PROBLEMS IN CLASS, WEEK 12.
Info: `www.math.lmu.de/~michel/WS12_MQM.html`

Problem 30. (The $f(x)g(\nabla)$ theorem.)

Notation: $\widehat{\ }$ denotes the Fourier transform, $^\vee$ denotes the inverse Fourier transform.

(i) Let $f, g \in L^\infty(\mathbb{R}^d)$. For every $\psi \in L^2(\mathbb{R}^d)$ define

$$\big(f(x)g(-i\nabla)\psi\big)(x) \;:=\; f(x)\,\big(g(2\pi\cdot)\,\widehat{\psi}\big)^\vee(x)\,. \qquad (*)$$

Prove that $(*)$ defines an element of $L^2(\mathbb{R}^d)$ and that the map $\psi \overset{T}{\longmapsto} f(x)g(-i\nabla)\psi$ is a bounded operator on $L^2(\mathbb{R}^d)$ with $\|T\| \leqslant \|f\|_\infty \|g\|_\infty$.

(ii) Let $f, g \in L^2(\mathbb{R}^d)$. Prove that $(*)$ defines a Hilbert-Schmidt map $\psi \overset{T}{\longmapsto} f(x)g(-i\nabla)\psi$ with $\|T\|_{\mathrm{HS}} = (2\pi)^{-d/2}\|f\|_2\,\|g\|_2$.

(iii) Let $f, g \in \overline{L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)}^{\|\,\|_\infty}$ (the closure in the L^∞-norm). This is the case, for instance, when $f, g \in L^\infty(\mathbb{R}^d)$ and $f(x), g(x) \to 0$ as $|x| \to \infty$. Prove that $(*)$ defines a compact operator $\psi \overset{T}{\longmapsto} f(x)g(-i\nabla)\psi$.