Proof of Weiestrass theorem via Bernstein polynomials

This is a concrete construction to prove that polynomials on $[0,1]$ are dense in $C([0,1])$.

To this aim define the polynomials $p_{n,k} : [0,1] \to \mathbb{R}$ by

$$p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}\,, \qquad \begin{array}{c} x \in [0,1]\,, \\ n,\, k \text{ integers} \\ \text{with } 0 \leqslant k \leqslant n. \end{array} \tag{1}$$

They satisfy the following identities:

$$\sum_{k=0}^{n} p_{n,k}(x) \;=\; 1 \tag{2}$$

$$\sum_{k=0}^{n} k\, p_{n,k}(x) \;=\; nx \tag{3}$$

$$\sum_{k=0}^{n} k(k-1)\, p_{n,k}(x) \;=\; n(n-1)x^2\,. \tag{4}$$

Indeed, (2) is just the Binomial Theorem $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k\, b^{n-k}$ applied to $a = x$ and $b = 1-x$, (3) follows from

$$\begin{aligned} \sum_{k=0}^{n} k\, p_{n,k}(x) \;&=\; \sum_{k=1}^{n} k \binom{n}{k} x^k (1-x)^{n-k} \;=\; nx \sum_{k=1}^{n} \frac{k}{n} \binom{n}{k} x^{k-1} (1-x)^{n-k} \\ &=\; nx \sum_{k=1}^{n} \binom{n-1}{k-1} x^{k-1} (1-x)^{n-k} \;=\; nx \sum_{r=0}^{n-1} \binom{n-1}{r} x^r (1-x)^{n-1-r} \;\overset{(2)}{=}\; nx\,, \end{aligned}$$

and (4) follows from

$$\begin{aligned} \sum_{k=0}^{n} k(k-1)\, p_{n,k}(x) \;&=\; \sum_{k=2}^{n} k(k-1) \binom{n}{k} x^k (1-x)^{n-k} \\ &=\; n(n-1)x^2 \sum_{k=2}^{n} \frac{k(k-1)}{n(n-1)} \binom{n}{k} x^{k-2} (1-x)^{n-k} \\ &=\; n(n-1)x^2 \sum_{k=2}^{n} \binom{n-2}{k-2} x^{k-2} (1-x)^{n-k} \\ &=\; n(n-1)x^2 \sum_{r=0}^{n-2} \binom{n-2}{r} x^r (1-x)^{n-2-r} \;\overset{(2)}{=}\; n(n-1)x^2\,. \end{aligned}$$

As a consequence, one has

$$\sum_{k=0}^{n} (k-nx)^2 p_{n,k}(x) \;=\; nx(1-x) \tag{5}$$

because

$$\sum_{k=0}^{n} (k-nx)^2 p_{n,k}(x) \;=\; \sum_{k=0}^{n} \big(k(k-1) - (2nx-1)k + n^2x^2\big)\, p_{n,k}(x) \;=\; nx(1-x)$$

where in the last step one uses (2), (3), and (4). Moreover, for any $\delta > 0$,

$$\sum_{k \text{ s.t. } |k-nx| \geqslant n\delta} p_{n,k}(x) \;\leqslant\; \frac{1}{4n\delta^2} \tag{6}$$

(in the inequality selecting k it is understood that k is an integer between 0 and n), because

$$\sum_{|k-nx|\geqslant n\delta} p_{n,k}(x) \;\leqslant\; \frac{1}{n^2\delta^2}\sum_{|k-nx|\geqslant n\delta}(k-nx)^2 p_{n,k}(x) \;\overset{(5)}{\leqslant}\; \frac{x(1-x)}{n\delta^2} \;\leqslant\; \frac{1}{4n\delta^2}$$

where last inequality is due to $x(1-x)\in[0,\frac{1}{4}]$ (since $x\in[0,1]$). All the preliminaries are now completed.

Take now $f\in C([0,1])$. Correspondingly, define the polynomials

$$B_n^{(f)}(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x)\,. \tag{7}$$

f is uniformly continuous, being a continuous function defined on a compact. That is, $\forall\varepsilon>0\;\exists\,\delta>0$ such that $|x-x'|\leqslant\delta$ implies $|f(x)-f(x')|\leqslant\varepsilon$ (uniformly in the choice of x,x'). Then one has

$$\begin{aligned}
|f(x)-B_n^{(f)}(x)| &\overset{(2)}{=} \Big|\sum_{k=0}^{n}\big(f(x)-f\big(\tfrac{k}{n}\big)\big)\,p_{n,k}(x)\Big| \;\leqslant\; \sum_{k=0}^{n}\big|f(x)-f\big(\tfrac{k}{n}\big)\big|\,p_{n,k}(x)\\
&= \sum_{|k-nx|\,\leqslant\, n\delta}\big|f(x)-f\big(\tfrac{k}{n}\big)\big|\,p_{n,k}(x) \;+\; \sum_{|k-nx|\,\geqslant\, n\delta}\big|f(x)-f\big(\tfrac{k}{n}\big)\big|\,p_{n,k}(x)\\
&\leqslant\; \varepsilon\sum_{k=0}^{n}p_{n,k}(x) \;+\; 2\,\|f\|_\infty\sum_{|k-nx|\geqslant n\delta}p_{n,k}(x)\\
&\overset{(6)}{\leqslant}\; \varepsilon \;+\; \frac{\|f\|_\infty}{2n\delta^2}\,.
\end{aligned}$$

Therefore, $\limsup_{n\to\infty}|f(x)-B_n^{(f)}(x)|\leqslant\varepsilon$, uniformly in x. Last, ε being arbitrary, one concludes $\sup\limits_{x\in[0,1]}|f(x)-B_n^{(f)}(x)|\xrightarrow{n\to\infty}0$. In other words,

$$\lim_{n\to\infty}\big\|f-B_n^{(f)}\big\|_\infty \;=\; 0$$

i.e., the polynomial $B_n^{(f)}$'s associated with f approximate f uniformly.

Remark: polynomials $B_n^{(f)}$ ("Bernstein polynomials") were introduced first by S. Bernstein in the paper *Démonstration du théorème de Weiestrass, fondeé sur le calcul des probabilités*, Comm. Soc. Math. Kharkow (2), 13 (1912), 1-2.

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