Institute of Mathematics, LMU Munich – Winter Term 2011/2012
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PROBLEM IN CLASS – WEEK 3

These additional problems are for your own preparation at home. They supplement examples and properties not discussed in class. Some of them will discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for the final exam. Further info at `www.math.lmu.de/~michel/WS11-12_FA2.html`.

Problem 9. Let $k \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ for some positive integer d. Use the notation $k(x,y)$ with $x, y \in \mathbb{R}^d$. Consider on $L^2(\mathbb{R}^d)$ the map $f \mapsto Tf$ defined by

$$(Tf)(x) := \int_{\mathbb{R}^d} k(x,y) f(y) \mathrm{d}y \qquad \text{for a.e. } x \in \mathbb{R}^d \tag{$*$}$$

(i) Show that $(*)$ defines a bounded operator $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ with operator norm at most $\|k\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}$.

(ii) Produce a sequence $\{T_n\}_{n=1}^\infty$ of finite rank operators in $\mathcal{B}(L^2(\mathbb{R}^d))$ such that $T_n \xrightarrow{n\to\infty} T$ in the operator norm, thus concluding that T is a compact operator.

(iii) Let $\{\varphi_n\}_{n=1}^\infty$ be an orthonormal basis of $L^2(\mathbb{R}^d)$. Show that

$$\sum_{n=1}^{\infty} \|Tf_n\|^2_{L^2(\mathbb{R}^d)} \;=\; \|k\|^2_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}$$

irrespectively of the choice of the orthonormal basis $\{\varphi_n\}_{n=1}^\infty$.

Problem 10. Let X, Y be Banach spaces and $T \in \mathcal{B}(X, Y)$.

(i) Show that T is invertible if and only if $\mathrm{Ran}T$ is dense in Y and T is "bounded below" in the sense that $\exists\, \varepsilon > 0$ such that $\|Tx\|_Y \geqslant \varepsilon \|x\|_X$ for all $x \in X$.

(Note: choosing Y to be just a normed space the same conclusion follows.)

(ii) Show that if T is invertible then $\sigma(T^{-1}) = \dfrac{1}{\sigma(T)} := \left\{ \dfrac{1}{\lambda} \in \mathbb{C} \,|\, \lambda \in \sigma(T) \right\}$.

(iii) Show that if $Y = X$, if T is invertible, and if $Tx = \lambda x$ for some $\lambda \neq 0$, then $T^{-1}x = \lambda^{-1}x$.

Problem 11. (Projections on a Banach space. Compare with Exercise 6, where orthogonal projections on a Hilbert space were discussed.)
Let X be a vector space. In this exercise $\oplus$ is the direct sum in the algebraic sense. (Note: in the Hilbert space case $\oplus$ is the orthogonal sum.) Recall that a projection on X (and *onto* $\mathrm{Ran}P$) is a linear map $P : X \to X$ such that $P^2 = P$.

(i) Show that if P is a projection on X then

$$X = \mathrm{Ker}P \oplus \mathrm{Ran}P\,.$$

(ii) Show that if P is a projection on X then $\mathbb{1}-P$ is a projection too with

$$\mathrm{Ker}(\mathbb{1}-P) = \mathrm{Ran}P\,, \qquad \mathrm{Ran}(\mathbb{1}-P) = \mathrm{Ker}P\,.$$

(iii) Show that for every subspace $X_0 \subset X$ there exists a projection onto X_0. (*Hint:* Zorn.)

Assume now in addition that X is a normed vector space.

(iv) Produce an example of a normed space X and a linear map $P : X \to X$ such that $P = P^2$ but P is not continuous.

(v) Show that if P is a bounded linear projection on X then both $\mathrm{Ker}P$ and $\mathrm{Ran}P$ are closed subspaces and either $P = \mathbb{O}$ or $\|P\| \geqslant 1$.

Last, let X be a Banach space. Note that given a subspace $X_0 \subset X$, the projection onto X_0 that exists by (iii) need not be continuous.

(vi) Show that if $X = X_0 \oplus X_1$ for some *closed* subspaces X_0, X_1 of X then there exists a bounded projection $P : X \to X$ such that $\mathrm{Ker}P = X_0$, $\mathrm{Ran}P = X_1$.

(vii) Assume that $X = X_0 \oplus X_1$ for some subspaces X_0, X_1 of X such that X_0 is closed and $\dim X_1 < \infty$. Pick another subspace of X, say $\widetilde{X}_1$, such that $X_0 \cap \widetilde{X}_1 = \{0\}$. Show that $\dim \widetilde{X}_1 \leqslant \dim X_1$, and that $\dim \widetilde{X}_1 = \dim X_1$ when $X = X_0 \oplus \widetilde{X}_1$.

(Note that (vii) proves Remark 1.2 stated in class, that is: if X_0 is a closed subspace of a Banach space X with $\mathrm{codim}X_0 < \infty$, i.e., if $X = X_0 \oplus X_1$ and $\mathrm{codim}X_0 = \dim X_1 < \infty$, then the codimension of X_0 does not depend on the choice of the complement subspace X_1.)

Problem 12. (Spectrum of self-adjoint operators.) Let $\mathcal{H}$ be a Hilbert space and let $A = A^*$ be a bounded, self-adjoint operator on $\mathcal{H}$. Show the following spectral properties of A.

(i) $\sigma(A) \subset \left[\inf_{\substack{x\in\mathcal{H} \\ \|x\|=1}} \langle x, Ax\rangle \,,\, \sup_{\substack{x\in\mathcal{H} \\ \|x\|=1}} \langle x, Ax\rangle \right] \subset \mathbb{R}$.

(ii) $\sigma_{\mathrm{r}}(A) = \emptyset$.

(iii) If $Ax = \lambda x$ and $Ay = \mu y$ with $\lambda \neq \mu$ then $\langle x, y\rangle = 0$.

(iv) If $\sigma(A) = \{0\}$ then $A = \mathbb{O}$.