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Mathematical aspects of Quantum Mechanics (Chapter VII)
by A. Michelangeli, G. Dell'Antonio (in preparation)

CHAPTER 7 . Weyl algebra. Bargmann, Fock, Segal representations. Second quantization. Strict quantization. Geometric Quantization

Appendices . A: Magnetic Weyl algebra B: Landau Hamiltonian

We have seen in the preceding Chapters that in the formulation of Quantum Mechanics by Heisenberg and Born for system with N degrees of freedom an important role was played by the possibility of finding 2N hermitian (or rather self-adoint) operators, denoted here by $Q_k, \; P_k \; k = 1..N$, which on a suitable domain satisfy the relations ,

$$[Q_k, Q_h] = [P_k, P_h] = 0, \qquad [Q_k, P_h] = i\delta_{k,h} \quad k, h = 1..N \qquad 7.1$$

We shall call (7.1) *Canonical Commutation Relations* .
In the construction of a quantum dynamics these operators play the same role as the coordinates in phase space, upon substitution of the Poisson Bracket with commutators.
We have remarked in Chapter 2 that these structures have the same algebraic properties.
In this setting, if the classical hamiltonian is $H_{class} = p^2 + V(q)$ then the operator $H_{Quant} = P^2 + V(Q)$ will be the generator (via Stone's theorem) of quantum dynamics.

Remark 7.1

It is worth remarking that the imaginary unit is needed in (7.1) because the commutator of two bounded hermitian operators is anti-hermitian.
This can be considered as a reflection of the fact that the symplectic structure of R^{2n} can be regarded as complex structure on C^n therefore in going from classical to quantum dynamics the canonical symplectic matrix J is substituted by multiplication by the imaginary unit i.

♣

A first difficulty in the analysis of relations (7.1) comes from the fact these relations cannot be satisfied by bounded operators.
If for given index k the operators P_k and Q_k were both bounded, from (7.1) it would follow

$$i\, n\, Q_k^{n-1} = Q_k^n P_k - P_k Q_k^n \quad \forall n$$

and then

$$n \|Q_k\|^{n-1} \leq 2 \|Q_k\|^n \|P_k\|$$

It follows that if $\|Q_k\| \neq 0$ then $n < 2\|Q_k\| \; \|P_k\|$ for any natural number n, a contradiction.
If for each value of the index at least one of the operators Q_k, P_k is unbounded, (7.1) must be written

$$Q_k P_h - P_h Q_k \subset i\delta_{h,k} I \qquad 7.2$$

i.e. on the domain of definition the operator on the left coincides with multiplication by $i\delta_{h,k}$.
In this weaker sense the solution of (7.2) *is not unique* (up to unitary equivalence).
We consider the case $N = 1$ and give three *inequivalent* solutions.

Solution A)

Hilbert space $\mathcal{H} = L^2(R, dx)$.

$$(Qf)(x) \equiv xf(x), \qquad D(Q) = \{f | f \in L^2, \quad xf(x) \in L^2\}$$

$$(Pf)(x) \equiv i\frac{df}{dx} \qquad D(P) = \{f | f \in L^2, \frac{df}{dx} \in L^2\}$$

Remark that both Q and P are self-adjoint (Q is multiplication by x, P is multiplication by p in the Fourier transform representation) and they have as common invariant dense domain the space $\mathcal{S}$.

This representation is irriducible.
Notice that $U_a \equiv exp\{iaQ\}$ and $V_b \equiv exp\{ibP\}$ with $a, b \in R$ are two one-parameter strongly continuous groups of unitary operators.
It is easy to verify that the following identity holds

$$U_a V_b U_a^* = V_b exp\{-iab\} \qquad 7.3$$

Remark that (7.2) is the differential form of (7.3).

Solution B)

Hilbert space $\mathcal{H} = L^2([0, 2\pi], d\theta)$
Q is a bounded operator defined on all of $\mathcal{H}$ by

$$(Qf)(\theta) = \theta f(\theta)$$

The domain $D(P)$ of the operator$P$ is made of all periodic function with square integrable derivative. On this domain P acts as $-i\frac{d}{d\theta}$.
Also in this case both the Q and P are self-adjoint, Q is bounded. Hovever Q *does leave the domain of P invariant* since if $f(2\pi) \neq 0$, Qf does not belong to the domain of P.
Therefore PQ is defined only on functions for which $f(0) = f(2\pi) = 0$.
This set is dense in $L^2([0, 2\pi)$ but the restriction of P to this set does not define uniquely a self-adjoint operator (see Chapter 9; to define a self-adjont operator one needs to impose boundary conditions).
It can be verified by a direct computation that equation (7.3) *is not satisfied:* the operator $U_a V_b U_a^* V_b^*$ *is not a multiple of the identity.*

Solution C)

In the space of continuous functions consider the *characters* ξ_λ of the group R, (i.e. $\xi_\lambda \equiv e^{i\lambda x} \quad \lambda \in R$.)
Denote by K the closure in the topology of L^∞ of their finite linear combinations $\sum_1^N c_i \xi_{\lambda_i}$.
K is the space of quasi periodic functions.
The quadratic form

$$(\phi, \psi) \equiv lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \bar{\phi}(t)\ \psi(t) dt$$

is well defined on K and defines a pre-scalar product.
Denote by $\mathcal{K}$ the Hilbert space completion of K in the topology of this scalar product.
The characters form a non-denumerable orthogonal basis in $\mathcal{K}$ and therefore $\mathcal{K}$ is non separable.
On $\mathcal{K}$ we define two families of unitary operators U_a and V_b, $a\ ,\ b; \in R$ by

$$V_a \xi_\lambda = \xi_{\lambda - a}, \qquad U_b\ \xi_k = e^{ixb} \xi_k$$

This operators satisfy (7.3) but the map $a \to V_a$ is not continuous in the strong topology of $\mathcal{K}$ and not even Lebesgue-measurable (i.e. all the functions $(\phi, V_a \phi) \quad \phi \in \mathcal{K}$ are not measurable as a function of a).
To verify this, notice that weak and strong measurability coincide for unitary maps, and that

$$V_0 \equiv I \qquad |(V_a - I)\xi_\lambda|_2 = \sqrt{2},\ a \neq 0$$

The representation C) is irreducible (an element of $B(\mathcal{K})$ which commutes with U_λ and V_a is a multiple of the identity).

It is not equivalent to A) because the map $a \to V_a$ is not Lebesgue-measurable (whereas in solution A) it is continuos).

Since there are many inequivalent representation of (7.2) we will not put (7.2) at the basis Quantum Mechanics.
Following H.Weyl we will privilege the relation (7.3) among unitary operators, requiring also that the map $b \to V_b$ be *Lebesgue-measurable* .
If the system has N degrees of freedom we require

$$U(a)V(b)U^*(a)V^*(b) = exp\{-i(a.b)\}, \qquad a, b \in R^N \qquad 7.4$$

with

$$U(a) = exp\{iaQ\}, \qquad V(b) = exp\{ibP\} \quad Q = \{Q_1, ..Q_N\} \quad P = \{P_1, ..P_N\}.$$

We shall prove that solution A) given above is the *unique* (modulo unitary equivalence) irreducible solution of (7.4) for which the map $a, b \in C^N \to U(a)$, $V(b)$ is Lebesgue-measurable.
If one does not require irreducibility any representation of (7.4) decomposes as a direct sum of identical copies of the irreducible ones.
This proof, originally due to Schroedinger and later refined by Weyl and von Neumann, leads to the identification of Schroedinger's and of Born-Heisenberg's representations of the C.C.R.
Let $z = a + ib$, $z \in C^N$, $a, b \in R^N$ and define

$$W(z) = exp\{-i\frac{(a,b)}{2}\}V(b)U(a) \qquad 7.5$$

Then (7.4) is equivalent to

$$W(z)W(z') = exp\{-\frac{i}{2}Im(z,z')\}W(z+z') \quad z \in C^N \qquad 7.6$$

We have denotes by (z, z') the scalar product in C^N (antilinear in the first element).
This shows that $z \to W(z)$ is a *projective unitary representation* of the multiplicative group C^N.
It is strongly continuous by construction.

Definition 7.1

We shall call *Weyl system* on the Hilbert space $\mathcal{H}$ a collection of operators which satisfy (7.6) and are continuous in z in the strong operator topology.

♢

Notice that the semidirect product the Weyl system with the group S^1 defines a group: the *Heisenberg group.*

Remark 7.2

As one sees from (7.5) in the definition of Weyl system the operators Q_k and P_k are not treated symmetrically.
One could have defined the Weyl operators $W(z)$ inverting the position of $U(a)$ and $V(b)$ in the definition.
It is easy to see that this would correspond to a unitary transformation.

♣

From Stone's theorem applied to the subalgebras corresponding to real and imaginary values of z (from (7.6) one sees that each of these subalgebras is commutative) it follows that there exist self-adjiont operators Q_k $P_k$which generate the corresponding N-parameter subgroups and which satisfy (7.1) on a common dense invariant domain.

Remark 7.3

Setting $x + iy = z \equiv \{x, y\}$ one has

$$Im(z, z') = \omega(\{x, y\}, \{x', y'\})$$

where ω is the standard symplectic two-form (recall that the complex structure of C^N is isomorphic to the symplectic structure of R^{2N}.

One can therefore write (7.6) as

$$W(z)W(z') = e^{-\frac{i}{2}\omega(z,z')}W(z + z') \quad z, z' \in C^N \tag{7.7}$$

♣

To prove uniqueness of the representation it is convenient to study first a more abstract algebraic structure, in analogy with what is done in the analysis of the representations of Lie groups through their group algebras.

Let $z \to W(z)$ be a Weyl system, Lebesgue - measurable in the weak sense.

For each function $f \in L^1(C^N)$ with norm $\|f\|_1$ we define an operator W_f as follows.

$$W_f \equiv \int dz f(z)W(z)$$

It is easy to see that W_f is a bounded operator with

$$||W_f|| \leq ||f||_1 \tag{7.8}$$

Therefore the linear map $f \to W_f$ is norm-contiuous.

The following identities are easy to verify

$$W_f + W_g = W_{f+g}, \qquad W_f^* = W_{\bar{f}}$$

$$W_f\, W_g = W_{f\times g}, \quad (f \times g)(z) \equiv \int dz' f(z - z')g(z')e^{\frac{i}{2}\omega(zz')} \tag{7.9}$$

The product $f \times g$ defined in (7.9) is often called *Moyal product* .

Lemma 7.1

The map $f \to W_f$ is injective.

♢

Proof

If $W_f = 0$, then for every $\phi \in \mathcal{H}$ one has

$$\int dz f(z)(\psi, W(z)\phi) = 0$$

Setting $\phi' = W(z_0)\phi$, $\psi' = W(z_0)\psi$, from

$$W(-z_0)W(z)W(z_0) = W(z)e^{\frac{i}{2}\omega(z,z_0)}$$

one derives that for every pair ψ', ϕ'

$$0 = \int dz f(z)(\psi', W(z)\phi')e^{\frac{i}{2}\omega(z,z_0)}$$

One concludes that the Fourier transform of $f(z)(\psi', W(z)\phi')$ vanishes.
For each value of z one can choose $\phi' = W(z)\psi'$ and therefore

$$< \phi', W(z)\psi' >=< \phi', \phi' >= 1$$

We conclude that $f = 0$.

♡

We can regard the W_f , with operator norm $\|W_f\|$, product law $W_f W_g = W_{f\times g}$ and conjugation given by (7.10) as a Banach algebra, without reference to the Weyl system.
Notice that (7.8) can also be written

$$||W_f|| \leq ||\hat{f}||_\infty$$

In fact it is easy to prove that the equality sign holds.
Since $L_\infty(X)$ is a C^*-algebra one can give to the Banach algebra generated by the collection of W_f a C^*-norm and regard their algebra as a C^* -algebra.
We will call this C^*-algebra *Weyl algebra* and we will denote it by the symbol $\mathcal{W}$.
We not indicate the number N of degrees of freedom, with the convention that $N < +\infty$ unless stated explicitly.

In each representation of the Weyl system the map $f \to W_f$ provides a correspondence between function in classical phase space and operators on a Hilbert space.
In Chapter 3 we have called *quantization* a procedure that associates to a function (in a suitable class) on phase space an operator on a Hilbert space $\mathcal{H}$. We shall come back at the end of this Chapter to the problem of quantization.
We shall call *Weyl quantization* the quantization performed according to Weyl's algebra.
Notice that if in our analysis if we substitute Weyl's algebra with the (abelian) algebra of the characters of the multiplicative group R^{2N} we obtain the correspondence $f \to \mathcal{F}(f) = \hat{f}$ (Fourier transform) and the product structure is mapped into convolution.
Therefore one may regard the corresondence $f \to W(f)$ as a *twisted convolution* or a *symplectic Fourier transform* .

This makes the following definition natural

Definition 7.2

We will call *Weyl quantization* the map $f \to W_\hbar(f)$ defined by

$$f \to W_\hbar(f) = \int f(z) W_\hbar(z) dz$$

♢

When we will came back to Weyl quantization in Chapter 11, we will use the more common notaton

$$W_\hbar(f) \equiv Op_\hbar^W(f)$$

Remark 7.4

Weyl's quantization, originally defined for continuous functions, can be extended to a large class of functions.
We shall discuss this it in some detail in Chapter 11.
It is also clear that a generalization of Weyl's quantization can be obtained by substituting to ω in (7.10) other symplectic structures.
We shall see an example in the Appendix, where we shall discuss the *magnetic Weyl algebra.*

♣

Remark 7.4

So far we have not taken into account that position and momentum have dimensions and that the product position times momentum has the dimension of an action.
This suggests to choose units such that $[Q^\hbar_k, P^\hbar_h] = i\hbar\delta_{k,h}$.
With this convention one has

$$W_\hbar(z) = e^{-i\frac{(a,b)}{\hbar}} V(b)U(a)$$

and the product that defines Weyl's algebra is

$$W_\hbar(z)W_\hbar(z') = W_\hbar(z+z')e^{-\frac{i}{2\hbar}\omega(z,z')} \qquad 7.10$$

From the point of view of mathematics it is natural to define as *semiclassical limit* of this algebraic structure the limit $\hbar \to 0$.
It is clear from (7.10) that the semiclassical limit is singular: the relation (7.10) contains a factor which has fast oscillations when $\hbar \to 0$.
We will discuss this limit in Chapter 8 and we shall analyse it further in Chapter 11.
In Chapter 11 we will see the relation, in the semiclassical limit, of the algebraic structure of the generators of one-parameter groups in the Weyl quantization with the Poisson algebra of the generators in Hamiltonian Mechanics.
The properties of oscillatory integrals play a crucial role in the formulation of the semi-classicl limit in the Schroedinger representation.
From the point of view of Physics $\hbar$ is a physical constant and its value is not at our disposal.
The *mathematical limit* $\hbar \to 0$ gives informations about those quantities that have the dimension of the action and take values large as compared to Plank's constant.

♣

Coming back to the Weyl algebra we recall (see Chapter 4) that every C^* algebra has a faithful representation as operator algebra on a Hilbert space $\mathcal{K}$.
If the C^* algebra is separable, the space $\mathcal{K}$ can be taken separable.
Given a representation π of $\mathcal{W}$ in $\mathcal{K}$ one can ask what are the conditions under which one can reconstruct Weyl's system.
To answer this question let us consider a sequence $\{f_n \in L^1(C^N)\}$ which converges in distributional sense to the measure concentrated in the point $z_0 \in C^N$.
Consider the sequence of operators $\pi(W_{f_n})$ on $\mathcal{K}$. If this sequence converges weakly, denote by $\pi(W(z_0))$ the limit operator.
If the limit exists for each subsequence, it is not hard to prove that the operators $\pi(W(z_0))$ determine a Weyl system.
We shall call *regular* those representations of the Weyl algebra that induce as above a Weyl system.
From the uniqueness theorem of representations of Weyl algebra we shall then derive the uniqueness theorem for the Weyl system.
It is worth remarking that the Weyl algebra *contains projection* .
Indeed, setting

$$f_0(z) = (2\pi)^{-N} exp\{-|z|^2/2\} \qquad 7.11$$

and using Weyl's relation one obtains

$$W_{f_0} = W^*_{f_0}, \qquad W_{f_0}W_{f_0} = W_{f_0}$$

Therefore W_{f_0} is a projection operator.

Since representation are homeomorphism, for every representation π, $\pi(W_{f0})$ is a projection operator.
Moreover from (7.10)

$$W_{f_0}W_fW_{f_0} = W_fW_{f_0} \qquad \forall f \tag{7.12}$$

We now construct the representations of the Weyl algebra.
As we have seen in Chapter 4, every state ρ determines a representation and *every representation is obtained in this way.*
Let us briefly recall this construction (GNS).
A state ρ of a C^* algebra $\mathcal{A}$ is by definition a linear positive functional continuous in the topology of $\mathcal{A}$.
Every state induces a pre-hilbert structure on $\mathcal{A}$ as follows

$$< a, b >\equiv \rho(a^*b) \tag{7.13}$$

Denote by $\mathcal{H}$ the Hilbert space obtained by completion.
Denote by $\tilde{a}$ the equivalence class of a. If $b \in \mathcal{A}$ define an operator $\hat{b}$ on $\mathcal{H}$ as follows

$$\hat{b}.\tilde{a} = \tilde{ba}$$

The operator $\hat{b}$ is well defined since $\rho(a*b*ba) \leq ||b||\rho(a*a)$.
Moreover $\hat{b}$ is closable and bounded and extends to a bounded operator on the entire space $\mathcal{H}$; we shall denote it by the same name.
From (7.13) follows $\hat{b}\hat{c} = \hat{bc}$.
The correspondence $a \to \hat{a}$ provides therefore a representation of $\mathcal{A}$ by means of bounded operators on the Hilbert space $\mathcal{H}_\rho$.
We shall denote by π_ρ the representation induced by ρ and with $P \equiv \pi_\rho(W_{f_0})$ the representative of the projection operator W_{f_0} in this representation.
For a generic C^*- algebra and a generic state the G.N.S. representation is not faithful, but for the Weyl algebra every representation obtained by this procedure is faithful.
This is due to the fact that the product of any two elements W_f e W_g of the Weyl algebra is, a part from a phase factor, an element of the Weyl algebra.
Since the Weyl algebra is separable also the space $\mathcal{H}_\rho$ is separable.
Let $\omega_j,\ j = 1, ..$ be an orthonormal basis of $P\mathcal{H}_\rho$.
Denote by $\mathcal{K}_i$ the subspace of $\mathcal{H}_\rho$ generated by the action of $\pi_\rho(W_f)$ applied to ω_i (for the sake of simplicity we omit the index ρ.)
If we prove

$$\oplus\mathcal{K}_i = \mathcal{H}_\rho \tag{7.14}$$

it follows that the representation π_ρ decomposes in the direct sum of faithful irreducible representations each of which has $\omega_j,\ j = 1, ..d_\rho$ as cyclic vector.
Indeed from

$$P\pi(W_f)P = P\int dzf(z)exp\{-|z|^2/4\}$$

it follows that $\pi_\rho(W_f)$ vanishes if and only if $f = 0$. From

$$< \pi_\rho(W_f)\phi_i, \pi_\rho(W_g)\phi_j >= \delta_{i,j}\int f(z)\bar{g}(z')exp\{i/2Im(z, \bar{z}')\}exp\{-1/2|z-z'|^2\}dzdz \tag{7.15}$$

one derives that the representation π_ρ is the direct sum of irreducible representations each in the Hilbert space generated by the action of $\pi_\rho(W_f)$ on ϕ_i.

They are all equivalent since the scalar product in (7.16) depends on the functions f and g *but not on the representation.*
We have proved

Theorem 7.1

All irreducible representations of the Weyl algebra are unitary equivalent to the Schroedinger representation.
As a consequence they are all regular , the operators $\pi(W(z))$ exist in every representation and define the same Weyl system; the map

$$z \to \pi_\rho(W(z))$$

is a strongly continuos map of C^N in the unitary operators of $\mathcal{H}_\rho$.

♢

Remark 7.5

In the Schroedinger representation in $\mathcal{H} = L^2(R^N)$ the elements of Weyl's algebra are compact operators; this is easy to verify because their integral kernels are known explicitely.

♣

Remark 7.6

The construction of the Weyl system holds for any even dimensional real vector space and for any non degenerate symplectic form ω.
Notice however that in the proof of uniqueness of the irreducible representation we have used the Weyl algebra, and this has required the use of Lebesgue measure (to introduce L^1 functions).
Lebesgue measure does non exist in R^∞ and neither exists in this space a (σ-continuous) measure that is quasi invariant (invariant modulo translations).
Therefore the uniqueness theorem *does not hold* for a Weyl structure in a system with infinitely many degrees of freedom (e.g. in the Theory of Quantized Fields or in Quantum Statistical Mechanics)

♣

In the Weyl system we shall call $\mathcal{K}$ *the base space* and shall *representation space* that Hilbert space on which $W(z)$ acts.

Remark 7.7

The symplectic structure $Im(z, z')$ is invariant for unitary maps in $\mathcal{K}$.
From the uniqueness theorem follows the existence of a correspondence Γ between $U(\mathcal{K})$ and $U(\mathcal{H})$.
It is easy to see that this correspondence preserves weak continuity, and then it induces a correspondence $\partial\Gamma$ between generators, i.e. between self-adjoint operators on $\mathcal{K}$ and self-adjoint operators on $\mathcal{H}$.
The application $\partial\Gamma$ extends by linearity to an application which we shall denote by the same name $\partial\Gamma\ : B(\mathcal{K}) \to B(\mathcal{H})$,

♣

Definition 7.3 : Second Quantization

The map $\partial\Gamma$ has functorial character and goes under the name *second quantization.*
This functor can be constructed also if $\mathcal{K}$ has infinite dimensions but then it depends on the representation of the Weyl system..

♢

If $t \to e^{itA}$ is a one-parameter strongly continuous group of unitary operators on $\mathcal{K}$ with generator A, Γe^{itA} is a strongly continuous group of unitary operators on $\mathcal{H}$ and $\partial\Gamma(A)$ is its generator.

The following theorem holds also if $\mathcal{K}$ is infinite dimensional.

Teorem 7.2 (Segal)

Let $\mathcal{K}$ a complex Hilbert space, W a Weyl system $\mathcal{K}$.
Let $\mathcal{H}$ the representation space W. Let A be an operator $\mathcal{K}$ with $A > 0$ and $Az \neq 0$, $\forall z \in \mathcal{K}$.
Let ω be a cyclic vector for W.
Suppose that there exists a one parameter group $\Gamma'(t)$ of unitary operators on $\mathcal{H}$ such that

a)

$\Gamma'(t)W(z)(\Gamma'(-t)) = W(e^{iAt}z)$

b)

$\Gamma'(t)\omega = \omega \qquad \forall t \in R$

c)

$\Gamma'(t) = e^{itH}, \qquad H \geq 0$

Then there exist unique a correspondence (second quantization) $\Gamma : U(\mathcal{K}) \to U(\mathcal{H})$ such that

$$\Gamma(e^{iAt}) = \Gamma'(t) \qquad 7.16$$

and moreover for any operator $B \geq 0$ on $\mathcal{H}$ one has $\partial\Gamma(B) > 0$ (recall that $\partial\Gamma(B)$ is the generator of the group $\Gamma(e^{iBt})$.

♢

Proof

Let

$$f(u) =< e^{-uH}W(z)\omega, W(z)\omega > \qquad u = s + it \qquad z \in K \qquad 7.17$$

the function f is bounded, holomorphic in $s > 0$, continuous in $s \geq 0$.
Denote by Φ this space of function and notice that they form an algebra.
Weyl's relations give

$$f(it) = e^{i/2Im(z_t,z)} < W(z_t - z)\omega, \omega >, \qquad z_t = e^{-iAt}z$$

The function $g(u) = e^{-1/2(e^{-uA}z,z)}$ belongs to Φ therefore also $gf \in \Phi$ and one has

$$(fg)(it) =< W(z_t - z)\omega, \omega > e^{-\frac{1}{2}Re(z_t,z)}$$

Substituting z with $-z$ one sees that also the function $< W(-z_t + z)\omega, \omega > e^{-\frac{1}{2}Re(z_t,z)}$ belongs to Φ and its boundary value is $\bar{f}\bar{g}$.
By taking adjoints, can construct a function which is bounded, holomorphic in $s < 0$, continuous $s \leq 0$ and has boundary value fg at $s = 0$.
We conclude that fg can be continued in the entire complex plane as an analytic function, and is therefore constant as a function of t..
Evaluating this function at zero its value is seen to be $e^{-(z,z)/2}$; therefore $< W(z_t - z)\omega, \omega >= e^{-|z_t - z|^2/4}$.
Since the kernel of A is the null vector, when z and t vary the vectors $z_t - z$ span a dense set in $\mathcal{K}$.
Therefore for every $z < W(z)\omega, \omega >= e^{-|z|^2/4}$.

It follows that for every unitary $U \in \mathcal{B}(K)$ the map

$$\sum a_i W(z_i)\omega \to \sum a_i W(Uz_i)\omega$$

is well defined and isometric on $D \equiv \cup_z \{W(z)\omega\}$. By density the map extends to a unitary operator $\Gamma(U)$.
By construction

$$\Gamma(U)W(z)\Gamma^*(U) = W(Uz), \qquad \Gamma(U)\omega = \omega$$

and therefore $U \to \Gamma(U)$ is a representation of $U(K)$, continuous because

$$< \Gamma(U)W(z)\omega, W(z)\omega >= e^{\frac{i}{2}Im(Uz,z)}e^{-|U(z)-z|^2/4}$$

In particular choosing $U = e^{-itA}$ one has for all z

$$\Gamma(e^{-iAt}W(z)\Gamma(e^{iAt} = \Gamma'(t)W(z)\Gamma'(-t))$$

and this proves (7.16).
To prove that if $B > 0$ as an operator on $\mathcal{K}$ then $\partial\Gamma(B) > 0$ it is sufficient to prove that if $B > 0$ then

$$\int < \Gamma(e^{-iBt})w, w' > g(t)dt = 0, \qquad \forall w, w' \in \mathcal{K} \qquad 7.18$$

if $g \in L^2(R), \quad \hat{g}(p) = 0, p < 0$.
By density it is sufficient to prove (7.18) for $w = W(z)\omega, \; w' = W(z')\omega$. Notice that

$$< \Gamma(e^{-iBt})W(z)\omega, W(z')\omega >= exp(-\frac{1}{4}(|z|^2 + |z'|^2) + 2(e^{-iBt}z, z')$$

and the exponential map preserves positivity.
Uniqueness follow from the cyclicity of ω.

Segal's theorem can be extended, with a more complicated proof, to the case in which $A \geq 0$ is self-adjoint and zero is a simple eigenvalue.

♡

Let us consider in the Schroedinger representation in $L^2(R^d)$ the positive self-adjoint operator

$$N = \sum_{k=1}^d N_k, \quad N_k = 1/2(P_k^2 + Q_k^2 - 1) = -\frac{1}{2}\Delta_k + \frac{1}{2}x_k^2 - \frac{1}{2}$$

The operators N_k satisfy on a dense domain

$$[N_h, P_k] = \delta_{k,h}Q_k \quad [N, Q_k] = -\delta_{h,k}P_k$$

The spectrum of N_k is non-degenerate and consists of the non negative integers. The eigenvector to the eigenvalue zero is $\frac{1}{2\pi}e^{-\frac{1}{2}x_k^2}$.
All continuous and bounded functions of functions N belong to the Weyl algebra.
It is convenient to introduce the operators

$$a_k = \frac{1}{\sqrt{2}}(Q_k - iP_k) \equiv \frac{1}{\sqrt{2}}(x_k + \frac{\partial}{\partial x_k}) \qquad k = 1, ..d$$

$$a_k^* = \frac{1}{\sqrt{2}}(Q_k + iP_k) \equiv \frac{1}{\sqrt{2}}(x_k - \frac{\partial}{\partial x_k}) \qquad k = 1, ..d$$

which satisfy in a dense domain (e.g. $D(N)$) the relations (that we still call of *canonical commutation relation*)

$$[a_k, a_h] = [a_k^*, a_h^*] = 0, \qquad [a_h, a_k^*] = \delta_{h,k} \tag{7.19}$$

The operators a_k e a_k^* have a dense common domain of definition, are adjoints and on $D(N)$ satisfy

$$N = \sum_{k=1}^{d} N_k, \quad N_k = a_k^* a_k, \quad [N_k, a_h] = -a_h \delta_{h,k}$$

The spectrum of the operators a_k is the real axis, while a_k^* have empty spectum.
All have a dense set of analytic vectors (in particular the analytic vectors of $\sqrt{N}$).
Often the operator N is denoted *number operator* ; in the Schroedinger representation it coincides with the hamiltonian of the harmonic oscillator.
The operators N_k are a complete system: an operator that commutes with all of them is a multiple of the identity.
Therefore there exists a canonical isomorphism of $\mathcal{H}$ with $(l^2)^{\otimes d}$ in which a complete orthonormal basis is given by the sequences of d non negative integers.
The eigenvalue which corresponds to the sequence $\{n_1, ...n_d\}$ the eigenvalue of N is $\sum_1^d n_k$.
To the non degenerate eigenvalue 0 di N corresponds the sequence $\{0, ...0\}$, and this vector coincides with the cyclic vector ω.
From (7.19) one derives

$$a_k\{n_1, ..n_d\} = \sqrt{n}_k\{n_1, , n_k - 1, ..n_d\}, \quad a_k^*\{n_1, ..n_d\} = \sqrt{n+1}\{n_1, , n_k + 1, ..n_d\}$$

Notice that $N_k^{-1/2} a_k$ and $N_k^{-1/2} a_k^*$ are bounded operators.
In the Schroedinger representation for a system with N degrees of freedom in $L^2(R^N)$ one has

$$\{n_1, ...n_d\} \to h_{n_1}(x_1)...h_{n_d}(x_N)e^{-\frac{1}{2}|x|^2}$$

where h_i is the i^{th} Hermite polynomial

Definition 7.4

The representation of the canonical commutation relations in the basis $\{n_1, n_d\}$ is the *Fock representation* .

♢

The functor Γ (second quantization) takes a particularly interesting form in the Fock representation.
If A is a complex-valued matrix of rank d we have

$$\Gamma(A) \equiv \{0, A, A \otimes I + I \otimes A, A \otimes I \otimes I + I \otimes A \otimes I + I \times I \times A + ...\} \tag{7.20}$$

In particular

$$\Gamma(0) = 1, \qquad \Gamma(e^t) = e^{-tN}, \quad lim_{t\to\infty}\Gamma(e^t)\phi = (\omega, \phi)\omega \tag{7.21}$$

Remark that the explicit form of the representation depends on the choice of the basis but a change of the basis leads to an *equivalent representation.*
Fock's representation is not much used in Non Relativistic Quantum Mechanics which studies systems with a fixed number of particles.
In this case Schroedinger's representation is more useful since it allows the use of techniques of Classical Functional Analysis (for example Sobolev inequalities , Lebesgue's dominated

convergence, positivity preserving semigroups...) that do not have a *natural* counterpart in Fock's representation.

We remark that if $\mathcal{K}$ is infinite dimensional one can still define a Fock representation but the representation will depend in general from the basis chosen.
A change of base associated to a matrix which is not of Hilbert-Schmidt cannot be realized with a unitary map, and therefore one ends up in an inequivalent Fock representation.
If one identifies $\mathcal{K}$ with $L^2(R, dG_k)$ where G_k is a Gauss measure one can construct Fock representations based on spaces of functions; typically in this case the representation space *is a space of distributions.*
But it should be stressed that in infinite dimensions *not every representation is of Fock type.*
In the case of systems with infinitely many degrees of freedom Fock's representation is an useful instrument for a perturbative analysis, using Duhamel's formula.
One should be aware that, due to non uniqueness, if the theory depends on a small parameter (coupling constant) one may be forced to use *different irreducible representation for different values of the parameter* so that perturbation expansion may present some difficulties.

REAL AND COMPLEX BARGMANN-SEGAL REPRESENTATIONS

The Weyl system admits also representations that use Gauss measures, and therefore can be extended to the infinite dimensional case.
Since in R^∞ there are Gauss measures which are inequivalent to each other, we will have inequivalent representations of the C.C.R.. In the case of a finite number of degrees of freedom the representations are all equivalent.
We describe two of them in the finite-dimensional setting, because of their relevance to the infinite dimensional case.

Let us note preliminarly that gaussian measure share with Lebesgue's the property of being cylindrical, and have the advantage of being probability measures. It is this property that allows an extension to infinite dimensions.

Complex Bargmann-Segal representation

In the infinite dimensional setting this is the weak normal distribution in the terminology used by I.Segal

The complex Segal-Bargmann representation diagonalizes the annichilation operators.
Even for systems which have a denumerable infinity of degrees of freedom it permits the use of techniques of Functional Analysis similar (but weaker) to the one that are used in the Schroedinger representation in the finite dimensional case.
The origin of the complex Bargmann-Segal representation can be traced to the remark of Fock who noticed that the commutation relations among a_k^* and a_h are satisfied by the operators z_k e $\frac{d}{dz_h}$ acting on the space $\mathcal{F}$ of entire functions with a suitable Gauss measure.
We shall come back to this point later.
To describe this representation and to verify its equivalence with the Schroedinger representation, consider the isomorphism F of Hilbert spaces

$$L^2(R^N, dx) \to \mathcal{B} \equiv L^2(C^N, dG)_{an} \qquad \phi(x) \to \psi(z) \qquad 7.22$$

where the target space is the Bargmann space $\mathcal{B}$ of functions analytic *in the sector* $z_k \geq 0,\ k = 1..N$ which are square-integrable for the Gauss measure

$$(\frac{1}{\pi})^n e^{-|z|^2} \Pi dx_i dy_i$$

Notice that the requirement that the functions be analytic in a quadrant *is necessary to have isomorphism.*
The correspondence is given by

$$F: \quad \phi(x) \to \psi(z) = (\frac{1}{\pi})^n \int_{R^n} e^{-1/2(z^2+|x^2|)+\sqrt{2}(z.x)} \phi(x) dx \qquad 7.23$$

The advantage of using $\mathcal{B}$ instead of the smaller space $\mathcal{F}$ is that one introduces a duality with the space $\mathcal{B}'$ obtained by complex conjugation and which is composed of functions analytic in the opposite sector.
On this dual space one can realize another representation of the Weyl algebra. Due to equivalence, there is a unitary map J between these representations.
On the other hand, taking boundary values, one has a unitary map between $\mathcal{B}$ and the space $\mathcal{B}_{real}$ of functions on R^N which are square integrable with respect to suitable Gauss measure. Since this is the space on which we will define the real Bargmann-Segal representation by uniqueness the there is a unitary map between the Complex and Real Bargmann-Segal representations.
In the real representation the operator which corresponds to multiplication by $z_k = q_k + ip_k$ when $q_k = 0$ is $\frac{\partial}{\partial x_k}$ and therefore $z \to \bar{z}$ correspond to hermitan conjugaton.
On a dense domain $Fa_k^* F^{-1}$ is multiplication by z and $Fa_k F^{-1}$ is the operator $\frac{\partial}{\partial z}$.
The operator $Fa_k^* F^{-1}$ cannot have eigenvectors because the equation $z\phi(z) = \lambda\phi(z)$ cannot be solved with $\phi(z)$ analytic.
The spectrum of the operator $Fa_k F^{-1}$ is the entire complex plane; in fact for every complex λ the equation $\frac{d\phi_\lambda}{dz)} = \lambda\phi_\lambda(z)$ has the solution $\phi_\lambda(z) = e^{\lambda z}$ which is analytic and square integrable with respect to Gauss measure.
The eigenvectors of $Fa_k F^{-1}$ are called *coherent states* and play a mayor role in Quantum Optics.
For $\lambda = 0$ the solution is $\phi_0(z) = C$ which corresponds to the vacuum in Fock space (and the the ground state of the harmonic oscillator in the Schroedinger representation.
It is easy to see that the vectors , obtained by repeated action of $Fa_k^* F^{-1}$ on ϕ_0

$$\frac{z_1^{m_1} .... z_n^{m_n}}{\sqrt{m_1!...m_n!}}, \qquad m_k \in N$$

form a complete orthonormal basis in $\mathcal{K}_n \equiv L^2_{an}(C^N, dz^N)$.
The vectors $\psi_a$are not orthogonal (the operators a_k are not self-adjoint) but provide a *complete system* in the sense that any vector ϕ can be expressed as an integral over coherent states

$$\phi(z) = \int e^{z\bar{w}} d\mu(w) \qquad d\mu(w) = (\frac{1}{\pi})^n \phi(w) e^{-|z|^2} \Pi dz_k \qquad 7.24$$

One sees from (7.24) that

$$(\frac{1}{\pi})^n \phi(w) e^{-|z|^2} \qquad 7.25$$

is a *reproducing kernel* for the state in this representation.
The inverse transformation F^{-1} is given as follows

$$(F^{-1}g)(z) = lim_{M\to\infty} \int_{|z|<M} \bar{A}(x,z) g(z) d\nu_n(z) \qquad 7.26$$

with $A(x,z) = e^{-\frac{1}{2}(z^2+x^2)+\sqrt{2}(z.x)}$.

Remark that for each $x \in R^n$ one has $A(x,z) \in \mathcal{K}_n$ but only for a dense subset of $\mathcal{K}_n$ the integral in (7.26) is absolutely convergent for $|x| \to \infty$.
For a generic vector in $\mathcal{K}_n$ convergence in (7.26) is understood in a weak sense.

As remarked before, a (different) complex Bargmann-Segal representation can be also constructed on the space $\mathcal{F}(C^N, d\mu)$ of entire functions on C^N, square integrable with respect to the Gaussian measure $d\mu = e^{-\frac{1}{2}|z|^2}dz$.
In this case the function

$$K(z,w) = (\frac{1}{2\pi})^N e^{\frac{1}{2}(z,w)} \qquad 7.27$$

is a reproducing Kernel i.e. for $f \in \mathcal{F}(C^N, d\mu)$

$$f(z) = \int_{C^N} K(z,w)f(w)d\mu \qquad 7.28$$

At the same time the function $K(z,w)$ is the integral kernel of the orthogonal projection P of $L^2(C^N, d\mu)$ onto $\mathcal{F}(C^N)$ (recall that the latter is composed of *entire functions).*
This structure permits the introduction of *Toeplitz Operators* defined in this particular case by

$$T_f : \mathcal{F} \to \mathcal{F} \quad T_f(g) = P(fg) \qquad 7.29$$

These operators (see e.g. [BC87] have a great relevance in Operator Theory and in Quantum Mechanics; their relation with the Weyl operators in the Bargmann-Segal representation can be seen to be

$$T_f = W_{\Theta f} \qquad \Theta f(z) = (\frac{1}{\pi})^N \int_{C^N} e^{-|z-w|^2} dw \qquad 7.30m$$

The difference between the operators T_f and W_f comes therefore from the difference in the space of representation; the map Θ is an isometry between the Fock space $\mathcal{F}$ and the Bargmann-Segal space $\mathcal{B}$.

The complex Bargman-Segal representation and the Fock representation play an important role in The Berezin-Wick quantization, that we will discuss brefly in Chapter 11; in this quantization the role of the *creation operators* is taken by Toeplitz operators.

Remark 7.8

The correspondence between $L^2(R^n, dx)$ and $\mathcal{K}_n$ can also be seen in terms of the symplectic structures.

$$\omega_n \equiv \sum dq_k \wedge dp_k \qquad i\mu_k \equiv \sum dz_k \wedge \bar{z}_k \qquad 7.31$$

defined respectively on $R^{2N} \equiv C^N$ and on C^{2N}.
Notice that $D_n \equiv \{\{z,w\} \in C^{2n}, w = \bar{z}\}$ is a symplectic sub-variety with respect to the two form $\sum dz_k \wedge dw_k$.
The corresponding symplectic reduction $C^{2n} \to C^n$ is given by

$$\{z, \bar{z}\} \to z$$

The linear symplectic transformation $(T^*R^n, \omega_n) \to (C^n, \mu_n)$ has as generating function $\Phi(x,z) = -i\ logA(x,z)$.
One can indeed verify the following identities

$$p_k = -\frac{\partial\psi}{\partial x_k} \qquad w_k = \frac{\partial\psi}{\partial z_k} \qquad k = 1, ...n$$

This symplectic transformation maps $\sum_k (p_k^2 + q_k^2)$ to $\sum_k z_k \bar{z}_k$ which is the classic counterpart of the map

$$\sum_k -(\frac{\partial}{\partial x_k})^2 + x_k^2 \to \sum_k z_k \frac{\partial}{\partial z_k} \qquad 7.32$$

♣

Real Bargmann-Segal representation

The real Bargmann-Segal representation is obtained using the isomorphism of Hilbert spaces (that depends on the positive matrix B)

$$L^2(R^N, dx) \to L^2(R^N, e^{-(x,Bx)} dx), \quad B > 0, \qquad \phi(x) \to \psi(x) = C_B e^{\frac{1}{2}(x,Bx)^2} \phi(x) \qquad 7.33$$

where C_B is a normalization constant.
This isomorphism induces on the canonical operators the map $X_k \to X_k, \quad P_k \to P_k - iB_{k,h} x_h$.
The real Bargman-Segal representation has been used by I.Segal in the infinite dimensional case to represent the quantum fields as linear functions on spaces of distributions (in the same way as, in the finite dimensional case, the coordinates x_k are linear functions on R^N).

Remark 7.9

To give a historical prospective on Segal's real wave representation it is worth noting that it has been introduced and employed for the quantization of the Klein-Gordon equation

$$\frac{\partial^2 u(t,x)}{\partial t^2} = \Delta u(t,x) - m u(t,x) \quad x \in R^d, \quad m \geq 0 \qquad 7.34$$

(for $m = 0$ this is the wave equation).
This hyperbolic equation admits a unique real solution (in suitable function spaces) if one chooses as intial data at time $t = 0$ the (real) function $u(0,x)$ and its gradient $\nabla u(0,x) \equiv v(0,x)$.
On the space of pairs of real - valued functions that describe these initial data there exists a natural (and singular) symplectic structure defined as

$$\omega(f,g) = \frac{1}{2} \int [f(x) \nabla g(x) - \nabla f(x) \; g(x)] dx, \quad x \in R^d \qquad 7.35$$

for pairs $\{f, g\} \in H^1(R^d) \otimes H^1(R^d)$
It can be extended to pairs in $H^{\frac{1}{2}}$ or to couples $f \in H^1, \; g \in L^2$.
This structure is invariant for the flow defined by equation (7.34) and under it the flow is hamiltonian with as hamiltonian the energy of the classical field.
In this formulation the space K is the Hilbert space of pairs of functions $\{f(x), \; g(x)\}$ with $f(x) \in H^1 \;\; g(x) \in L^2$..
The second quantization in this case corresponds (see e.g. [S92] to the Schroedinger representation but has as configuration space, instead of R^N, the space of the real distribution-valued solutions of the equation (7.34).

♣

The connection with the Fock representation is seen by the following *formal argument:* chosen two bases $f_n, g_m \in \mathcal{S}$ orthonormal with respect to the L^2 scalar product, define the linear functions on $\mathcal{S}'$ (coordinates on $\mathcal{S}'$.
Notice that the Fourier transform is an isometry of $\mathcal{S}'$).

$$q_n = \int f_n(x) \phi(x) dx, \qquad p_m = \int g_m(x) \pi(x) dx$$

Then the real wave representation gives $\{q_n\}, \{p_m\}$ as operators which satisfy, on a suitable domain, the relations

$$[q_n, p_m] = i\delta_{n,m}, \quad [q_n, q_m] = [p_n, p_m] = 0, \quad n, m = 1, 2,$$

The complex Segal representation has coordinates on $\mathcal{S}'$ given by $q_n + ip_n$. This are coordinates of a *lagrangian manifold* that evolves in time according to the equation of the classical field. In this respect the complex representation is of lagrangian rather than hamiltonian nature.

The real and complex Segal representations depends on the choice if the Gaussian measure (*weak normal distribution* in the terminology of I.Segal).
We shall study conditions under which there is equivalence.
The representation space has as coordinates linear functionals i $\xi(z),\ z = \{f \in \mathcal{S}, g \in \mathcal{S}\}$; for a detailed analysis see e.g. [Gr67]
Remark that a Gaussian measure is completely characterized by the mean and the variance; but even if the space K has finite dimension *there does not exists* a *canonical space* in which the measure is realized.

Remark 7.10

We shall see an example of this in Chapter 14 when we shall discuss Brownian motion and the process of Ohrstein-Uhlembeck (that can be considered as a Quantum Field Theory in zero space-dimension).
We shall there come back the arbitrariness and restrictions in the choice of the measure space if we want to restrict attention to choices that allow as measurable functions suitable functions of the coordinates.
For example in the theory of Relativistic Quantized Fields are of relevance the polynomials and the exponential functions which can be *informally written* as $\phi(x)^p$ and $e^{i\phi(x)}$. t
These expression are informal since the coordnate $\phi(x)$ is a distribution.
In Quantum Field Theory are of relevance the polynomials in the evaluation of the coordinates on regular functions, that one writes informally as

$$\int \phi(x) f(x) dx, \int \pi(x) g(x) dx \quad \phi,\ \pi \in \mathcal{S}', \quad f, g \in \mathcal{S} \qquad 7.36$$

One can ask whether it is possible, by taking limits in measure, to choose $f,\ g$ in less regular function spaces.
This depends on the dimension of the space in which is defined the Klein-Gordon equation, since it is linked to the different value of k for which the immersion of H^k in L^2 is a trace class operator, where

$$H^k = \{\phi(x) \in L^2(R^d), (-\Delta + |x|^2 + 1)^k \phi \in L^2(R^d)\} \qquad 7.37$$

♣

We shall discuss now briefly the condition under which a linear symplectic map in the space $\mathcal{K}$ is realized by a unitary transformation in $\mathcal{H}$.
Consider the family of unitary transformations $z \to e^{itA} z, \quad A = A^*$.
In the finite dimensional case for each value of t the Gauss measure is transformed into an equivalent one.
Using in $\mathcal{K}$ the base of the eigenvectors of A it easy to see that the Radon-Nykodym derivative is

$$(det(A^*A))^{-1}) exp\{-Tr[(A^*A))^{-1} - I)]\} \qquad 7.38$$

This considerations are valid also in the case $\mathcal{K}$ has infinite dimensions, *under the condition that (7.38) be well defined.*
As a consequence the two measures are certainly equivalent (and therefore the map is implemented by a unitary operator and the representations are equivalent) if the operator $(A^*A))^{-1} - I$ is trace class.
However this condition *is not necessary* .
A necessary and sufficient condition is that $((A^*A))^{-1} - I)$ be of class Hilbert-Schmidt and therefore $A = I + B$ where B is a Hilbert-Schmidt operator.
This can be seen as follows. Notice that any H.S. operator B can be written as the limit, in the H.S. topology (a Hilbert space topology) of operators $B_N = \sum_{n=1}^N b_n \Pi_n$ where b_n are eigenvalues of B and π_n are one-dimensional projection operators.
It is easy to verify that the limit

$$lim_{N\to\infty} e^{-TrB_N} det B_N \qquad 7.39$$

is finite; it is often denoted by $e^{Tr_1 B}$.
Taking $B = (A^*A)^{-1} - I$ proves the statement.

Remark 7.11

In the real Bargmann-Segal representation this has the following interpretation: two gaussian measures with densities formally written as

$$D_\lambda = C_\lambda e^{-\sum_n \lambda_n x_n^2}, \qquad D_\mu = C_\mu e^{-\sum_n \mu_n x_n^2} \qquad 7.40$$

where C is a normalization factor and λ_n (respectively μ_n) are the eigenvalues of the operators L (resp. M) are equivalent if and only if $L - M$ is a Hilbert-Schmidt operator.
If $L - M$ is not trace-class, the series $e^{\sum_k \lambda_k x_k^2 - \sum_k \mu_k x_k^2}$ does not converge in general on the support of the Gaussian with density measure D_λ; however there exists a sequence of real numbersi c_n (connected to the normalization constans C_λ and C_μ) such that the series

$$\sum_n ((\lambda_n - \mu_n) x_n^2 - c_n)$$

converges almost surely wit respect to the Gauss measure with density D_μ

♣

We have remarked that the one-parameter groups of symplectic linear maps on $\mathcal{K}$ are realized by one parameter groups of unitary operators on $\mathcal{H}$.
Suppose that the group is determined by the solution of the liner homogeneous equation

$$\dot{z} = JBz$$

where B is a symmetric matrix.
Making use of the Fock representation it is easy to see that $\partial\Gamma(B)$ (the corresponding generator of the unitary group on $\mathcal{H}$) is the self-adjoint operator

$$\partial\Gamma(B) \equiv \sum_{k,j} a_k^* B_{k,j} a_j \qquad 7.41$$

The operator $\partial\Gamma(B)(N+I)^{-1}$ is bounded and therefore the analytic vectors for N are analytic vectors for all $\partial\Gamma(B)$.

It is easy to verify that the following relation holds

$$[\partial\Gamma(B_1), \partial\Gamma(B_2)] = \partial\Gamma([B_1, B_]) + Tr(B_1 B_2) \qquad 7.42$$

One can also express $\partial\Gamma(B)$ as bilinear expression in the operators in P_i and Q_k and the resulting commutation rules are similar to (7.42)
In case $\mathcal{K} = R^3$ this leads to a simple expression for the generators in $\mathcal{H}$ of the rotation group.

Before discussing in some detail the rotation group, we remark that if G is a group of linear symplectic transformations on $\mathcal{K}$ and $U(g) = \Gamma(U(g))$, in general the operators $\Gamma(U(g))$ do not provide a representation of G.
This is true only if G is semisimple.
A simple example is the Galilei group, which has time-translations as abelian invariant subgroup; we shall discuss it shortly.

In the case of the rotation group in R^3 we denote by $j_k,\ k = 1,2,3$ the generators of the rotations around the axes, and use the notation $\hat{j}_k \equiv \partial\Gamma(j_k)$.
One has

$$\hat{j}_k = i \sum_{h,j} \epsilon_{k,j,l} P_j Q_l \qquad 7.43$$

where $\epsilon_{k,j,l}$ is Ricci' symbol.
The analytic vectors of N are a set of analytic vectors for the $\hat{j}_k$ and in this domain the following relations are statisfied

$$[\hat{j}_k, \hat{j}_l] = i\epsilon_{k,l,m}\hat{j}_m \qquad 7.44$$

From (7.37) follows that each $\hat{j}_k$ commutes with $\hat{j}^2 = \sum_k (\hat{j}_k)^2$ and therefore for each value of k one can diagonalize simultaneously $\hat{j}^2$ e $\hat{j}_k$.
With the notation

$$L_\pm \equiv \hat{j}_1 \pm i\hat{j}_2, \quad L_3 \equiv \hat{j}_3 \quad L^2 \equiv \hat{j}^2$$

one has, on the analytic vectors of N

$$[L_3, L_\pm] = \pm L_\pm, \qquad L^2 = L_3^2 + L_3 + L_- L_+ \qquad 7.45$$

In the Schroedinger representation (and therefore in any other representation of the Weyl system) one has $e^{2\pi i L_3} = I$ (rotations around any one axis of an angle multiple of 2π is represented by the identity operator)
Therefore the eigenvalues of L_3 must be a subset of the integers.
We shall denote by the symbol m the eigenvalues of L_3, with $g(l) \in N$ the eigenvalues of L^2 and with $|l, m>$ the corresponding common eigenvalues.
From (7.45) one derives

$$L_3 L_\pm |l, m >= (m \pm 1|l, m >, \qquad L^2 L_\pm |l, m >= g(l) L_\pm |l, m > \qquad 7.46$$

and from (7.46)

$$L_\pm |l, m >= \sqrt{g(l) - m(m+1)}|l, m \pm 1 > \qquad 7.47$$

From (7.47) noticing that $L_-.L_+$ is a positive operator (since $L_- = L_+^*$) one derives that $g(l)$ must have the form $g(l) = l(l+1)$ and that the joint eigenvalues of L_3 and L^2 are $l, m; m \in \{-l, ..l\}$.
In order to have more explicit formulas it is convenient to refer to the Schroedinger representation.

On the domain of the harmonic oscillator one has

$$\hat{j}_k f(x) = i \sum_{h,l} \epsilon_{k,h,l} x_h \frac{\partial f}{\partial x_l} \qquad 7.48$$

It is convenient to use the isomorphism $L^2(R^3) \simeq L^2(R^+) \times L^2(S^3, d\mu)$ (description in spherical coordinates) where μ is the invariant measure on the sphere of radius one.
In these new coordinates the operators $\hat{j}_k$ take the form

$$\hat{j}_k = I \otimes J_k$$

where J_k has the same expression as $\hat{j}_k$ but now as an operator on $L^2(S^3, d\mu)$.
Using spherical coordinates the common eigenvalues of J_3, J^2 to the eigenvalues m, l take the form $|l, m >\equiv Y_l(\theta) e^{im\phi}$ where $Y_l(\theta)$ are the spherical harmonics.

We now treat briefly the Galilei group in the Schroedinger representation.
The Galilei group is a ten parameter Lie group; its defining representation, in the enlarged phase space for a material point of mass m is
i)

$$x \to x + a, \quad p \to p, \qquad x,\ p \in R^3,$$

ii)

$$x \to x + vt, \quad p \to p + mv \qquad v \in R^3$$

iii)

$$x \to Rx, \quad p \to Rp \qquad R \in O(3)$$

iv)

$$x \to x, \quad p \to p \quad t \to t + \tau \quad \tau \in R$$

In this notation x is the cartesian coordinate, p is the momentum.
The elements of the abelian subgroup ii) are called is "boosts").
The subgroup iv) is time translations; it is an abelian non-compact invariant subgroup ; therefore the Galilei group in s not semisimple and its representations by means of unitary are in general *projective* .
For each value of t the function

$$K(x, p, t) \equiv pt - mx \qquad 7.49$$

generates the symplectic subgroup ii) with parameter $v \in R^3$. The operators K_m satisfy

$$\{K_i, p_j\} = m\delta_{i,j}, \qquad \{K_i, x_j\} = t\delta_{i,j} \qquad \{x_i, p_j\} = \delta_{i,j}$$

and together with

$$\{x_i, t\} = \{t, p_j\} = \{t, K_j\} = 0$$

define the structure of Galilei group as a Lie group.
Weyl quantization substitutes the functions q_k, p_k, K_k with the operators $\hat{q}_k$, $\hat{p}_k$, $\hat{K}_k$ which satisfy the commutation relations (t and m are parameters)

$$[\hat{K}_i, \hat{p}_j] = im\delta_{i,j}, \qquad [\hat{K}_i, \hat{x}_j] = it\delta_{i,j} \qquad [\hat{x}_i, \hat{p}_j] = i\delta_{i,j} \qquad 7.50$$

Using (7.46) it is not difficult to give the explicit expression of the unitary operators $U_t(v, m)$ implementing the maps $x \to x + vt,\ p \to p + mv$.

In the representation in which the operators $\hat{x}_k$ are diagonal one obtains for any function $g \in L^2(R^3)$

$$(U_t(v,m)g)(x) = exp\{-it\frac{mv^2}{2}\}exp\{-i(x.mv)\}exp\{i(p.v)\}g(x) \qquad 7.51$$

where con $p_k \equiv -i\frac{\partial}{\partial x_k}$.
Using (7.51) one can verify that indeed this is a projective representation of the Galilei group. Notice that

$$2\sigma((0,mv),(tv,0)) = mt|v^2|$$

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THE FORMALISM OF QUANTIZATION. STRICT QUANTIZATIONS

Generally speaking, a *quantization* is a linear map that associates to a function f of a suitable class on a function space X an operator $W_\hbar(f)n$ on a Hilbert space $\mathcal{H}$.
If the function is real valued, in general it is required that Q_f be selfadjoint.
It is required that *some* functional relations are preserved under quantizaton.
We will see that there is *no quantization that preserves all functional relations.*
In definition (7.8) we have defined as *Weyl Quantization* the map that to each functon f (of a suitable class) on phase space $R^d \times R^d$ associates the operator $Q^W(f)$ acting on $L^2(R^d)$.
Other quantization have been constructed (see the Berezin-Wick and Toepliz quantizations described in Chapter 11) for which the operators act on a Hilbert space of analytic functions.
The Weyl quantization and the Berezin-Wick quantization are *strict quantizations* of a Poisson structure in the following sense.

Definition 7.7

A *Poisson structure* (or *Poisson algebra)* is a triple $(X\ ,\times,\ \{.,.\})$ where X is a real vector space ,$\times$ is a bilinear associative and commutative map $X \times X \to X$ (called product) and $\{.,.\}$ is an antisymmetric map from $X \times X$ to X that is for every $f \in X$ a derivation both with respect to $\times$ and with respect to $\{.,.\}$.
Therefore for each two elements $f\ ,\ g\ h\ \in X$ one has

i)
$f \times g = g \times f, \quad (f \times g) \times h = f \times (g \times h)$

ii)
$\{f,g\} = -\{g,f\}$

iii)
$\{f,gh\} = \{f,g\}h + f\{g,h\}$ (Leibnitz's rule)

iv)
$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$ (Jacobi identity)

♣

Notice that iii) and iv) are the requirement that the Poisson bracket act as a derivation with respect to both product structures.

A *quantization by deformation* associates to each value of the parameter $0 < \hbar \le \hbar_0$, and to each $f \in X$ an element of a C^* algebra $\mathcal{A}$ (quantum observables) in such a way that the algebraic structure of the quantum observables converges *in a suitable sense* when $\hbar \to 0$ to the structure described by the Poisson algebra.

Often X is the space of all functions on a symplectic manifold $\mathcal{M}$; in this case one applies the quantization procedure only to a subset $X_0 \subset X$ (for example to functions of class C_0^∞ on $\mathcal{M}$).

Definition 7.8

Let X be a Poisson Algebra densely contained in the self-adjoint part of an abelian algebra $\mathcal{A}_0$.

If I is a subset of R^+ which has zero as only accumulation point, a *strict quantization* of the Poisson algebra $(X\ , \times \{.,.\})$ is a family of maps $\mathcal{Q}^\hbar$, $\hbar \in I$ from $\mathcal{A}_0$ to the real elements of a family $\mathcal{A}^\hbar$ of C^*-algebras, with norm $\|.\|_\hbar$, which satisfies the following conditions

(a) *linearity*

$\mathcal{Q}^\hbar$ is linear for each value of $\hbar$ and $\mathcal{Q}^0$ is the inclusion.

(b) *Rieffel condition*

If $a \in \mathcal{A}_0$ the map $I \ni \hbar \to \mathcal{Q}^\hbar(a) \in R^+$ is continuous.

(c) *von Neumann's condition*

For $a, b \in \mathcal{A}_0$ one has

$$lim_{\hbar\to 0}\|\mathcal{Q}^\hbar(a)\ _J\ \mathcal{Q}^\hbar(b) - \mathcal{Q}^\hbar(a\times b)\|_\hbar = 0$$

where the suffix J denotes the Jordan product

$$\mathcal{Q}^\hbar(a)\ _J\ \mathcal{Q}^\hbar(b) \equiv \frac{1}{2}[\mathcal{Q}^\hbar(a)\ \mathcal{Q}^\hbar(b) + \mathcal{Q}^\hbar(b)\ \mathcal{Q}^\hbar(a)] \qquad 7.52$$

(d) *Dirac's condition*

For $a\ , b \in \mathcal{A}_0$ one has

$$lim_{\hbar\to 0}\|\frac{1}{2\hbar}[\mathcal{Q}^\hbar(a)\ \mathcal{Q}^\hbar(b) - \mathcal{Q}^\hbar(b)\ \mathcal{Q}^\hbar(a)] - \mathcal{Q}^\hbar(\{a,b\})\| = 0$$

(e) *completeness condition*

$\mathcal{Q}^\hbar(\mathcal{A}_0)$ is dense $\mathcal{A}^\hbar_{real}$ for $\hbar \in I$.

♣

Remark 7.12

The notation *strict quantization* is introduced to distinguish it from the *formal quantization* obtained by formal power expansion in the parameter $\hbar$.

♣

Recall that a *Poisson manifold* is a smooth manifold $\mathcal{M}$ that admits on $C^\infty_{loc}(\mathcal{M})$ a Poisson structure in which the product is the the standard product of functions on $\mathcal{M}$.

Definition 7.9

A complete quantization of a Poisson manifold $\mathcal{M}$ is a choice of a subalgebra $\mathcal{A}_0$ of $C^\infty(\mathcal{M}_r)$ and a strict quantization of this subalgebra.

♣

Under favorable circumstances the liner maps $\mathcal{Q}^\hbar$ are morphisms for each value of $\hbar$ and define for each value of $\hbar$ a structure of modified product.

Definition 7.10

A strict quantization is called *strict deformation quantization* if $\mathcal{Q}^\hbar(\mathcal{A}_0)$ is for each value of $\hbar$ a subalgebra of $\mathcal{A}^\hbar$ and the map is injective.

If this is the case, we can define the product $\mathcal{A}_0 * \mathcal{A}_0 \to \mathcal{A}_0$ in such a way that

$$\mathcal{Q}^\hbar(A \ * \ B) = \mathcal{Q}^\hbar(A) \, . \, \mathcal{Q}^\hbar(B)$$

♣

Weyl quantization is a strict deformation quantization (corresponds to *deform* the product of two functions). Another strict deformation quantization is Berezion-Wick quantization that we will introduce in Chapter 11.

Let us consider the important special case in which the Poisson structure is realized in a space of functions (e.g. C^∞) on the classical phase space $T^*(R^d)$ and the corresponding quantum structure is realized by means self-adjoint operators on $\mathcal{H} = L^2(R^d)$.
Analogous considerations can be done in the case $X = T^*(T^d)$.
Let $\{x_k\}$ be cartesian coordinates in R^d. To simplify notations we write $\mathcal{Q}^\hbar(A) \equiv \hat{A}$ neglecting the parameter $\hbar$.
We want to find a correspondence between classical observables A and quantum observables $\hat{A}$ which satisfies

a)
$A \leftrightarrow \hat{A}$ is linear

b)
$x_k \leftrightarrow \hat{x}_k$,where $\hat{x}_k$ is multiplication by x_k

c)
$p_k \leftrightarrow \hat{p}_k \equiv -i\hbar\frac{\partial}{\partial x_k}$

d)
The correspondence $A \leftrightarrow \hat{A}$ is such that if f is continuous then $\hat{f}(x) = f(\hat{x})$
$\hat{f}(p) = (\mathcal{F}f)(\hat{x})$ where $\mathcal{F}$ denotes Fourier transform.

e)
$L_\zeta \leftrightarrow \hat{L}_\zeta$ con $\zeta = (\alpha, \beta)$ $\alpha, \ \beta \in R^d$. Here L_ζ is the symplectic generator of the translations in the direction ζ and $\hat{L}_\zeta$ is the generator of the group of unitary operators $t \to W(t\zeta) = W_\zeta(t)$ defined by

$$(w_\zeta(t)\phi)(x) = e^{\frac{i}{2}(t\alpha;x+\frac{1}{2}t^2 \ \beta)}\phi(x + t \ \beta)$$

(i.e. the one parameter group associated by the Weyl algebra to the direction z).

It is worth noticing that, through suitable limit procedures, a) , e) imply b) , c).

Remark 7.13
Through the correspondence $A \leftrightarrow \hat{A}$ linear symplectic transformations are mapped to unitary transformations.
This is not true in general for non linear symplectic transformations, except the ones that obtained as lift of tranformation of coordinates in R^d.

♣

One can prove that conditions a), .. e) *determine completely* the correspondence $A \leftrightarrow \hat{A}$, and that it cannot be extended to generic functions in phase space.
One has indeed the following theorem

Theorem (van Hove [vH51]
Let $\mathcal{G}$ the class of C^∞ functions on phase space that generate global one-parameter groups. We shall denote by Φ_g the group generated by $g \in G$.

There does not exist a map $g \leftrightarrow \hat{g}$, with $\hat{g}$ self-adjoint , such that

$$\hat{p}_k = i\frac{\partial}{\partial x_k}, \quad \hat{x}_k = .x_k, \quad (a\ g + b\ h)\hat{} = a\hat{g} + b\hat{h}$$

$$(\{g,h\})\hat{} = i[\hat{g},\hat{h}],$$

$$\Phi^f_s.\Phi^g_t.\Phi^f_{-s}.\Phi^g_{-t} \Rightarrow e^{is\hat{f}}\ e^{i\hat{g}}\ e^{-is\hat{f}}e^{-it\hat{g}} \qquad 7.54$$

♢

An interesting problem in the theory of quantization is the following.
Let β be a quantization, let $H(q,p)$ be a hamiltonian on R^{2d} and let $\beta(H) \equiv \hat{H}$. Let H be self-adjoint and let $U(t)$ be the corresponding unitary group.
Let $A \in D(\beta)$ and let A_t its classical evolution according to evolution

$$\frac{dA}{dt} = \{H, A\}$$

and suppose that $A_t \in D(\beta)$, $\forall t$.
A natural question to ask is what is the relation between $\beta(A_t)$ e and $U(t)\beta(A)U(-t)$, i.e. what is the obstruction to the commutativity of the following diagram

$$A \rightarrow^{\Phi_{cl}} A_t \rightarrow^{\beta} \hat{A}_t \rightarrow^{\Phi_q} (\hat{A})_t \rightarrow^{\beta^{-1}} A \qquad 7.55$$

where Φ_{cl} and Φ_q are respectively the flux associated to the classic hamiltonian H and the one associated to the quantum hamiltonian $\hat{H}$.
One can try to estimate

$$\|\beta(A)_t - \beta(A_t)\| \qquad 7.56$$

or

$$|[\beta(A)_t - \beta(A_t)]\psi| \qquad 7.57$$

for a suitable dense set of vectors ψ.
A dual problem, that privileges the role of Quantum Mechanics, is the estimate of

$$|\tilde{A}_t - A_t|_\infty, \qquad |\tilde{A}_t - A_t|_{L^p} \qquad 7.58$$

where the function $\tilde{A}$, *if it exists* is defined by

$$\beta(\tilde{A}_t) = \beta(A)_t$$

The next problem is to introduce a small parameter $\hbar$, which codifies the difference between the two formalisms (Classical and Quantum), and to require that the quantization be defined for any $0 < \hbar \leq \hbar_0$.
In particular we expect that the right hand side of (7.56) , (7.57) , (7.58) be infinitesimal in $\hbar$ and that

$$[\beta_\hbar(A), \beta_\hbar(B)] = \frac{1}{\hbar}\beta(\{A,B\}) + 0(1) \qquad 7.59$$

(recall that in the definition of the Weyl agebra there is a phase that becomes $e^{\frac{c}{\hbar}}$.
One can hope also to find an asymptotic expansion in $\hbar$ (non convergent in general) for the right hand side of (7.56) , (7.57), (7.58).

This means finding functions A_k, $k = 0, 1, ..$ with $A_0 = A$ such that for each integer N

$$||\beta_\hbar(A)_t - \sum_0^N \hbar^k \beta_\hbar(A_k)(t)|| \leq c_n \hbar^{N+1} \qquad 7.60$$

where c_N are suitable constants.
The series will be only asymptotic if one does not prove that $sup_N c_N < \infty$.

The answer to these questions depends on the specific correspondence $A \leftrightarrow \beta_\hbar(A)$. We will see in Chapter 11 some examples of quantization by deformation.
We remark that also in a strict quantization the series (7.58) ned not converge for a large class of classical observables A; in this case the map $A \rightarrow \beta_\hbar(A)$ is well defined for every $0 < \hbar \leq \hbar_0$ but is not analytic.

GEOMETRIC QUANTIZATION

We end this Chapter with a brief description of another form of quantization, called *Geometric Quantization* .
This approach, initiated by B.Kostan and B.Souriau in the early 70', has had a remarkable development in the following years and is still object of intense research.
For an introduction on can consult [GS94] [Ki01]; for later developments one can see [H10].

While Deformation Quantization has its origin in Heisenberg's formulation of Quantum Mechanics, Geometric Quantization has its origin in Schroedinger's formulation. It aims at constructing, starting with phase space of a mechanical system, a Hilbert space in which a quantum mechanical theory can be formulated.
In this sense, Bargmann's and Segal's complex representation is a paradigm of Geometric Quantization and so is the construction of Wigner functions that we will consider in Chapter 11.
We will consider a simple example of Geometric Quantization in the appendix A of Chapter 8, in the framework of the analysis of the semiclassical limit through the W.K.B. method.
As remarked, the purpose of Geometric Quantization is to associate a Hilbert space $\mathcal{H}(\mathcal{M})$ to a symplectic manifold $\{M, \omega\}$ where ω is a closed two-form.
The Hilbert space $\mathcal{H}(\mathcal{M})$ is constructed viewing $\mathcal{M}$ as a fibered manifold $\mathcal{V}$, with complex-valued fibers on which a connection is defined with curvature ω.
Recall that, denoting by $\Gamma(V)$ the collection of smooth sections of $\mathcal{V}$, a *connection* ∇ is a map

$$\nabla : \ \Gamma(V) \rightarrow \Omega^1(M) \otimes \Gamma(V) \qquad 7.61$$

(Ω^1 is the collection of 1-forms σ on $\mathcal{M}$) which satisfies for any smooth function f

$$\nabla(\sigma_1 + \sigma_2) = \nabla\sigma_1 + \nabla\sigma_2, \qquad \nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma \qquad 7.62$$

We shall assume that the fibered manifold is *locally trivializable* (reducible to a product manifold) by a changement of coordinates in a neighborhood of each point of $\mathcal{M}$.
It is then possible to represent in each point of $\mathcal{M}$ the connection by a one-form Θ.
With this notation the *curvature* of the connection is given by $\Omega = d\Theta$. A connection is *flat* if $\Omega = 0$.
It is easy to verify that this is independent of the trivialization chosen.
The collection of all sections turn out a *too large set.*
For example, if $\mathcal{M} = X \otimes T^+X$, where X is the configuration space of a mechanical system, if the connection is flat and if one takes the collection of all smooth sections, the resulting Hilbert

space is $L^2(\mathcal{M}, dl)$ where dl is Lebesgue measure, whereas the Hilbert space of Quantum Mecanics is $L^2(X, dx)$ (notice that this *is not a subspace of the former)*
It is then necessary to consider only a subset of sections of $\Gamma(V)$. This choice goes under the name of *Polarization* (roughly speaking, a choice of conjugate variables).
Various polarization have been chosen. One posible choice is the *Kaeler Polarization* detrmined by a choice of complex structure for $\mathcal{M}$ (since $\mathcal{M}$ is locally a symplectic manifold, its symplectic structure defines locally a complex structure).
The Kaeler Polarization is then the choice of olomorphic leaves.
Another choice is the polarization defined by the selection as leaves of *lagrangean manifolds.*We shall call this *Real Polarization .*
One should remark the analogy of these choices with the real and the complex Bargmann-Segal representation of the Weyl system.
For example, in the case of the hydrogen atom, one can consider the set Ω_- of points in phase space in which the energy is strictly negative. Ω_- is provided with the standard symplectic structure.
In this case the base manifold is compact, the symplectic fibers are smooth (except the origin) and one can consider only the *Bohr-Sommerfeld fibers* i.e the fibers for which a globally flat section can be defined (so that a complete set of action-angle variables can be defined).
One proves [GS84] that the set of Bohr-Sommerfeld fibers is discrete, and this leads to the *Bohr-Sommerfeld Quantization.*
From the point of view of semiclassical analysis this can be interpreted as a procedure that replaces, for the construction of the Hilbert space, the space of function over Ω_- with a space of functions defined over the collection of all smooth Bohr-Sommerfeld orbits.
The definition of Bohr-Sommerfeld fibers can be extended to other systems ad this extends the definition of *Bohr-Sommerfeld Quantization.*
Remark that the symplectic manifold on which the quantization is performed is locally isomorphic to a toroidal manifold and the fibers are defined by a *momentum map.*
For further detais about this interesting field of research on can consult [GS94][Ki01] .

References for Chapter 7

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APPENDIX : The magnetic Weyl algebra

In this appendix we describe the *Magnetic Weyl algebra* , a modification of the Weyl algebra useful for the description of particles in a magnetic field.
We have seen that the Weyl algebra is a structure adapted to *quantize* hamiltonian Mechanics with phase space R^{2N} using the standard symplectic form.
In classical Mechanics when treating charged particles in a magnetic field it may be convenient, instead of modifying the hamiltonian through a redefinition of momentum (minimal coupling), to leave the hamiltonian invariant and modify the Poisson Brackets into *Magnetic Poisson Brackets.*
In the same way in Quantum Mechanics in the treatment of non relativistic particles interacting with an electromagnetic field, it may turn to be convenient to make use of a modified form of Weyl Algebra,the *Magnetic Weyl Algebra* .
This permits often to clarify topological effects which are due to the presence of the magnetic field.

Let us briefly recall the Poisson structure associated to a symplectic manifold.
In particular consider the configuration space $\mathcal{M}$ of a particle which we identfy with R^N, $N \geq 2$. Each fiber of the tangent space is a copy of R^N and each fiber of the cotangent spacecan be also identified with R^N.
We shall denote by $q \equiv \{q_1, .., q_N\}$ a system of coordinates relative to orthogonal axes and by $\{p_1, .., p_n\}$ coordinates relative to orthogonal axes in the fibers of the cotangent space.
A symplectic structure on $\mathcal{M}$ is a closed non-degenerate two-form $\Sigma \in \Omega^2\mathcal{M}$ (the space of two-forms on $\mathcal{M}$).
We suppose always that Σ has C^∞ coefficients with respect to the standard two-form.
Remark that, being non degenerate, Σ uniquely defines an isomorphism $\beta : \ \Omega^1 \to \Xi(\mathcal{M})$ (the fibered space of the vector fields on $\mathcal{M}$.) Defining

$$\{f, g\}_\Sigma = \Sigma(\beta(df)\beta(dg)) \qquad 7A.1$$

the symplectic manifold acquires a Poisson structure.
In the case $\mathcal{M} = R^N$ and without magnetic field the symplectic structure most commonly used is

$$\sigma \ : \ \Xi \times \Xi \to R, \quad \sigma[(q, p), (q', p')] = q'.p - q.p' \qquad 7A.2$$

where $q, \ q' \in R^N$ are orthogonal coordinates in R^N and $p, \ p' \in R^N$ are coordinates relative to the axes parallel to $dq_1, \ ..dq_N$.
We treat only the case of one particle in R^3 subject to an external magnetic field B; the case of several particles is analyzed in a similar way.
The dynamics of a non-relativistic particle of mass m in R^3 in field of scalar potential V and subject to a magnetic field $B(q)$ is given by the hamiltonian

$$H(p, q) = \frac{1}{2m}(p + eA(q))^2 + V(q), \qquad rot \ A = B \qquad 7A.3$$

together with the symplectic form (7A.2).
Notice that the magnetic field defines a two-form $\hat{B}$ through (in local coordinates) $\hat{B}_{i,j}(q) = \frac{1}{2}\epsilon_{i,j,k}B_k(q)$ where $\epsilon_{i,j,k}$ is the totally antisymmetric Ricci symbol.
This is an instance of Hodge duality in a manifold of dimension three. .
The same dynamics can be equivalently described with the hamiltonian $H'(p, q) = \frac{1}{2m}p^2 + V(q)$ but then the symplectic two-form must be modified adding the closed two-form $\hat{B}$.

The resulting form is closed and non degenerate together with the old one defines a new Poisson structure.
This representation has an intrinsic ambiguity in that two vector potentials $A(q)$, $A'(q) \in R^3, \; q \in R^3$ give rise locally to the same two-form $\hat{B}(q)$ if $A'(q) - A(q) = \nabla\phi(q)$ where $\phi(q)$ is a C^1 function (local gauge invariance).
We shall call *gauge group* this group of transformations.
In local coordinates the magnetic field $B(q)$ is represented by the antisymmetric tensor $rotA$ and therefore corresponds to the antisymmetric two-form $(\partial_i \, A_k - \partial_k \, A_i) dq_k \wedge dq_i$ which is by construction invariant for local gauge transformations.
This formalism can be extended to the case of generic smooth manifolds $\mathcal{M}$ and that one can consider cases in which the magnetic field is represented by a two form which is closed but not exact.
For example one can consider the case of the magnetic field of an infinite rectilinear wire with constant electric current.
The corresponding two form is closed but not exact, and originates topological effects (Bohm-Aharanov effect).
In the phase space $R^3 \times R^3$ with natural coordinates $q_k, \; p_h, \; h, k = 1, 2, 3$ the equations of motion

$$\dot{q}_k = \frac{1}{m} p_k, \quad \dot{p}_k = \frac{e}{m} \sum_{h=i}^{3} B_{k,h}(q) \; p_h \qquad 7A.4$$

are associated to the hamiltonian (7A.3) through the symplectic form $\sum_k dp_k \wedge dq_k$; they can also be associated to the hamiltonian

$$H_0(p, q) = \frac{1}{2m} p^2 + V(q)$$

through the symplectic form

$$\sum_k dp_k \wedge dq_k + \frac{e}{m} \sum_{h,k=1}^{3} B_{i,k} \; dp_h \wedge dp_k$$

The corresponding Poisson Brackets are

$$\{f, g\}_B = \sum_{h,k=1}^{3} \left(\frac{\partial f}{\partial p_k} \frac{\partial f}{\partial q_h} - \frac{\partial f}{\partial q_k} \frac{\partial f}{\partial p_h} + \frac{e}{m} B_{k,h} \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial p_k} \right) \qquad 7A.5$$

Notice that the equation of motion (7A.4) *are gauge invariant* (they only depend on the magnetic field) while the hamiltonian varies through the addition of a total derivative.
It seems preferable to introduce a quantization map invariant under gauge tranforamtions. This leads to the intruduction of the *Magnetic translations* and to the *Magnetic Weyl Algebra* [Z68] .
We want to find a symplectic transformations α that extend maps $x \to x + q, \; x \in R^3$

$$\alpha_x \{q, \; p\} = \{q + x, \; p + \tau_x(q, p)\} \qquad 7A.6$$

. The group property implies $\tau_{x+y}(q, p) = \tau_x(q, p) + \tau_y(q + x, p + \tau_x(q.p))$.
The condition to be symplectic is

$$\alpha_x^*(\sigma_B) - \sigma_B = 0$$

It is easy to show that this identity reads

$$T_{-x;i,j}(q,p)dq_i \wedge dq_j + S_{-x;i,j}(q-px)dq_j \wedge dp_i = 0$$

where

$$T_{x;i,j}(q,p) = \frac{\partial}{\partial q_i}\tau(x,k) - \frac{\partial}{\partial q_k}\tau(x,j) + eB(q)_{j,k} - eB(q+x)_{j,k} \quad S_x = \frac{\partial}{\partial p_j}\tau_{x,i}(q,p)$$

From this one derives $\frac{\partial \tau(q,p)}{\partial p_k} = 0, \; k = 1,2,3$ and moreover

$$\frac{\partial}{\partial q_j}(\tau_{q,p})_k - \frac{\partial}{\partial q_k}(\tau_x(q,p))_j + eB(q)_{j,k} - eB(q+x)_{j,k} = 0 \qquad 7A.7$$

To determine τ we must invert the differential relation (7A.7) which can be written, taking into account that the magnetic field is (at least locally) an exact differential form

$$\frac{\partial}{\partial t}\alpha_{-tx}(q,p)_{t=0} = -(x, e(DA(q)).x))$$

where the one-form A satisfies $dA = B$ (in coordinates $(rotA)_{i,j} = B_{i,j}$).
This will the origin of the gauge ambiguity.
The solution $A(q)$ of (7A.7) is

$$\tau_x(q) = eA(q+x) - A(q) \qquad rotA = B \qquad 7A.8$$

and is defined modulo the addition of the gradient of a function (gauge ambiguity).
It follows that magnetic translations are *defined modulo a gauge transformation.*
Recall that in Hamiltonian Mechanics the *momentum* μ_Ξ associated by a symplectic form to a vector field Ξ (and therefore to an infinitesimal transformation in configuration space) is by definition the contraction of the symplectic form with the field Ξ.
The *momentum map* associated to the magnetic translation is $\mu_A(q,p) = p - eA(q)$.
Gauss's theorem implies that the integral of a one-form A along a closed path is equal to the flux of the magnetic field across a surface that has the given path as boundary.
The magnetic translations along a closed path may therefore generate a non trivial homotopy if the form which represents the magnetic field is closed but not exact.
This will play a role in the quantization.
Weyl's quantization in its algebraic structure does not pose serious problems since the symplectic form that defines the Weyl system is simply substituted by the magnetic symplectic form.
Its description in the Schroedinger representation requires a choice of gauge, in accordance with the fact that the classical hamiltonian (and therefore its quantum counterpart) depends on the choice of gauge. The representation of the state will depend on this choice, but all the expectation values are *gauge independent.*
The choice of a gauge in which to describe the Schroedinger equation (and therefore to use the formalism of partial derivatives and the Fourier transform) corresponds in the geometric quantization to a choice of a local Lagrangian manifold.
In the presence of a magnetic field $B(x)$ the Weyl product takes the following form in dimension N [MP04]

$$(f \ *_B^\hbar \ g)(\xi) = (\frac{2}{\hbar})^{2N} \int d\eta \int d\zeta e^{-\frac{2i}{\hbar}\sigma(\eta,\zeta) - \frac{i}{\hbar}\int_{\mathcal{T}(q,y,z)} B(x;\eta,\zeta)d\xi} f(\xi-\eta)g(\xi-\zeta) \qquad 7A.9$$

(recall that for each value of x, $B(x)$ is a two form; we denote by $B(x;\eta,\zeta)$ its value, where η,ζ are in the tangent space at x).
In 7A.9 we have denoted by $\mathcal{T}(q,y,z)$ the projection on configuration space of the triangle $\mathcal{T}(\xi,\eta,\zeta)$ with vertices the vectors ξ,η,ζ (i.e the symplectic area of the triangle; this referenceto the *symplectic area* is a feature common to Geometric Quantization (see [We94] [BC86])
Only the projection on configuration space enters because the two form is modified only in that space; the particular form (7A.9) is most naturally derived in its infinitesimal form and extended by parallel tranport.
The product described by (7A.9) is usually called *magnetic Weyl -Moyal product.*
It is associative, non-commutative and satisfies

$$(f \ *_B^\hbar \ g)^* = (g \ *_B^\hbar \ f) \qquad 7A.10$$

We can give (7A.9) a more convenient form.
Using Fourier transform it can be seen that

$$(f \ *^\hbar \ g)(\xi) = \int_\Xi d\eta \int_\Xi d\zeta e^{-\frac{i}{\hbar}\sigma(\xi-\eta,\xi-\zeta)} f(\eta)g(\zeta)$$

where Ξ is phase space and $\xi,\ \eta,\ \zeta \in T(\Xi) \equiv \Xi$.
The *Weyl-Moyal magnetic* product is now

$$(f \ *_B^\hbar \ g)(\xi) = \int_\Xi d\eta \int_\Xi d\zeta e^{-\frac{i}{\hbar}(\sigma+eB)(\xi-\eta,\xi-\zeta)} f(\eta)g(\zeta) \qquad 7A.11$$

If the magnetic field s of class C^∞ the Weyl-Moyal magnetic product is a map $\mathcal{S}(R^N)\times\mathcal{S}(R^N)$ to $\mathcal{S}(R^N)$, and can be extended by duality to a continuous map $\mathcal{S}(R^N)\times\mathcal{S}'(R^N)$ in $\mathcal{S}'(R^N)$ trough

$$(F_B^\hbar f,g) = (F, f *_B^\hbar fg), \quad (f *_B^\hbar F, g)* = (F, f *_B^\hbar g), \quad F \in \mathcal{S}'(R^N),\ f \in \mathcal{S}(R^N) \qquad 7A.12$$

and, again by duality and with a limit procedure to a continuous map $\mathcal{S}'(R^n)\times\mathcal{S}'(R^N)$ in $\mathcal{S}'(R^N)$ which satisfies

$$(F *_B^\hbar G, f)) = (G, F *_B^\hbar f), \quad F,\ G \in \mathcal{S}', \quad f \in \mathcal{S} \qquad 7A.13$$

The resulting extended magnetic Weyl algebra is useful to compose quantum observables in a gauge invariant way.
When we will analyze in Chapter 11 the Wigner transform we will see that it is sometimes convenient to use a representation in term of functions on $\mathcal{M}\times\mathcal{M}$ rather then on $\mathcal{M}\times\mathcal{M}^*$ where $\mathcal{M}$ is configuration space.
In the case $\mathcal{M} = R^N$ that we are considering one obtains this more convenient form by taking the Fourier transform with respect to the second variables, keeping in mind that $p_k = i\hbar\frac{\partial}{\partial q_k}$.
With the notation $\phi = (I\otimes\mathcal{F})f$ where $\mathcal{F}$ is Fourier transform the multipliction lax for the magnetic Weyl algebra becomes

$$(\phi *_B^\hbar \psi)(q,x) = \int_{R^N} dy\phi(q - \frac{\hbar}{2}(x-y), y))\ \psi(q + \frac{\hbar}{2}y, x-y)e^{-\frac{i}{\hbar}\Phi_B(q,;x;y)} \qquad 7A.14$$

wher $\Phi_B(q;x;y)$ is the magnetic flux through the triangle with vertices in the points $\{q - \frac{\hbar}{2}x\ ,\ q - \frac{\hbar}{2}x + \hbar y\ ,\ q + \frac{\hbar}{2}x\}$.

This product depends only on the magnetic field.
If the two-form representing the magnetic field is exact in any representation on the Hilbert $L^2(R^N)$ it is convenient to make use of a one form a related to the two-form B by $B = da$.
In local coordinates the one-form a is represented by a *vector potential* $A(x)$ related to B through $B(x) = rot\ A(x)$
This relation *has not a unique inverse* and the solutions differ from each other by a gradient (at least locally)

$$A'(x) = A(x) + \nabla\phi \qquad 7A.15$$

where $\phi(x)$ is a scalar field.
We shall see that the representations which correspond to different choices of A are all unitarily equivalent and we will give the unitary operator which implements the equivalence.
The need to introduce the vector potential the has its origin in the fact that in the Schroedinger representation there exist unitary operators which commute with the elements of the magnetic Weyl algebra.
This is the quantum counterpart of classical gauge invariance.
The analysis of this probem could be done in the general context of projective representations of algebras defined by a *twisted product* (as are the Weyl algebra and the magnetic Weyl algebra).
In the following we consider only the special case of the Schroedinger representation of the magnetic Weyl algebra on the configuration space R^3.
Let $A = A_k(x)dx_k$ be a one-form in R^3 $dA = B$. In coordinates

$$B_{i,j} = h(x,y)_{k,j} dx_k \wedge dx_j, \quad h(x,y)_{k,j} = \frac{\partial A_k(x)}{dx_j} - \frac{\partial A_j(x)}{dx_k} \qquad 7A.16$$

If $x, y \in R^3$ define $\Gamma_A[x,y]$ to be the integral of the one-form A on the segment

$$[a,b] \equiv \cup_{s\in[0,1]}[sx + (1-s)y]$$

Due to Stokes's theorem, if we denote by $\Omega_B(q_1,q_2,q_3)$ the flux of the two-form B across the triangle defined by the points q_1, q_2, q_3 one has

$$\Omega_B(q, q+\hbar x, q+\hbar x+\hbar y) = \Gamma_A([q, q+\hbar x])\ \Gamma_A[q+\hbar x, q+\hbar x+\hbar y]\ (\Gamma_A[q+\hbar x+\hbar y])^{-1} \quad 7A.17$$

Setting

$$\omega_b^\hbar(q,x,y) = e^{-\frac{i}{\hbar}\Omega^\hbar(q,x,y)} \quad \lambda_A^\hbar(q,x) = e^{-\frac{i}{\hbar}{}_A\Gamma_A([q,q+\hbar x])}$$

one obtains

$$\omega^\hbar(q,x,y) = \lambda_A^\hbar(q,x)\lambda_A^\hbar(q+\hbar x, y))(\lambda_A^\hbar(q, x+y)^{-1} \qquad 7A.18$$

Setting

$$[U_A^\hbar(x)\phi](q) = \lambda_A^\hbar(q,x)\phi(q+\hbar x), \quad V^\hbar(p)\hat\phi(k) = \phi(k+p) \qquad 7A.19$$

one verifies that these unitary operators generate a representation determined by A, of the magnetic Weyl algebra associated to the magnetic two-form B.
Using (7A.19) it is easy to verify that if one modifies the one- form A in $A' = A + d\Phi$ where Φ is a sufficiently regular scalar field one obtains

$$e^{\frac{i}{\hbar}\Phi(q)}[U_A^\hbar(x)\phi](q) = [U_{A'}^\hbar(x)\phi](q)e^{\frac{i}{\hbar}\Phi(q)}$$

Notice that the magnetic Weyl algebra is described by the following relationsé

$$[Q_j, Q_k] = 0 \quad [Q_k, \Pi_{A,j}^\hbar] = i\hbar\delta_{i,j} \quad [\Pi_{A,j}^\hbar, \Pi_{A,k}^\hbar] = i\hbar B_{k,j} \qquad 7A.20$$

Denoting by $e^{iQ.p}$ the group that induces translations in Fourier transform and with

$$U_A^\hbar(x) \equiv e^{ix\Pi_A^\hbar} \equiv e^{-\frac{i}{\hbar}\Gamma_A([Q,Q+\hbar x])} e^{ix.P}$$

the group of magnetic translations in the configuration space one has

$$U_A^\hbar(x) U_A^\hbar(x') = \omega_B^\hbar(Q; x, x') U_A^\hbar(x+x')$$

and a unitary representation of the magnetic Weyl group is given by the unitary operators

$$W_A^\hbar(q,p) \equiv e^{-i\sigma(q,p;Q,\Pi_A^\hbar)} = e^{-\frac{i}{2}q.p} e^{-iQ.p} U_A^\hbar(x) \qquad 7A.21$$

Remark 7A.1

In a planar system with constant perpendicular magnetic field the relation (7A.20) takes the form

$$[Q_1, Q_2] = 0 \quad [\Pi_1, \Pi_2] = \hbar B \quad [Q_k, \Pi_j] = \delta_{k,j} \qquad 7A.22$$

Setting $K_j = P_j - \frac{1}{\hbar B}\epsilon_{j,k} Q_k$ one has

$$[K_j, K_m] = \frac{1}{\hbar B}\epsilon_{k,j}, \quad [\Pi_j, \Pi_k] = \hbar B \epsilon_{j,k} \quad [K_j, \Pi_k] = 0 \qquad 7A.23$$

It is to be noted that the pair $K_1,\ K_2$ generates a Heisenberg algebra , and the same is true for the pair Π_1, Π_2.
These two algebras commute. They can be represented in the space $L^2(R^2, dx_1 dx_2)$ by

$$K_1 \equiv .x_1, \quad \Pi_2 \equiv .x_2 \quad K_2 \equiv \hbar B \frac{\partial}{\partial x_1}, \quad \Pi_1 \equiv \frac{1}{\hbar B}\frac{\partial}{\partial x_2}$$

From (7A.23) one can trace back the symplectic transformation between this representation and the one in the space $L^2(R^2, dQ_1 dQ_2)$.

♣

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As an example of the Fock (or of the complex Bargmann-Segal representation, we will discuss briefly the case of a particle in R^3 subjected to a magnetic field $B(x)$ oriented along an axis, that we will choose to be $\hat{3}$ [F28] [LL77]
We shall start with the case in which the magnetic field has constant strength B_0.
The hamiltonian, in suitable units, can be presented as

$$H_0 = -(\nabla + iA_0)^2, \quad A_0 = \frac{B_0}{2}\{-x_1, x_2, 0\} \quad B_0 > 0 \qquad 7B.1$$

We have chosen a suitable gauge; one has $\nabla A_0 = B_0\hat{3}$. The notation H_0 is used to distinguish the case of constant magnetic field.
This hamiltonian corresponds to free motion along the axis $\hat{3}$; therefore we will consider only the motion in the $\{\hat{1}, \hat{2}\}$ plane.
It is convenient to introduce the complex notation

$$z = x_1 + ix_2, \qquad \partial = \partial_z = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}) \quad \bar{\partial} = \partial_{\bar{z}} \qquad 7B.2$$

It is convenient also to introduce the operators $P_0^{\pm} = H_0 \pm B$.
They describe in $L^2(R^3) \otimes C^2$ the dynamics a particle with magnetic moment 1 and spin $\frac{1}{2}$ under the influence of the magnetic field (often called Pauli system).
Define the operators

$$Q_0 = -2i\bar{\partial}_z - A_0 \qquad \bar{Q}_0 = -2\partial_z \qquad 7B.3$$

These operators can be written, with $\psi = \frac{B_0}{4}|z|^2$ as

$$Q_0 = -2ie^{-\psi_0}\bar{\partial}e^{\psi_0}, \qquad \bar{Q}_0 = Q_0^* = -2ie^{\psi_0}\partial e^{-\psi_0} \qquad 7B.4$$

Notice that ψ_0 solves $\Delta\psi = B_0$ i.e. is a potential for B_0.
One verifies that

$$[Q_0, \bar{Q}_0] = 2B_0 I \qquad 7B.5$$

i.e. that the pair $Q_0, \bar{Q}_0$ satisfy canonical commutation relations.
One verifies also that the operators

$$P_0^+ = Q_0\bar{Q}_0, \qquad P_0^- = \bar{Q}_0 Q_0 \qquad 7B.6$$

are orthogonal projection operators with $P_+ + P_- = I$ and that

$$H_0 = Q_0.\bar{Q}_0 - BI = \bar{Q}_0.Q_0 + BI \qquad 7B.7$$

The vectors in the subspace $\mathcal{H}_-$ on which P_+ projects satisfy

$$P_0^+ u = 0 \Rightarrow Q_0 u = e^{-\psi_0}\bar{\partial}(e^{\psi_0}u) = 0 \qquad 7B.8$$

This implies that the function $f \equiv e^{\psi_0}u$ is entire analytic in C.
On the other hand $U = e^{-\psi_0}f \in L^2(R^2)$.
The vectors in $\mathcal{H}_-$ are therefore in one-to -one correspondence with the elements of the space

$$\mathcal{F} \equiv \{f(z) : \ (\hat{\partial}f(z)).e^{-\frac{B_0|z|^2}{4}} \in L^2\} \qquad 7B.9$$

which is Fock space.
From (7B.5) one derives also the the spectrum of the operator H_0 is

$$SpH_0 = \{(2q+1)B_0, \quad q \in \mathbf{N}\} \qquad 7B.10$$

and the multiplicity of each eigenvalue is infinite (because P_+ project on a space of infinite dimension).
One verifies the the projection operator P_k on the k^{th} subspace $\mathcal{L}_k$ has kernel

$$K_k(z,w) = e^{\frac{1}{4}(wz-|z|^2-|w|^2)}\Pi_{j=1}^k L_k(\frac{1}{2}(z:j-w_j)^2) \qquad 7B.11$$

where L_k is the Laguerre polynomial of order k

$$L_k(y) = \frac{e^y}{n!}\frac{d^n}{dy^n}(e^{-y}y^n)$$

We consider now briefly the case in which the magnetic field is not constant, but still directed along $\hat{3}$ [Iw83] [MR03][RT08] [ST03]
The motion along $\hat{3}$ is still free motion, and we study only the motion on the $\hat{1}, \hat{2}$ plane .
Define as before $z = x_1 + ix_2$, $\partial = \partial_1 - i\partial_2$ and introduce as before the potential ϕ solution of

$$\Delta\phi_B = B, \quad \Delta = -\bar{\partial}.\partial \qquad 7B.12$$

We consider $B(x)$ as a small perturbation of B_0 and write $B(x) = B_0 + \lambda b(x)$ with $b(x)$ of compact support and λ a small parameter.
We write the solution of (7B.12) as

$$\phi_B(z) = \frac{B_0}{4}|z|^2 + \phi_\lambda(z) \qquad 7B.13$$

and the vector potential $A(x)$ as

$$A(x) = A_0(x) + \lambda a(x), \quad A_0(x) = \{-x_2 - \lambda a_2(x),\ x_1\}$$

Notice that if the flux Ψ of $b(x)$ (i.e$\int b(x)dx$) is not zero, $a(x)$ cannot decay wit the distance R more than $\frac{1}{R}$ since $\int_0^{2\pi} a(R\hat{n}).nd\theta = \Psi$.
The hamiltonian is

$$H_\lambda = (i\nabla + A)^2, \quad A = A_0 + \lambda a \qquad 7B.14$$

The parameter λ is small, and we can try to expand H in powers of λ.
To first order there is a term $(A_0.a)$ which does not vanish at infinity if the total magnetic flux of $b(x)$ is not zero (recall that $a(x)$ in that case cannot decay more than as R^{-1}.)
In spite of this one proves that $H_\lambda - H_0$ is compact relative to H_0 and therefore by Weyl's theorem the essential spectrum of H_λ is the same as that of H_0 i.e. $\{(2n+1)B_0$.

Also in this case one can define

$$Q = -2i\bar{\partial} - A = -2ie^{-\phi}\bar{\partial}e^{\phi}$$

$$\bar{Q} = -2i\partial - \bar{A} = -2ie^{\phi}\partial e^{-\phi}$$

One has

$$[Q, \bar{Q}] = 2B_0 + 2b(x) \quad x \in R^2 \qquad 7B.15$$

Notice that the algebra generated by Q and $\bar{Q}$ is no longer the Heisenberg algebra.
One can still define the operators

$$P^+ = Q.\bar{Q} \qquad P^- = \bar{Q}.Q$$

but they are no longer projetion operators. Still one has

$$P^- - P^+ = 2B_0 + 2b(x), \qquad H = Q.\bar{Q} - (B_0 + b(x) = \bar{Q}.Q + 0 + b(x) \qquad 7B.16$$

The spectrum of H is still contained in $[B, \infty)$ and the eigenspace to the lowest eigenvalue the set of functions u for which $P^- u = 0$ i.e $Qu = o$.
This implies

$$\bar{\partial} e^{\psi} u = 0 \Rightarrow u = f e^{-\psi} \qquad 7B.17$$

. Therefore the eigenspace to the lowest eigenvalue is made of entire functions which belong to $L^2(C)$ when multiplied by $e^{-\psi}$.
This is a Fock space relative to the (non gaussian measure) $e^{-\psi(z)}$ where $\psi(z)$ is the *potential* of $B_0 + b(z)$.
It is now not easy to construct a complete orthonormal set. In general the n^{th} eigenfunction in the Fock representation is given by an entire function $\xi_n(z)$ (not a polynomial).
And it is non longer easy to find the remaining part of the spectrum of H.
Since the perturbation is relatively compact, the points $\{(2n+1)B_0\}$ belong to the essential spectrum of H but in general they are no longer eigenvalues. The eigenvalues $\lambda_{n,k}$ have $(2n+1)B_0$ as limit point, and under suitable conditions $\lambda_{n,k}$ converge super-exponentially to $(2n+1)B_0$ as $k \to \infty$. [RS11]
There is so far no complete theory to determine the location of the eigenvalues of H.

Remark 7B.1
The corresponding Landau problem on a torus gives further problems since the requirement that the eigenfunctions be single-valued restricts B_0 to have quantized values of the flux across the torus.
The same problem appears when one adds to the Landau Hamiltonian a potential (scalar or vector) which is periodic. Let $\mathcal{C}$ a corresponding cell. The requirement that the eigenvector be single valued requires also here that the flux of B_0 across $\mathcal{C}$ be quantized.
In this case the fact that the $U(1)$bundle over $\mathcal{C}$ (corresponding to the fact that on each point the phase of the wave function can be changed) can be non trivial leads to interesting topological problems.

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