LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

Winter Term 2011-2012

Functional Analysis II – Final Test, 4.02.2012

Funktionalanalysis II – Endklausur, 4.02.2012

Name:/*Name*: ______

Matriculation number:/*Matrikelnr.:* ______ **Semester:**/*Fachsemester:* ______

Degree course:/*Studiengang:* ❑ Bachelor PO 2007 ❑ Lehramt Gymnasium (modularisiert)
❑ Bachelor PO 2010 ❑ Lehramt Gymnasium (nicht modularisiert)
❑ Diplom ❑ Master ❑ TMP ❑ ______

Major:/*Hauptfach:* ❑ Mathematik ❑ Wirtschaftsm. ❑ Informatik ❑ Physik ❑ Statistik ❑ ______

Minor:/*Nebenfach:* ❑ Mathematik ❑ Wirtschaftsm. ❑ Informatik ❑ Physik ❑ Statistik ❑ ______

Credits needed for:/*Anrechnung der Credit Points für das:* ❑ Hauptfach ❑ Nebenfach (Bachelor/Master)

Extra solution sheets submitted:/*Zusätzlich abgegebene Lösungsblätter:* ❑ Yes ❑ No

problem	1	2	3	4	5	6	$\sum$
total points	10	10	10	10	10	10	60
scored points							

homework bonus		final test performance		total performance		FINAL MARK	

INSTRUCTIONS:

- This booklet is made of sixteen pages, including the cover, numbered from 1 to 16. The test consists of six problems. Each problem is worth the number of points specified in the table above. 50 points are counted as 100% performance in this test. You are free to attempt any problem and collect partial credits.
- The only material that you are allowed to use is black or blue pens/pencils and one hand-written, two-sided, A4-paper "cheat sheet" (Spickzettel). You cannot use your own paper: should you need more paper, raise your hand and you will be given extra sheets.
- Prove all your statements or refer to the standard material discussed in class.
- Work individually. Write with legible handwriting. You may hand in your solution in English or in German. Put your name on every sheet you hand in.
- You have 120 minutes.

GOOD LUCK!

Fill in the form here below only if you need the certificate (Schein).

Dieser Leistungsnachweis entspricht auch den Anforderungen
nach § Abs. Nr. Buchstabe LPO I
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UNIVERSITÄT MÜNCHEN

ZEUGNIS

Der / Die Studierende der __________
Herr / Frau __________ *aus* __________
geboren am _____ *in* __________ *hat im* **WiSe** _____ *–Halbjahr* **2011-2012**
meine Übungen **zur Funktionalanalysis II** __________

mit __________ *besucht.*
Er / Sie hat __________
schriftliche Arbeiten geliefert, die mit ihm / ihr besprochen wurden. __________

MÜNCHEN, *den* **4 Februar 2012**

Name

PROBLEM 1. (10 points)

Consider the bounded linear operator $T : L^2[0,1] \to L^2[0,1]$ defined by

$$(Tf)(x) := \int_0^1 \big[\, 4\,(\cos 2\pi(x-y))^3 - 3\cos 2\pi(x-y)\big]\, f(y)\, \mathrm{d}y \qquad \text{for a.e. } x \in [0,1]\,.$$

(i) Prove that T is compact and self-adjoint.

(ii) Write a spectral decomposition of T, i.e., produce an orthonormal system $\{e_n\}_n$ of $L^2[0,1]$ and a collection $\{\lambda_n\}_n$ of non-zero numbers such that $T = \sum_n \lambda_n \langle e_n, \cdot\rangle\, e_n$.

(iii) Argue for which $\lambda \in \mathbb{C}$ the equation $Tf = e^{2012x} + \lambda f$ (as an identity in $L^2[0,1]$) admits solutions $f \in L^2[0,1]$. Prove your statement.

SOLUTION:

SOLUTION TO PROBLEM 1 (CONTINUATION):

Name

PROBLEM 2. (10 points)

Let X be a Banach space and let $T : X \to X$ be a bounded linear map. Denote by T' the Banach adjoint of T and by $\sigma_{\rm p}(T)$ and $\sigma_{\rm r}(T)$ respectively the point spectrum and residual spectrum of T.

(i) Prove that $\lambda \in \sigma_{\rm r}(T) \Rightarrow \lambda \in \sigma_{\rm p}(T')$.

(ii) Prove that $\lambda \in \sigma_{\rm p}(T) \Rightarrow$ either $\lambda \in \sigma_{\rm p}(T')$ or $\lambda \in \sigma_{\rm r}(T')$.

SOLUTION:

SOLUTION TO PROBLEM 2 (CONTINUATION):

Name

PROBLEM 3. (10 points)

Let A be a bounded, self-adjoint linear operator on a Hilbert space $\mathcal{H}$. Assume that $[0,1] \subset \sigma(A)$ and that A admits a cyclic vector in $\mathcal{H}$. Denote by $\{E_\Omega\}_\Omega$ the projection-valued measure associated with A. Compute $\|E_\Omega A\|$ when

(i) $\Omega = [\frac{1}{4}, \frac{1}{2})$,

(ii) $\Omega = [\frac{1}{4}, \frac{1}{3}) \cup ((\frac{1}{3}, \frac{1}{2}] \cap \mathbb{Q})$.

SOLUTION:

SOLUTION TO PROBLEM 3 (CONTINUATION):

Name

PROBLEM 4 (10 points).

Let A be a bounded, self-adjoint linear operator on a Hilbert space $\mathcal{H}$. Denote by $\{E_\Omega\}_\Omega$ the projection-valued measure associated with A. Set $\Omega_n := \{\lambda \in \mathbb{R} : |\lambda| \geqslant \frac{1}{n}\}$, $n \in \mathbb{N}$, and assume that E_{Ω_n} has finite rank $\forall n \in \mathbb{N}$, that is, $\dim R(E_{\Omega_n}) < \infty$ $\forall n \in \mathbb{N}$. ($R(T)$ denotes the range of the operator T.) Prove that A is compact.

SOLUTION:

SOLUTION TO PROBLEM 4 (CONTINUATION):

Name

PROBLEM 5. (10 points)

On the Hilbert space $L^2[0,1]$ consider the operator A whose domain and action are

$$\begin{aligned}\mathcal{D}(A) &:= \{\, f \in C^2([0,1]) \,:\, f(0) = f(1)\,, f'(0) = f'(1) \,\}\,,\\ (Af)(x) &:= -f''(x) \qquad \forall x \in [0,1]\,.\end{aligned}$$

(i) Prove that A is symmetric.

(ii) Find an orthonormal basis of $L^2[0,1]$ consisting of eigenfunctions of A.

(iii) Prove that A is essentially self-adjoint and find $\sigma(\overline{A})$ (the spectrum of the closure of A).

SOLUTION:

SOLUTION TO PROBLEM 5 (CONTINUATION):

Name

PROBLEM 6. (10 points)

Let A be a densely defined (possibly unbounded), self-adjoint operator in a Hilbert space $\mathcal{H}$. Denote by $\{E_\Omega\}_\Omega$ the projection-valued measure associated with A.

Let $\psi_1, \dots, \psi_N$ be N linearly independent vectors in the domain of A and let $\mu \in \mathbb{R}$ be such that

$$\langle \psi, A\psi \rangle \ < \ \mu \|\psi\|^2$$

for any non-zero element $\psi \in \operatorname{span}\{\psi_1, \dots, \psi_N\}$.

Show that $\dim \mathrm{R}(E_{(-\infty,\mu]}) \geqslant N$. ($R(T)$ denotes the range of an operator T.)

SOLUTION:

SOLUTION TO PROBLEM 6 (CONTINUATION):