PROBLEM IN CLASS – WEEK 8

These additional problems are for your own preparation at home. They supplement examples and properties not discussed in class. Some of them will discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for the final exam. Further info at `www.math.lmu.de/~michel/WS11-12_FA2.html`.

Problem 29. Let $\mathcal{H}$ be a Hilbert space and $A = A^* \in \mathcal{B}(\mathcal{H})$.

(i) Show that $A \leqslant \|A\|\,\mathbb{1}$

(ii) Show that if $A \geqslant \mathbb{O}$ then $\sigma(A) \subset [\,0, \|A\|\,]$.

(iii) Show that if $\sigma(A) \subset [0, R]$ for some $R > 0$ then $\mathbb{O} \leqslant A \leqslant R\,\mathbb{1}$.

Problem 30. Let $\mathcal{H}$ be a Hilbert space. Show that for an operator $T \in \mathcal{B}(\mathcal{H})$ the following conditions are *equivalent*:

(a) T is invertible.

(b) T^* is invertible.

(c) Both T and T^* are bounded away from zero ($\|Tx\| \geqslant c\|x\|$ $\forall x \in \mathcal{H}$ and for some $c > 0$).

(d) Both T and T^* are injective and $\operatorname{Ran} T$ is closed.

Problem 31. Let $\mathcal{H}$ be a Hilbert space.

(i) Show that if $T \in \mathcal{B}(\mathcal{H})$ is invertible, then the partial isometry V_T in its polar decomposition $T = V_T|T|$ is a unitary operator.

(ii) Let $S \in \mathcal{B}(\mathcal{H})$ with $\|S\| < 1$. Show that for every unitary operator U on $\mathcal{H}$ there exist unitaries U_1 and V_1 such that $S + U = U_1 + V_1$.

Problem 32. (The Russo-Dye-Gardner theorem.)

Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$.

(i) Show that if $\|T\| < 1$, say $\|T\| < 1 - \frac{2}{n}$ for some integer $n \geqslant 3$, then there are unitary operators $U_1, \dots, U_n$ on $\mathcal{H}$ such that $T = \frac{1}{n}(U_1 + U_2 + \cdots + U_n)$. (*Hint:* Problem 31 (ii).)

(ii) Produce a counterexample to the conclusion in (i) when $\|T\| = 1$.