HOMEWORK ASSIGNMENT no. 13, issued on Wednesday 25 January 2012
Due: Wednesday 1 February 2012 by 2 pm in the designated "FA2" box on the 1st floor
Info: www.math.lmu.de/~michel/WS11-12_FA2.html

Each exercise sheet is worth a full mark of 40 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.

Exercise 49. On the Hilbert space $L^2[0,1]$ consider the densely defined operators A_D and A_N whose domain and action are

$$\begin{cases}\mathcal{D}(A_D) = \{\psi \in C^2([0,1]) \,|\, \psi(0) = \psi(1) = 0\} \\ (A_D\psi)(x) = -\psi''(x) \quad \text{for a.e. } x \in [0,1]\end{cases}, \quad \begin{cases}\mathcal{D}(A_N) = \{\psi \in C^2([0,1]) \,|\, \psi'(0) = \psi'(1) = 0\} \\ (A_N\psi)(x) = -\psi''(x) \quad \text{for a.e. } x \in [0,1]\,.\end{cases}$$

(i) Show that both A_D and A_N are symmetric.

(ii) Show that A_D is essentially self-adjoint and find $\sigma(\overline{A_D})$.

(iii) Show that A_N is essentially self-adjoint and find $\sigma(\overline{A_N})$.

(iv) Show that the operator $A_{D,N}$ on $L^2[0,1]$ with domain $\mathcal{D}(A_{D,N}) = \mathcal{D}(A_D) \cap \mathcal{D}(A_N)$ and action $(A_{D,N}f)(x) = -\psi''(x)$ for a.e. $x \in [0,1]$ is symmetric and has at least two distinct self-adjoint extensions.

Exercise 50. Consider the operator A_0 on the Hilbert space $L^2[0,2\pi]$ given by

$$A_0 f = -\mathrm{i}f'\,, \qquad \mathcal{D}(A_0) = \{f \in C^1([0,2\pi]) \,|\, f(0) = f(2\pi) = 0\}\,.$$

Recall from Problem 49 that A_0 is symmetric and admits non-trivial self-adjoint extensions (the latter follows also from von Neumann's theorem, Problem 48). Produce *all* self-adjoint extensions of A_0 (i.e., for each of them give domain and action).

Exercise 51. (Essential self-adjointness is *not* preserved in the strong operator limit.)

Let A be a symmetric, non essentially self-adjoint operator on a Hilbert space $\mathcal{H}$ such that A has a self-adjoint extension $\widetilde{A}$. (For example, the operator $A_{D,N}$ in Exercise 49 (iv), or the operator A_0 in Problem 49/Exercise 50.)

(i) Let P_n, $n \in \mathbb{N}$, be the spectral projection of $\widetilde{A}$ corresponding to the interval $[-n, n]$. Show that each $P_n \widetilde{A} P_n$ is a bounded, self-adjoint operator on $\mathcal{H}$.

(ii) Show that each $P_n \widetilde{A} P_n$ constructed in (i) is an essentially self-adjoint operator on $\mathcal{D}(A)$, the domain of A.

(iii) Show that $P_n \widetilde{A} P_n \varphi \xrightarrow{n \to \infty} A\varphi \ \forall \varphi \in \mathcal{D}(A)$.

Comment: thus, the strong limit of the essentially self-adjoint operators $P_n \widetilde{A} P_n$'s is not essentially self-adjoint.

Exercise 52. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of densely defined self-adjoint operators on a Hilbert space $\mathcal{H}$ and let A be another self-adjoint operator on $\mathcal{H}$. Assume that

$$\lim_{n \to \infty} \left\| e^{itA_n} \varphi - e^{itA} \varphi \right\| = 0 \qquad \forall \varphi \in \mathcal{H}, \qquad \forall t \in \mathbb{R}.$$

Show that

$$\lim_{n \to \infty} \left\| R_z(A_n)\varphi - R_z(A)\varphi \right\| = 0 \qquad \forall \varphi \in \mathcal{H}$$

where $R_z(A_n) = (z\mathbb{1} - A_n)^{-1}$, $R_z(A) = (z\mathbb{1} - A)^{-1}$ for an arbitrary $z \in \mathbb{C} \setminus \mathbb{R}$.
(*Hint:* represent the resolvent $R_z(A)$ with an integral involving e^{itA}, then use the spectral theorem.)

This is the last homework assignment. Congrats for having survived 104 homework exercises and problems in class!