Functional Analysis II

Institute of Mathematics, LMU Munich – Winter Term 2011/2012
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HOMEWORK ASSIGNMENT no. 11, issued on Wednesday 11 January 2012
Due: Wednesday 18 January 2012 by 2 pm in the designated "FA2" box on the 1st floor
Info: `www.math.lmu.de/~michel/WS11-12_FA2.html`

Each exercise sheet is worth a full mark of 40 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.

Exercise 41. (Operator convex functions.)

A continuous, real-valued function f on an interval I is said OPERATOR CONVEX (on the interval I) if $f(\lambda A + (1-\lambda)B) \leqslant \lambda f(A) + (1-\lambda) f(B)$ for any two bounded, self-adjoint operators A, B on a Hilbert space $\mathcal{H}$ with spectrum in I and for every $\lambda \in [0,1]$.

(i) Show that for a continuous, real-valued function f on an interval I to be operator convex is equivalent to $f\Big(\dfrac{A+B}{2}\Big) \leqslant \dfrac{f(A)+f(B)}{2}$ for any two bounded, self-adjoint operators A, B on a Hilbert space $\mathcal{H}$ with spectrum in I.

(*Hint:* prove first $f(\frac{A_1+A_2+\cdots+A_n}{n}) \leqslant \frac{f(A_1)+f(A_2)+\cdots+f(A_n)}{n}$ for any $n=2^m$, $m \in \mathbb{N}$, and hence for any positive integer n, where $A_1, \dots, A_n$ are self-adjoint. In such an inequality take some of the A_j's equal to A and the others equal to B so to obtain the operator convex condition for $\lambda \in \mathbb{Q} \cap [0,1]$.)

(ii) Show that the function $f(t) = t^2$ is operator convex on every interval.

(iii) Show that the function $f(t) = t^3$ is *not* operator convex on $[0, \infty)$.

(*Hint:* disprove with a counterexample for 2×2 matrices.)

(iv) Show that the function $f(t) = |t|$ is *not* operator convex on any interval that contains an open neighbourhood of zero. (*Hint:* Problem 23 (i).)

(v) Show that the function $f(t) = t^{-1}$ is operator convex on $(0, \infty)$.

(*Hint:* prove the identity $\frac{A^{-1}+B^{-1}}{2} - \big(\frac{A+B}{2}\big)^{-1} = \frac{(A^{-1}-B^{-1})(A^{-1}+B^{-1})^{-1}(A^{-1}-B^{-1})}{2}$.
Alternative strategy: show first convexity in the (commutative!) case where A and B are replaced by $\mathbb{1}$ and $A^{-1/2}BA^{-1/2}$ and then use Exercise 37 (i) to obtain the general result.)

Exercise 42. (Spectral resolution of the position operator.)

Consider the position operator on $L^2[0,1]$, i.e., the map $A : L^2[0,1] \to L^2[0,1]$, $(A\psi)(x) = x\psi(x)$ a.e. in $[0,1]$. (Recall from Problem 35 that $A = A^*$, $\|A\| = 1$, $\sigma(A) = [0,1]$.)

(i) Give the explicit action of the operator $f(A)$ on $L^2[0,1]$ where $f:[0,1]\to\mathbb{C}$ is a given bounded, Borel-measurable function. Use the measurable functional calculus to answer this question (see (iii) below, instead).

(ii) Exhibit the projection-valued measure $\{E_\Omega\}_\Omega$ associated with A, that is, give the explicit action of E_Ω on $L^2[0,1]$ for each Borel set $\Omega\subset\sigma(A)$.

(iii) Conversely, given the projection-valued measure $\{E_\Omega\}_\Omega$ associated with A determined in (ii), construct $f(A)$ (i.e., give its explicit action) using the spectral resolution for A.

(Note that this Exercise, together with Problem 35, completes the proof of Example 2.19(c) stated in class.)

Exercise 43. (The restriction to a spectral subspace.)

Let $\mathcal{H}$ be a Hilbert space, A be a self-adjoint operator in $\mathcal{B}(\mathcal{H})$ and $\{E_\Omega\}_\Omega$ be the projection-valued measure associated with A.

(i) Show that the subspace $\operatorname{ran} E_\Omega$ is invariant under A for any Borel set $\Omega\subset\sigma(A)$.

(ii) Show that if Ω is a closed Borel set in $\sigma(A)$ then $\sigma\big(A\big|_{\operatorname{ran} E_\Omega}\big)\subset\Omega$.

(*Hint:* spectral theorem, multiplication operator form.)

Exercise 44. (More applications of spectral theorem: unitary group; norm of the resolvent.)

Let $\mathcal{H}$ be a Hilbert space and A be a self-adjoint operator in $\mathcal{B}(\mathcal{H})$.

(i) Show that the operator $U(t)=e^{\mathrm{i}tA}$ constructed with the functional calculus for A is a unitary operator for all $t\in\mathbb{R}$ and that

$$U(t)^*=U(-t)\,,\qquad U(t)U(s)=U(t+s)\qquad \forall t,s\in\mathbb{R}\,.$$

(ii) Prove that the operator-valued function $t\mapsto U(t)$ defined in (i) is differentiable with respect to the operator norm topology and $U'(t)=\mathrm{i}AU(t)=\mathrm{i}U(t)A$ $\forall t\in\mathbb{R}$.

(iii) Let $\lambda\notin\sigma(A)$. Show that $\|(\lambda-A)^{-1}\|=\dfrac{1}{d(\lambda,\sigma(A))}$.

(*Hint:* you need only one direction, the other being given by Problem 7 (v).)