Institute of Mathematics, LMU Munich – Winter Term 2011/2012
Prof. T. Ø. Sørensen Ph.D, A. Michelangeli Ph.D

HOMEWORK ASSIGNMENT no. 12, issued on Wednesday 18 January 2012
Due: Wednesday 25 January 2012 by 2 pm in the designated "FA2" box on the 1st floor
Info: `www.math.lmu.de/~michel/WS11-12_FA2.html`

Each exercise sheet is worth a full mark of 40 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.

Exercise 45. Let A be the integral operator on $L^2[0,1]$ defined by $(Af)(x) = \int_0^1 \min(x,y)f(y)\mathrm{d}y$ for a.e. $x \in [0,1]$.

(i) Prove that A is bounded and self-adjoint.

(ii) Reduce A to the form of a multiplication by a function, that is, produce a measure space $(\mathcal{M},\mu)$, an isomorphism $U : L^2[0,1] \to L^2(\mathcal{M},\mathrm{d}\mu)$, and a bounded measurable function $F:\mathcal{M} \to \mathbb{R}$ such that UAU^* acts on $L^2(\mathcal{M},\mathrm{d}\mu)$ as the operator of multiplication by F.

Exercise 46 (Cyclic vectors.)

Consider the self-adjoint operators A and B on $L^2[-1,1]$ where A is the multiplication by the function $x \mapsto x$ and B is the multiplication by the function $x \mapsto x^2$.

(i) Show that the function $f(x) = 1$ is a cyclic vector for A.

(ii) Show that the function $f(x) = \theta(x)$ (the Heaviside function) is not a cyclic vector for A.

(iii) Show that B does not have cyclic vectors.
(*Hint:* if f is cyclic, consider g given by $g(x) = \overline{f(-x)}\,\mathrm{sgn}\,x$ if the L^2-space is over $\mathbb{C}$, and $g(x) = f(-x)\,\mathrm{sgn}\,x$ if the L^2-space is over $\mathbb{R}$.)

(iv) Show that $L^2[-1,1] \cong \mathcal{H}_1 \oplus \mathcal{H}_2$ for two Hilbert subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ each of which has a cyclic vector for B.

(v) (*All* cyclic vectors for the position operator on $[-1,1]$.)
Show that $f \in L^2[-1,1]$ is a cyclic vector for A if and only if $f(x) \neq 0$ almost everywhere.

You are asked to answer question (i) and (ii) without using (v), but by direct check instead.

Exercise 47. (Unbounded multiplication operator. Unbounded position operator.)

Let X be a metric space and μ be a positive measure on the Borel σ-algebra of X such that $\mu(\Lambda) < \infty$ for any bounded Borel set $\Lambda \subset X$. Let $\phi : X \to \mathbb{C}$ be a (possibly unbounded) measurable function. Consider the linear map M_ϕ on $L^2(X, \mathrm{d}\mu)$ whose domain and action are defined by

$$\begin{aligned} \mathcal{D}(M_\phi) &:= \{f \in L^2(X, \mathrm{d}\mu) \,|\, \phi f \in L^2(X, \mathrm{d}\mu)\} \\ (M_\phi f)(x) &:= \phi(x) f(x) \quad \mu\text{-a.e.} \end{aligned}$$

(i) Show that $\mathcal{D}(M_\phi)$ is dense in $L^2(X, \mathrm{d}\mu)$.

(ii) Show that $M_\phi^* = M_{\overline{\phi}}$ (in particular, M_ϕ is self-adjoint $\Leftrightarrow$ ϕ is real-valued).

(iii) Show that $\sigma(M_\phi) = \operatorname{ess\,ran} \phi := \big\{\lambda \in \mathbb{C} \,|\, \forall \varepsilon > 0 \;\, \mu(\{x \in X \,|\, |\lambda - \phi(x)| < \varepsilon\}) > 0\big\}$, the "essential range" of ϕ.

(iv) Show that λ is an eigenvalue of M_ϕ $\Leftrightarrow$ $\mu(\{\phi^{-1}(\lambda)\}) > 0$.

Consider now the position operator q on $\mathbb{R}$, i.e., the operator M_ϕ on $L^2(\mathbb{R}, \mathrm{d}x)$ (i.e., with the Lebesgue measure $\mathrm{d}x$) defined as above with $\phi(x) = x$.

(v) Show that q is self-adjoint, has no eigenvalue, and $\sigma(q) = \mathbb{R}$.

(*Hint:* there are some pitfalls with respect to the bounded case (Problems 35 and 36) owing to domain issues, otherwise the solution is the same.)

Exercise 48. Let A be a symmetric operator on a Hilbert space $\mathcal{H}$ such that its domain $\mathcal{D}(A)$ contains an orthonormal basis $\{\psi_n\}_{n=1}^\infty$ of $\mathcal{H}$ consisting of eigenvectors for A.

(i) Show that A is essentially self-adjoint.

(ii) Show that $\sigma(\overline{A})$ is the closure of the set of the eigenvalues of A.