is bounded. Prove that T is bounded. [*Hint*: closed graph theorem]