Functional Analysis II

Institute of Mathematics, LMU Munich – Winter Term 2011/2012
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PROBLEMS IN CLASS – WEEK 12

These additional problems are for your own preparation at home. They supplement examples and properties not discussed in class. Some of them will discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for the final exam. Further info at `www.math.lmu.de/~michel/WS11-12_FA2.html`.

Problem 45. (Properties of the adjoint of a densely defined operator.)

Let A and B be two *densely defined* operators on a Hilbert space $\mathcal{H}$. Show the following.

(i) $(\alpha A)^* = \overline{\alpha} A^* \ \forall \alpha \in \mathbb{C}$.

(ii) If $\mathcal{D}(A) \cap \mathcal{D}(B)$ and $\mathcal{D}(A^*) \cap \mathcal{D}(B^*)$ are dense in $\mathcal{H}$, then $\mathcal{D}(A+B) = \mathcal{D}(A) \cap \mathcal{D}(B)$, $\mathcal{D}(A^*+B^*) = \mathcal{D}(A^*) \cap \mathcal{D}(B^*)$, and $(A+B)^* \supset A^* + B^*$.

(iii) If $\mathcal{D}(AB)$ is dense, then $(AB)^* \supset B^* A^*$.

(iv) If $A \subset B$ then $A^* \supset B^*$.

(v) If A is self-adjoint, A has no symmetric extension.

(vi) $\operatorname{Ker} A^* = (\operatorname{Ran} A)^\perp$. (Compare with the bounded case: Problem 25 (i).)

Problem 46. Let A be a densely defined operator on a Hilbert space $\mathcal{H}$. Show the following.

(i) If A is injective and $\operatorname{Ran} A$ is dense in $\mathcal{H}$ then $(A^{-1})^* = (A^*)^{-1}$.

(ii) If A is closable and $\overline{A}$ is injective, then $\overline{A^{-1}} = \overline{A}^{\,-1}$.

(iii) If A is self-adjoint and injective, then A^{-1} is self-adjoint too.

Problem 47. (The domain of the adjoint can be quite small.)

On the Hilbert space $L^2(\mathbb{R})$ consider the defined operator T with domain $\mathcal{D}(T) = C_0^\infty(\mathbb{R})$ and action $(Tf)(x) = \left(\int_{-\infty}^{+\infty} \frac{f(t)}{\sqrt{1+|t|}} \, \mathrm{d}t \right) e^{-x^2}$. Find T^* (i.e., give its domain and action).

Problem 48. (von Neumann's theorem. Standard Schrödinger operator.)

(i) Let A be a symmetric operator on a Hilbert space $\mathcal{H}$. Assume that there exists a map $C : \mathcal{H} \to \mathcal{H}$ with the following properties:

1. C is anti-linear (i.e., $C(\alpha x + \beta y) = \overline{\alpha}\, Cx + \overline{\beta}\, Cy \ \forall x, y \in \mathcal{H}$, $\forall \alpha, \beta \in \mathbb{C}$),
2. C is norm-preserving (i.e., $\|Cx\| = \|x\| \ \forall x \in \mathcal{H}$),
3. $C^2 = \mathbb{1}$,
4. the domain of A is invariant under C (i.e., $C\mathcal{D}(A) \subset \mathcal{D}(A)$),
5. $AC = CA$ on $\mathcal{D}(A)$.

(Nomenclature: a map C with the properties 1., 2., 3. is called a CONJUGATION. Thus, the assumption reads: there is a conjugation C on $\mathcal{H}$ that leaves $\mathcal{D}(A)$ invariant and commutes with A.)

Show that A has self-adjoint extensions.

(*Hint:* check that C is an isometry between $\mathrm{Ker}(A + \mathrm{i}\mathbb{1})$ and $\mathrm{Ker}(A - \mathrm{i}\mathbb{1})$.)

(ii) Consider the operator H on $L^2(\mathbb{R}^d)$ whose domain and action are

$$\mathcal{D}(H) = C_0^\infty(\mathbb{R}^d)$$
$$(H\psi)(x) = -(\Delta\psi)(x) + V(x)\psi(x) \quad \text{for a.e. } x \in \mathbb{R}^d$$

where $\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$ and V is a given real-valued function in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$. Show that H is symmetric and has at least one self-adjoint extension.

(*Hint:* apply (i).)