Functional Analysis II

Institute of Mathematics, LMU Munich – Winter Term 2011/2012
Prof. T. Ø. Sørensen Ph.D, A. Michelangeli Ph.D

PROBLEM IN CLASS – WEEK 7

These additional problems are for your own preparation at home. They supplement examples and properties not discussed in class. Some of them will discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for the final exam. Further info at `www.math.lmu.de/~michel/WS11-12_FA2.html`.

Problem 25. Let $\mathcal{H}$ be a Hilbert space.

(i) Show that $\operatorname{Ker} T^* = (\operatorname{Ran} T)^\perp$ for every $T \in \mathcal{B}(\mathcal{H})$.

(ii) Show that $\mathcal{H} = \overline{\operatorname{Ran} T} \oplus \operatorname{Ker} T^*$ for every $T \in \mathcal{B}(\mathcal{H})$.

(iii) Show that if $N \in \mathcal{B}(\mathcal{H})$ is normal then $\operatorname{Ker} N = \operatorname{Ker} N^* = (\operatorname{Ran} N)^\perp = (\operatorname{Ran} N^*)^\perp$.

Problem 26. Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$. Show that

$$\frac{1}{2}\,\|T\| \;\leqslant\; \sup_{\substack{x\in\mathcal{H}\\ \|x\|=1}} |\langle x, Tx\rangle| \;\leqslant\; \|T\|\,.$$

(Note that in Exercise 28 you proved that the second inequality becomes an identity when T is self-adjoint.)

Problem 27. Let $\mathcal{H}$ be a Hilbert space.

(i) Assume that $A = A^* \in \mathcal{B}(\mathcal{H})$ with $\|A\| \leqslant 1$. Show that the operator $U := A + \mathrm{i}\sqrt{\mathbb{1} - A^2}$ is unitary and $A = \frac{1}{2}(U + U^*)$.

(ii) Show that any $T \in \mathcal{B}(\mathcal{H})$ can be decomposed as $T = a_1U_1 + a_2U_2 + a_3U_3 + a_4U_4$ where each U_j is a unitary operator and the coefficients $a_j \in \mathbb{C}$ satisfy $|a_j| \leqslant \|T\|/2$. (*Hint*: Problem 20 and part (i) above.)

Problem 28. Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{B}(\mathcal{H})$.

(i) Show that if $\lambda \in \sigma(T)$ then there exists a sequence of unit vectors $\{x_n\}_{n=1}^\infty$ such that either $\|Tx_n - \lambda x_n\| \to 0$ or $\|T^*x_n - \overline{\lambda}x_n\| \to 0$ as $n \to \infty$.

(Note that this reverts the conclusion of Exercise 18 (i).)

(ii) Show that if T is normal then $\lambda \in \sigma(T)$ if and only if there exists a sequence of unit vectors $\{x_n\}_{n=1}^\infty$ such that $\|Tx_n - \lambda x_n\| \to 0$ as $n \to \infty$. (*Hint*: Exercise 18 (i) for one direction, part (i) above for the other direction.)

(Note that this generalises the conclusion of Exercise 18 (iii).)

(iii) Show that if T is self-adjoint (in fact, also if T is normal, in view of Problem 37 (ii)) and λ is an isolated point in $\sigma(T)$ then λ is an eigenvalue of T. (*Hint*: continuous functional calculus.)