Functional Analysis II

Institute of Mathematics, LMU Munich – Winter Term 2011/2012
Prof. T. Ø. Sørensen Ph.D, A. Michelangeli Ph.D

PROBLEMS IN CLASS – WEEK 13

These additional problems are for your own preparation at home. They supplement examples and properties not discussed in class. Some of them will discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for the final exam. Further info at `www.math.lmu.de/~michel/WS11-12_FA2.html`.

Problem 49. (Momentum operator on $[0, 2\pi]$.)

Consider the operators A_0 and A on the Hilbert space $L^2[0, 2\pi]$ given by

$$\begin{aligned} A_0 f &= -\mathrm{i} f', \qquad \mathcal{D}(A_0) = \{ f \in C^1([0, 2\pi]) \,|\, f(0) = f(2\pi) = 0 \}, \\ A f &= -\mathrm{i} f', \qquad \mathcal{D}(A) = \{ f \in C^1([0, 2\pi]) \,|\, f(0) = f(2\pi) \}. \end{aligned}$$

(i) Show that both A_0 and A are symmetric and that $A_0 \subset A$.

(ii) Find A_0^*.

(iii) Find $\overline{A_0}$.

(iv) Find A^* and show that A is essentially self-adjoint.

(v) Show that A_0 has no eigenvalues.

(vi) Show that A admits an orthonormal basis of eigenvectors.

Problem 50. Let T be a densely defined operator on a Hilbert space $\mathcal{H}$. Show that if $\sigma(T) \subsetneq \mathbb{C}$ then T is necessarily closed.

Problem 51. (Yet another motivation for unboundedness: canonical commutation relations cannot be satisfied by bounded operators.)

Let P and Q be two densely defined operators on a Hilbert space $\mathcal{H}$ such that $\mathcal{D}(PQ) \cap \mathcal{D}(QP)$ is dense in $\mathcal{H}$ and on such a dense $QP - PQ = \mathrm{i}\mathbb{1}$. Show that either P or Q or both must be unbounded.

Problem 52. (Stone's theorem.)

Preliminary remark: if A is a densely defined self-adjoint operator on a Hilbert space $\mathcal{H}$ and $\forall t \in \mathbb{R}$ one defines $U(t) := e^{\mathrm{i}tA}$ with the functional calculus, then $\{U(t)\}_{t \in \mathbb{R}}$ is a unitary group.

The proof is the same for bounded or unbounded A, see Exercise 44 (i). If A is bounded, such a group is differentiable in the *norm* operator topology, with $U'(t) = \mathrm{i}AU(t)$ (Exercise 44 (ii)), whereas if A is unbounded the same proof gives that $\{U(t)\}_{t\in\mathbb{R}}$ is a *strongly* continuous group and $U'(t) = \mathrm{i}AU(t)$ holds in the *strong* operator topology.

Assume now that $\{U(t)\}_{t\in\mathbb{R}}$ is a strongly continuous unitary group on a Hilbert space $\mathcal{H}$, i.e., each $U(t)$ is unitary, $U(t+s) = U(t)U(s)$ $\forall t, s \in \mathbb{R}$, and $\forall \psi \in \mathcal{H}$ $U(t)\psi \to U(t_0)\psi$ if $t \to t_0$.

(i) Let $\mathcal{D} \subset \mathcal{H}$ be the subspace of all finite linear combinations of vectors $\varphi_f \in \mathcal{H}$ of the form $\varphi_f = \int_{-\infty}^{+\infty} f(t)U(t)\varphi\,\mathrm{d}t$ for some $\varphi \in \mathcal{H}$ and some $f \in C_0^\infty(\mathbb{R})$, where the integral can be taken to be a Riemann integral since $U(t)$ is strongly continuous. Show that $\mathcal{D}$ is dense in $\mathcal{H}$.

(ii) For $\varphi_f \in \mathcal{D}$ define $A\varphi_f := -\mathrm{i}\,\varphi_{-f'}$. Show that A is symmetric.
(*Hint:* compute $\lim_{s\to 0} \frac{U(s)-\mathbb{1}}{s}$ on each φ_f.)

(iii) Show that both A and $U(t)$ leave $\mathcal{D}$ invariant and commute on $\mathcal{D}$.

(iv) Show that A is essentially self-adjoint.
(*Hint:* if u is a solution to $A^*u = \pm\mathrm{i}u$, consider the function $t \mapsto \langle U(t)\varphi, u\rangle$ $\forall \varphi \in \mathcal{D}$.)

(v) Show that $U(t) = e^{\mathrm{i}t\overline{A}}$.
(*Hint:* set $w(t) := U(t)\varphi - V(t)\varphi$, $\varphi \in \mathcal{D}$, where $V(t) := e^{\mathrm{i}t\overline{A}}$. Compute $\frac{d}{dt}\|w(t)\|^2$.)

This is the last sheet of problems in class. Congrats for having survived 104 homework exercises and problems in class!