QUESTION TIME (7 Dec 2011)

Alessandro

① X, Y Banach. $T \in \mathcal{B}(X,Y)$, injective, with bdd inverse on $\mathrm{Ran}\,T$, and $\mathrm{Ran}\,T$ dense in Y.
Why can one define T^{-1} on the whole Y?
[MANUEL]

Answer: it's an application of the Bounded Extension Principle:
$T^{-1}: \mathrm{Ran}\,T \to X$ is bdd and $\overline{\mathrm{Ran}\,T} = X$, whence T^{-1} can be uniquely extended to $Y \to X$ with the same norm.

Explicitly: let $y \in Y$, then $y \overset{\|\,\|}{\longleftarrow} y_n = Tx_n \in \mathrm{Ran}\,T$. $\{x_n\}_n$ is Cauchy in X because $\{y_n\}_n$ is Cauchy in Y and T^{-1} is bdd $\mathrm{Ran}\,T \to Y$.
$(\|x_n - x_m\|_X \leq \|T^{-1}\|\,\|y_n - y_m\|_Y)$. Thus (completeness) $x_{n_k} \to x \in X$.
Such x is uniquely identified by y. Indeed if $y \longleftarrow \tilde{y}_n = T\tilde{x}_n$
then $\tilde{x}_{n_k} \to \tilde{x} \in X$ and $\|x - \tilde{x}\|_X \leq \|x - x_{n_k}\| + \|x_{n_k} - \tilde{x}_{n_h}\| +$
$+ \|\tilde{x}_{n_h} - \tilde{x}\| \leq \|x - x_{n_k}\| + \|T^{-1}\|\,\|y_{n_k} - \tilde{y}_{n_h}\| + \|\tilde{x}_{n_h} - \tilde{x}\| \longrightarrow 0$
as $k, h \to \infty$, whence $x = \tilde{x}$. Set $Sy := x$.

Standard check: S is linear.
S is bdd b/c $\|Sy\| = \|x\| = \lim_{k\to\infty} \|x_{n_k}\| = \lim_{k\to\infty} \|T^{-1}y_{n_k}\| \leq \|T^{-1}\| \lim_{k\to\infty} \|y_{n_k}\| \leq \|T^{-1}\|\,\|y\|$.
S agrees with T^{-1} on $\mathrm{Ran}\,T$ (obvious). $S(\underbrace{Tx}_{y \in \mathrm{Ran}\,T}) = x$ $\forall x \in X$ is obvious.
$T(Sy) = Tx = T(\lim x_{n_k}) = \lim Tx_{n_k} = \lim TT^{-1}y_{n_k} = \lim y_{n_k} = y$.
Thus S inverts T on the whole Y.

(2) In class $T \in B(X,Y)$ was defined to be <u>compact</u> if $\overline{T\mathring{B}_1}$ is compact in Y $(\mathring{B}_1 = \{x \in X, \|x\| < 1\})$ whereas Alessandro uses $\overline{TB_1}$ $(B_1 = \{x \in X, \|x\| \leq 1\})$.

Is this the same?

[MANUEL, MARIUS]

<u>Answer</u> : sure!

Assume $\overline{TB_1}$ compact in Y. Since $\overline{T\mathring{B}_1} \subset \overline{TB_1}$ obviously and thus $\overline{T\mathring{B}_1}$ is a closed subset of a compact, then $\overline{T\mathring{B}_1}$ too is compact.

Vice versa: assume $\overline{T\mathring{B}_1}$ compact.

Take $y \in \overline{TB_1}$, ie $y \longleftarrow Tx_n$, $\|x_n\| \leq 1$.

Set $\tilde{x}_n := \frac{n-1}{n} x_n$. $\tilde{x}_n \in \mathring{B}_1$ b/c $\|\tilde{x}_n\| = \frac{n-1}{n}\|x_n\| < 1$.

Then
$$\|y - T\tilde{x}_n\| \leq \|y - Tx_n\| + \|Tx_n - T\tilde{x}_n\|$$
$$\leq \|y - Tx_n\| + \frac{1}{n}\underbrace{\|x_n\|}_{\leq 1} \xrightarrow{n \to \infty} 0$$

whence $y \in \overline{T\mathring{B}_1}$.

Conclusion : $\overline{T\mathring{B}_1} = \overline{TB_1}$.

(3) Examples of projections on a Hilbert space that are not orthogonal? [MARIUS]

Answer Enjoy first Problem in class no. 11 (part (iv)).

Otherwise, consider e.g. $P = \begin{pmatrix} 0 & 0 \\ a & 1 \end{pmatrix}$: $P^2 = \begin{pmatrix} 0 & 0 \\ a & 1 \end{pmatrix} = P$

but $P \neq P^*$.

(4) The spectrum of V^*V $(V : L^2[0,1] \to L^2[0,1]$, Volterra$)$ looks like the spectrum of the Laplacian in a box. Is there any "physical" link between Volterra and Schrödinger? [MARIUS]

Answer $\sigma(V^*V) = \left\{ \dfrac{1}{\pi^2 (n+\frac{1}{2})^2} \right\}_{n=0}^{\infty} \cup \{0\}$ ($\to$ Exercise 11)

so it's rather reminiscent of $\dfrac{1}{\sigma(-\Delta)}$. Which is understandable,

because $V^*Vf = \lambda f \Rightarrow -\Delta f = \frac{1}{\lambda} f$ (in 1 dim).

In other words, informally speaking, V is the anti-derivative

$V^*V \sim (-\Delta)^{-1}$.

(5) Is there a Weyl criterion for the <u>continuous</u> spectrum? [MARIUS]

<u>answer</u> I recall that $\sigma_{cont}(T)$ has different (non-equivalent) definitions in the literature. Let's take here the def. given in class:

$$\sigma_{cont}(T) = \left\{ \lambda \in \mathbb{C} \;\middle|\; \begin{array}{l} (\lambda - T) \text{ is injective} \\ (\lambda - T) \text{ is not surjective} \\ \mathrm{Ran}(\lambda - T) \text{ dense in } X \end{array} \right\}.$$

Let $\lambda \in \sigma_{cont}(T)$.

Since $(\lambda - T)$ is injective, then it is invertible on its range:
$\exists\, (\lambda - T)^{-1} : \mathrm{Ran}(\lambda - T) \to X$. Necessarily $\|(\lambda - T)^{-1}\| = \infty$, because otherwise $(\lambda - T)^{-1}$ could be extended to a bdd operator on the whole X, contradicting the assumption that $(\lambda - T)$ is not surjective.

Therefore $\forall n \in \mathbb{N}\ \exists\, y_n = (\lambda - T)x_n \in \mathrm{Ran}(\lambda - T)$ such that

$$\frac{\|(\lambda - T)^{-1} y_n\|}{\|y_n\|} \geq n. \quad \text{This implies} \quad \frac{\|(\lambda - T)x_n\|}{\|x_n\|} \leq \frac{1}{n}.$$

Set $z_n := \dfrac{x_n}{\|x_n\|}$. Then $\{z_n\}_{n\in\mathbb{N}}$ is a <u>Weyl sequence</u> for T at λ because $\|z_n\| = 1$ and $\|Tz_n - \lambda z_n\| \longrightarrow 0$ as $n \to \infty$.