PROBLEM IN CLASS – WEEK 10

These additional problems are for your own preparation at home. They supplement examples and properties not discussed in class. Some of them will discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for the final exam. Further info at `www.math.lmu.de/~michel/WS11-12_FA2.html`.

Problem 37. (Measurable functional calculus for commuting self-adjoint operators. Measurable functional calculus for *normal* operators.)

(i) Let $A_1, \dots, A_n$ be pairwise commuting, bounded, self-adjoint operators on a Hilbert space $\mathcal{H}$. Set $\Sigma := \sigma(A_1) \times \cdots \times \sigma(A_n) \subset \mathbb{R}^n$. Show that there exists a unique $*$-homomorphism $\Psi : \mathcal{M}_B(\Sigma) \to \mathcal{B}(\mathcal{H})$ ($\mathcal{M}_B(\Sigma)$ is the space of bounded, Borel-measurable functions on Σ) such that

1. $\mathbf{1} \xmapsto{\Psi} \mathbb{1}$,
2. $\pi_i \xmapsto{\Psi} A_i$, $i = 1, \dots, n$, where π_i is the function $\mathbb{R}^n \ni (x_1, \dots, x_n) \xmapsto{\pi_i} x_i$,
3. if $f_k, f \in \mathcal{M}_B(\Sigma)$, $f_k(x) \xrightarrow{k\to\infty} f(x)$ $\forall x \in \Sigma$, and $\|f_k\|_\infty \leqslant C < \infty$, then $\Psi(f_k) \xrightarrow{k\to\infty} \Psi(f)$ in the strong operator topology,

and show that Ψ has the additional properties

4. $\Psi(\overline{f}) = \Psi(f)^*$,
5. $\|\Psi(f)\| \leqslant \|f\|_\infty$,
6. $\Psi(f)B = B\Psi(f)$ for any $B \in \mathcal{B}(\mathcal{H})$ commuting with $A_1, \dots, A_n$.

(*Hint:* define Ψ on step functions $f(x) = \sum c_{k_1,\dots,k_n} \mathbf{1}_{E_1}(x_1) \cdots \mathbf{1}_{E_n}(x_n)$ (with $E_j \in \sigma(A_j)$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$) by $\Psi(f) := \sum c_{k_1,\dots,k_n} \mathbf{1}_{E_1}(A_1) \cdots \mathbf{1}_{E_n}(A_n)$ where $\mathbf{1}_{E_j}(A_j)$ is given by the functional calculus for A_j.)

(ii) Let $\mathcal{H}$ be a Hilbert space and let $A \in \mathcal{B}(\mathcal{H})$ be a normal operator. Show that there exists a unique $*$-homomorphism $\Psi : \mathcal{M}_B(\sigma(A)) \to \mathcal{B}(\mathcal{H})$, $f \mapsto \Psi(f) \equiv f(A)$, such that

1. $\mathbf{1} \xmapsto{\Psi} \mathbb{1}$,
2. $f(A) = A$ on the function $f(z) = z$, $z \in \sigma(A)$,
3. $\overline{f}(A) = f(A)^*$,
4. if $f_k, f \in \mathcal{M}_B(\sigma(A))$, $f_k(x) \xrightarrow{k\to\infty} f(x)$ $\forall x \in \sigma(A)$, and $\|f_k\|_\infty \leqslant C < \infty$, then $f_k(A) \xrightarrow{k\to\infty} f(A)$ in the strong operator topology,

and show that Ψ has the additional properties

5. $\|f(A)\| \leqslant \|f\|_\infty$,
6. $f(A)B = Bf(A)$ for any $B \in \mathcal{B}(\mathcal{H})$ commuting with A and A^*.

(*Hint:* use part (i) for the real and the imaginary part of A, see Problem 20.)

Problem 38. (Obstruction to a functional calculus for generic operators.)

Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{B}(\mathcal{H})$ (not necessarily normal or self-adjoint). Let $f \in C(\sigma(T))$, $f(x) = \overline{x}x = |x|^2$ and define $f(T) := T^*T$. Show that a spectral mapping theorem in this case, i.e., $\sigma(f(T)) = f(\sigma(T))$, fails to hold for generic T.

Problem 39.

(i) Let $\{E_\Omega\}_{\Omega\in\Sigma_B(\mathbb{R})}$ be a projection-valued measure on a Hilbert space ($\Sigma_B(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$). Show that the three properties that define $\{E_\Omega\}_{\Omega\in\Sigma_B(\mathbb{R})}$, i.e.,

1. each E_Ω is an orthogonal projection on $\mathcal{H}$,
2. $E_\emptyset = \mathbb{O}$, $E_\mathbb{R} = \mathbb{1}$,
3. $E_{\bigcup_{n=1}^\infty \Omega_n} = \sum_{n=1}^\infty E_{\Omega_n}$ strongly, for pairwise disjoint $A_1, A_2, \dots \in \Sigma_B(\mathbb{R})$,

imply

4. $E_{\Omega_1}E_{\Omega_2} = E_{\Omega_2}E_{\Omega_1} = E_{\Omega_1\cap\Omega_2}$ $\forall \Omega_1, \Omega_2 \in \Sigma_B(\mathbb{R})$.

(Note that this completes Definition 2.13 stated in class.)

Problem 40. Recall that one defines the integral of a bounded Borel function f on $\mathbb{R}$ with respect to a projection-valued measure $\{E_\Omega\}_{\Omega\in\Sigma_B(\mathbb{R})}$ on a Hilbert space by $\int_\mathbb{R} f\mathrm{d}E = \lim_{n\to\infty}\int_\mathbb{R} f_n\mathrm{d}E$ where $\{f_n\}_{n=1}^\infty$ is a sequence of step functions that approximates f uniformly ($\|f - f_n\|_\infty \to 0$). Show that such a limit is unique, i.e., independent of the approximating sequence of step functions.

(Note that this completes Step 3 in the proof of Theorem 2.15 discussed in class.)