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PROBLEM IN CLASS – WEEK 5

These additional problems are for your own preparation at home. They supplement examples and properties not discussed in class. Some of them will discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for the final exam. Further info at `www.math.lmu.de/~michel/WS11-12_FA2.html`.

Problem 17. Find all the solutions (λ, f), with $\lambda \in \mathbb{C}$ and $f \in L^2[0,1]$, to the integral equation

$$\int_0^1 f(y)\,\cos 2\pi(x-y)\,\mathrm{d}y \;=\; \lambda\, f(x)$$

considered as an identity in $L^2[0,1]$. (*Hint:* rewrite the equation in the form $(\lambda - T)f = 0$ for a suitable compact operator T and use the spectral decomposition of T.)

Problem 18. Recall the spectral radius formula

$$r(T) \;:=\; \sup_{\lambda\in\sigma(T)} |\lambda| \;=\; \lim_{n\to\infty} \|T^n\|^{1/n}$$

valid for any linear bounded operator T on a Banach space.

(i) Show that if a numerical sequence $\{a_n\}_{n=1}^\infty$ is such such that $a_n \in \mathbb{R}$ and $a_{n+m} \leqslant a_n + a_m$ for all n and m, then the limit $\displaystyle\lim_{n\to\infty} \frac{a_n}{n}$ exists and it is equal to $\displaystyle\inf_{n\in\mathbb{N}} \frac{a_n}{n}$.

(ii) Deduce that

$$\lim_{n\to\infty} \|T^n\|^{1/n} \;=\; \inf_{n\in\mathbb{N}} \|T^n\|^{1/n}\,.$$

Note: the spectral radius formula was proved by Gelfand in 1941 (I. M. Gelfand, *Normierte Ringe*, Mat. Sbornik **9** (1941), 3-24). Curiously, the identity $\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\in\mathbb{N}} \frac{a_n}{n}$, which can be derived from Problem 98 in G. Pólya, G. Szegő, *Aufgaben und Lehrsätze aus der Analysis I*, Springer, Berlin, 1925, p. 17, was re-discovered in the framework of the operator spectral theory only around 1952 (see F. Riesz, B. Sz. Nagy, *Leçons d'analyse fonctionnnelle*, Akad. Kiadó, Budapest, 1952, pp. 420-421, and A. F. Ruston, *Fredholm Theory in Banach Spaces*, Cambridge Univ. Press, 1986, p. 222).

Problem 19.

(i) Give an example of a bounded operator T on a Hilbert space $\mathcal{H}$ for which the spectral radius $r(T)$ is strictly less than $\|T\|$.

(ii) Let $R > 0$. Give an example of a 2×2 matrix with spectrum $\{0\}$ and norm greater than or equal to R (considering the matrix as an operator on $\mathbb{C}^2$ as usual).

Problem 20. Let $\mathcal{H}$ be a Hilbert space.

(i) Show that any $A \in \mathcal{B}(\mathcal{H})$ can be decomposed in a unique way as $A = R_A + \mathrm{i}\, I_A$ where $R_A = R_A^*$ and $I_A = I_A^*$ in $\mathcal{B}(\mathcal{H})$ (the "real" and "imaginary" part of A).

(ii) Show that A is normal if and only if $[R_A, I_A] = \mathbb{O}$.

(iii) Show that A is unitary if and only if A is normal and $R_A^2 + I_A^2 = \mathbb{1}$.