Functional Analysis II

Institute of Mathematics, LMU Munich – Winter Term 2011/2012
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PROBLEM IN CLASS – WEEK 2

These additional problems are for your own preparation at home. They supplement examples and properties not discussed in class. Some of them will discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for the final exam. Further info at `www.math.lmu.de/~michel/WS11-12_FA2.html`.

Problem 5. Let X be a Banach space. Show that any operator $S : X \to X$ of the form $S = \mathbb{1} - T$, where T is compact, has closed range.
(*Hint:* show that you can factor out $\mathrm{Ker}S$ and prove $\|Sx\| \geqslant c\|x\|$ $(c > 0)$ on the complement.)

Problem 6. Notation: given $q, r \in (1, \infty)$ and d positive integer, by $L^q_t(\mathbb{R}, L^r(\mathbb{R}^d_x))$ we denote the Banach space of functions $f(t, x)$, $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, such that $f(t, \cdot) \in L^r(\mathbb{R}^d)$ for almost all $t \in \mathbb{R}$ and $t \mapsto \|f(t, \cdot)\|_{L^r(\mathbb{R}^d)}$ is in $L^q(\mathbb{R})$ – this is a customary notation in PDE theory. By q' we denote the Hölder dual index of q, that is, $\frac{1}{q} + \frac{1}{q'} = 1$.

Let $t \mapsto U(t)$, $t \in \mathbb{R}$, be a one-parameter group of unitary operators on $L^2(\mathbb{R}^d)$, that is, each $U(t) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is unitary and $U(0) = \mathbb{1}$, $U(t + s) = U(t)U(s)$ $\forall\, t, s$. Assume that U satisfies the following property:

$$\|U(\cdot)f\|_{L^q_t(\mathbb{R},L^r(\mathbb{R}^d_x))} \leqslant C\,\|f\|_{L^2(\mathbb{R}^d)} \qquad \forall f \in L^2(\mathbb{R}^d) \tag{$*$}$$

for some constant $C > 0$. (An important application is when U is the free Schrödinger propagator, $U(t) = e^{it\Delta}$. In this case $(*)$ goes under the name of *Strichartz inequality*, a fundamental tool for the proof of uniqueness of solutions to PDE.) Deduce from $(*)$ the "dual" inequality

$$\Big\| \int_{-\infty}^{+\infty} U(-t)F(t, \cdot)\,\mathrm{d}t \Big\|_{L^2(\mathbb{R}^d)} \leqslant C\,\|F\|_{L^{q'}_t(\mathbb{R},L^{r'}(\mathbb{R}^d_x))} \qquad \forall F \in L^{q'}_t(\mathbb{R}, L^{r'}(\mathbb{R}^d_x))\,, \tag{$**$}$$

that is, the function $x \mapsto (\int_{-\infty}^{+\infty} U(-t)F(t, \cdot)\,\mathrm{d}s)(x)$ is in $L^2(\mathbb{R}^d)$ and its L^2-norm satisfies the bound $(**)$.

(*Hint:* just a straightforward application of the definition of the Hilbert adjoint of $U(t)$ and of the Banach adjoint of $T : L^2(\mathbb{R}^d) \to L^q_t(\mathbb{R}, L^r(\mathbb{R}^d_x))$, $f \mapsto U(\cdot)f$.)

Problem 7. (Useful identities and inequalities involving resolvents.) Let X be a Banach space. As usual, given $T \in \mathcal{B}(X)$, $\rho(T)$ is the resolvent set for T and $R_\lambda(T)$, $\lambda \in \rho(T)$, is the resolvent of T at λ, that is, $R_\lambda(T) = (\lambda\mathbb{1} - T)^{-1}$. Prove the following:

(i) $R_\lambda(T) - R_\mu(T) = (\mu - \lambda) R_\lambda(T) R_\mu(T) \quad \forall \lambda, \mu \in \rho(T)$

(ii) $R_\lambda(T) - R_\lambda(S) = R_\lambda(T)(T - S) R_\lambda(S) \quad \forall \lambda \in \rho(T) \cap \rho(S)$

(iii) $R_\lambda(T) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n R_{\lambda_0}(T)^{n+1}$ for $|\lambda - \lambda_0| < \|R_{\lambda_0}(T)\|^{-1}$ and $\lambda, \lambda_0 \in \rho(T)$

(iv) $R_\lambda(T) = \sum_{n=0}^{\infty} \lambda^{-1-n} T^n$ for $|\lambda| > \|T\|$

(v) $\|R_\lambda(T)\| \geqslant (\mathrm{dist}(\lambda, \sigma(T)))^{-1}$ $\forall \lambda \in \rho(T)$

(vi) If X is a Hilbert space and $T = T^*$ then $\|R_\lambda(T)\| \leqslant |\mathfrak{Im}(\lambda)|^{-1}$ $\forall \lambda \in \mathbb{C} \setminus \mathbb{R}$

Problem 8. Consider the operator $T : \ell^1(\mathbb{N}) \to \ell^1(\mathbb{N})$, $T(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$.

(i) Show that T is bounded with $\|T\| = 1$.

(ii) Determine T' (domain and action).

(iii) Show that $\sigma(T) = \{\lambda \in \mathbb{C}, |\lambda| \leqslant 1\}$.

(iv) Show that $\sigma_{\mathrm{p}}(T) = \{\lambda \in \mathbb{C}, |\lambda| < 1\}$.

(v) Show that $\sigma_{\mathrm{p}}(T') = \emptyset$.

(vi) Show that $\sigma_{\mathrm{r}}(T) = \emptyset$.

(vii) Show that $\sigma(T') = \{\lambda \in \mathbb{C}, |\lambda| \leqslant 1\}$.

(viii) Show that $\sigma_{\mathrm{r}}(T') = \{\lambda \in \mathbb{C}, |\lambda| \leqslant 1\}$.