Functional Analysis II

Institute of Mathematics, LMU Munich – Winter Term 2011/2012
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PROBLEM IN CLASS – WEEK 11

These additional problems are for your own preparation at home. They supplement examples and properties not discussed in class. Some of them will discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for the final exam. Further info at `www.math.lmu.de/~michel/WS11-12_FA2.html`.

Problem 41. (Fuglede's theorem.)

Show that if S and T are operators in $\mathcal{B}(\mathcal{H})$ (where $\mathcal{H}$ is a Hilbert space) and T is normal, then $ST = TS$ implies $ST^* = T^*S$.

(*Hint:* using the assumption and the functional calculus for normal operators ($\rightarrow$ Problem 37 (ii)) show that $e^{-\lambda T^*} S\, e^{\lambda T^*}$ is bounded uniformly in $\lambda \in \mathbb{C}$, then apply Liouville's theorem to its matrix elements.)

Problem 42. Show that a $N \times N$ hermitian matrix has cyclic vectors if and only if all its eigenvalues are distinct.

Problem 43. Consider the following statement for an operator $A = A^* \in \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space: $\mathbb{O} \leqslant A \leqslant \mathbb{1}$ if and only if $A^2 \leqslant A$.

(i) Prove the statement without the spectral theorem (only with Hilbert space techniques).

(ii) Prove the statement using the spectral theorem.

Problem 44. (Riesz projection.)

Let A be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$ and Λ be a non-empty compact subset of $\sigma(A)$. Consider in the complex plane a closed, piecewise smooth, positively oriented curve Γ such that the intersection between $\sigma(A)$ and the region enclosed by Γ is Λ, and also $\Gamma \cap \sigma(A) = \emptyset$. (Hence Λ is separated by a gap from the rest of the spectrum of A.) Show that

$$\chi_{\Lambda}(A) \;=\; \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} \frac{1}{z-A}\, \mathrm{d}z$$

where χ_{Λ} is the characteristic function of Λ.