Functional Analysis II

Institute of Mathematics, LMU Munich – Winter Term 2011/2012
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PROBLEM IN CLASS – WEEK 1

These additional problems are for your own preparation at home. They supplement examples and properties not discussed in class. Some of them will discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for the final exam. Further info at `www.math.lmu.de/~michel/WS11-12_FA2.html`.

Problem 1. (Basic examples of compact operators.)

(i) Show that finite rank operators in $\mathcal{B}(X,Y)$ (X, Y normed spaces) are compact.

(ii) Show that the identity operator on an infinite dimensional Banach space is not a compact operator. (*Hint:* Show that Riesz lemma implies that the unit ball is not compact.)

(iii) Show that the bounded linear functionals on a Banach space X are all compact, i.e., $X' \subset \mathcal{K}(X,\mathbb{C})$.

(iv) Show that the operator $T : C[0,1] \to C[0,1]$, $(Tf)(x) := \int_0^1 K(x,y)f(y)\,\mathrm{d}y$, where $K : [0,1]\times[0,1] \to \mathbb{C}$ is continuous, is compact. (*Hint:* Ascoli-Arzelà.)

Problem 2.

(i) Show that the composition of a compact operator with a bounded operator is always compact.

(ii) Property (i) does not prevent the composition of two non-compact operators to be compact though. Give examples of this phenomenon, that is, construct a non-compact, bounded operator T such that T^2 is compact.

(iii) Let $T \in \mathcal{K}(X)$ and assume $\dim X = \infty$. Show that $0 \in \sigma(T)$. (*Hint:* use (i).)

(iv) Show that a compact operator maps weakly convergent sequences into norm convergent sequences. (*Hint:* Uniform Boundedness principle.)

(v) Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$ and let $T : \mathcal{H} \to V$ be a compact operator, where V is a normed vector space. Show that $\|Te_n\|_V \to 0$ as $n \to \infty$.

Problem 3. (In infinite dimensions, injectivity is "exceptional" for compacts.)
Let $T : X \to Y$ be a compact operator between Banach spaces and assume $\dim X = \infty$. Construct a *non-injective* compact operator arbitrarily close in norm to T (this proves that non-injective compact operators are dense in $\mathcal{K}(X,Y)$). (*Hint:* Exercise 5(ii) + Hahn-Banach.)

Problem 4. (Approximation of compacts with finite rank operators.)

(i) Let X, Y be Banach spaces and consider the subpace $\mathcal{K}(X,Y) \subset \mathcal{B}(X,Y)$ of compact operators from X to Y. Show that $\mathcal{K}(X,Y)$ is closed in $\mathcal{B}(X,Y)$, i.e., show that if $\{T_n\}_{n=1}^{\infty}$ is a sequence of compact operators in $\mathcal{K}(X,Y)$ such that $T_n \xrightarrow[n\to\infty]{\|\,\|} T \in \mathcal{B}(X,Y)$, then T is compact.

(ii) Let X, Y be Banach spaces. Prove that every limit (in the operator-norm topology) of finite rank operators from X to Y is compact.

(iii) Let $\mathcal{H}$ be a separable Hilbert space and let $T : \mathcal{H} \to \mathcal{H}$ be a compact operator on it. Prove that T is the norm limit of a sequence of finite rank operators, in the following explicit sense. If $\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$ and $P_N : \mathcal{H} \to \mathcal{H}$ (N positive integer) is the orthogonal projection onto the span of $\{\varphi_1, \dots, \varphi_N\}$, then $T \circ P_N \xrightarrow[N\to\infty]{\|\,\|} T$.

Note: property (iii) reverses (ii) in the case of separable Hilbert spaces. For long the question whether the same "**approximation property**" holds in general Banach spaces remained a major open problem and in 1936 Mazur (a former Ph.D. student of Banach) offered a live goose as the prize for solving it. It was only in 1972 that Enflo first solved it in the negative, constructing a separable Banach space X for which $\mathcal{K}(X)$ is not the closure of finite rank operators. A live goose was then indeed awarded by Mazur to Enflo in a ceremony that was broadcast throughout Poland...