Functional Analysis II

Institute of Mathematics, LMU Munich – Winter Term 2011/2012
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PROBLEM IN CLASS – WEEK 9

These additional problems are for your own preparation at home. They supplement examples and properties not discussed in class. Some of them will discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for the final exam. Further info at `www.math.lmu.de/~michel/WS11-12_FA2.html`.

Problem 33. {The spectral radius is semi-continuous but not norm continuous. To add.}

Problem 34. {The spectrum is norm continuous for normal operators. To add.}

Problem 35. (Position operator on $[0, 1]$)

Consider the linear map A on $L^2[0, 1]$ defined $\forall f \in L^2[0, 1]$ by $(Af)(x) := xf(x)$ a.e. in $[0, 1]$.

(i) Show that $\|A\| = 1$.

(ii) Show that $A = A^*$.

(iii) Show that A has no eigenvalues (i.e., $\sigma_{\mathrm{p}}(A) = \emptyset$).

(iv) Show that A has spectrum $\sigma(A) = [0, 1]$.

(Note that this completes the first part of the proof of Example 2.19(c) stated in class.)

Problem 36. (Multiplication operator on a finite measure space.)

Let (X, μ) be a measure space with $\mu(X) < \infty$. (Here μ is a positive measure. For concreteness, $X \subset \mathbb{R}^d$ with the Borel σ-algebra and μ is a positive, finite Borel measure over X.) Let $\phi : X \to \mathbb{C}$ be a bounded measurable function. Consider the linear map M_ϕ on $L^2(X, \mathrm{d}\mu)$ defined $\forall f \in L^2(X, \mathrm{d}\mu)$ by $(M_\phi f)(x) := \phi(x)f(x)$ for a.e. $x \in X$.

(i) Show that $\|M_\phi\| = \|\phi\|_\infty$.

(ii) Show that $M_\phi^* = M_{\overline{\phi}}$ (in particular, M_ϕ is self-adjoint $\Leftrightarrow$ ϕ is real-valued).

(iii) Show that $\sigma(M_\phi) = \operatorname{ess\,ran} \phi := \big\{\lambda \in \mathbb{C} \,|\, \forall \varepsilon > 0 \ \mu(\{x \in X \,|\, |\lambda - \phi(x)| < \varepsilon\}) > 0\big\}$, the "essential range" of ϕ.

(iv) Show that λ is an eigenvalue of M_ϕ $\Leftrightarrow$ $\mu(\{\phi^{-1}(\lambda)\}) > 0$.