Completeness

Completeness: i.e., all Cauchy sequences converge.

• In a way, completeness is "automatic", just push one button and switch it on:

Theorem. If $(\mathcal{M}, d)$ is an incomplete metric space, it is possible to find a complete metric space $\widetilde{\mathcal{M}}$ so that $\mathcal{M}$ is isometric to a dense subset of $\widetilde{\mathcal{M}}$. The completion $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ is unique.

• **Abstractly** speaking, the completion $\widetilde{\mathcal{M}}$ is made of the equivalence classes of all Cauchy sequences $(x_1, x_2, x_3, \dots)$ in $\mathcal{M}$ under the equivalence relation $(x_1, x_2, x_3, \dots) \sim (y_1, y_2, y_3, \dots) \Leftrightarrow d(x_n, y_n) \to 0$. In the isometric embedding $\mathcal{M} \to \widetilde{\mathcal{M}}$ each $x \in \mathcal{M}$ is mapped into the equivalence class $[(x, x, x, \dots)]$. A Cauchy sequence $\{\mathbf{x}^{(N)}\}_N$ in $\widetilde{\mathcal{M}}$, $\mathbf{x}^{(N)} = [(x_1^{(N)}, x_2^{(N)}, x_3^{(N)}, \dots)]$, converges to $\mathbf{x} = [(x_1^{(1)}, x_2^{(2)}, x_3^{(3)}, \dots)]$.

• **In concrete applications** $\mathcal{M}$ is a subset of an already complete metric space $(\widetilde{\mathcal{M}}, d)$ and taking its completion amounts to **take its closure** in $(\widetilde{\mathcal{M}}, d)$. That is, some Cauchy sequence in $\mathcal{M}$ (with respect to the metric induced by d) do not converge in $\mathcal{M}$ while they converge to a limit point in $\widetilde{\mathcal{M}} \setminus \mathcal{M}$, and the completion consists of just adding to $\mathcal{M}$ such limit points. Like adding to "black" marbles some further "red" marbles.

• Actually the abstract theorem is automatic and fairly cheap, but almost useless. We never work with equivalence classes of objects, too big headache! (We like $\sqrt{2}$ to be a point on the real line, not an equivalence class.) The key is to **identify the completion** with something that is much easier to deal with. Like to identify your points with black marbles to which you may add red marbles.

• Completeness allows one to **create new structures** that are more powerful and they canonically include the previous ones. Like $\mathbb{R}$ from $\mathbb{Q}$, or $L^2[0,1]$ from $C[0,1]$. The Cauchy property can be checked directly from the sequence without any reference to the limit itself. This idea allows one to create new objects as limits of Cauchy sequences even though the limit is not checked directly.

• Completeness, as the distinguishing property of infinite-dimensional Banach spaces, enters at a very early stage of the theory to guarantee the **existence of a non-empty interior** of the Banach space. This is the **Baire category theorem**: if $X = \bigcup_{n=1}^{\infty} A_n$ is a complete metric space then at least one $\overline{A_n}$ must have a non-empty interior. Thus, pictorially, a Banach space is indeed a sufficiently "fat" boy, he doesn't play with those "skinny" kids of topology or set theory. (Banach spaces are sets of **second category**, not of **first category**.) Not only: you can "slice" him in countably many pieces (poor boy!) and at least one slice is still suitably "fat".

• This allows to prove a number of cornerstones in Banach space theory: **uniform boundedness** principle, **open mapping** theorem, **inverse mapping** theorem, **closed graph** theorem. They all rely on Banach being sufficiently "fat". (Notice: Hahn-Banach doesn't require completeness – it requires Zorn though.)

• In the Hilbert space framework, completeness is the key ingredient for the existence of the projection of a point x onto a closed subspace V of $\mathcal{H}$ (the **projection theorem**: $\mathcal{H} = \overline{V} \oplus V^{\perp}$) and therefore for the **Riesz lemma** and all what follows from Riesz (e.g., definition and properties of the **adjoints**).

• Thus, quantum mechanically, Physics does not "dictates directly" the spaces of wave functions to be complete (in fact one always deals with a dense of wave functions, doesn't need all them), but completeness is needed for the existence of the main objects of the theory. For instance, the fact that to every ket $|\psi\rangle$ there's one bra $\langle\psi|$ (Riesz lemma).

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