Institute of Mathematics, LMU Munich – Winter Term 2011/2012
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HOMEWORK ASSIGNMENT no. 9, issued on Wednesday 14 December 2011
Due: Wednesday 21 December 2011 by 2 pm in the designated "FA2" box on the 1st floor
Info: `www.math.lmu.de/~michel/WS11-12_FA2.html`

Each exercise sheet is worth a full mark of 40 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.

Exercise 33. (Spectrum of a partial isometry.)

(i) Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{B}(\mathcal{H})$ with $\|T\| \leqslant 1$. Show that $\mathbb{O} \leqslant TT^* \leqslant \mathbb{1}$ (in the sense of operators) and that the operator $M : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$, $x \oplus y \mapsto (Tx + Sy) \oplus 0$ is a partial isometry, where $S := \sqrt{\mathbb{1} - TT^*}$. (*Hint:* Problem 24.)

(*Note*: for you own curiosity, identify the initial and final spaces of the partial isometry M [you should be able to do that] and see that they are quite nasty. In fact, the definition of partial isometries is deceptively simple and the standard examples [orthogonal projections, unitary operators, etc.] continue the deception, but the structure of a partial isometry can be quite complicated!)

(ii) Let K be a compact subset of the closed unit disc $\{\lambda \in \mathbb{C} \,|\, |\lambda| \leqslant 1\}$ such that $0 \in K$. Show that there exists a partial isometry U on a Hilbert space $\mathcal{H}$ with spectrum K.

(*Hint:* part (i) and Exercise 18 (iv).)

Exercise 34. (Examples of polar decompositions.)

(i) Let $\mathcal{H}$ be a Hilbert space, $\eta, \xi \in \mathcal{H}$ with $\|\eta\| = \|\xi\| = 1$, and $T = |\xi\rangle\langle\eta| : \mathcal{H} \to \mathcal{H}$ (i.e., T is the operator defined by $Tx := \langle \eta, x\rangle \xi$ for all $x \in \mathcal{H}$). Find the polar decomposition $T = U_T|T|$ of T, i.e., give the partial isometry U_T and the absolute value operator $|T|$.

(ii) Find the polar decomposition $R = U_R|R|$ of the right shift operator $R : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, $R(x_1, x_2, x_3, \dots) := (0, x_1, x_2, \dots)$ for all $x = (x_1, x_2, x_3, \dots) \in \ell^2(\mathbb{N})$.

(iii) Find the polar decomposition $L = U_L|L|$ of the left shift operator $L : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, $L(x_1, x_2, x_3, \dots) := (x_2, x_3, x_4 \dots)$ for all $x = (x_1, x_2, x_3, \dots) \in \ell^2(\mathbb{N})$.

(iv) Find the polar decomposition $V = U_V|V|$ of the Volterra integral operator $V : L^2[0,1] \to L^2[0,1]$, $(Vf)(x) := \int_0^x f(y)\mathrm{d}y$ for almost all $x \in [0,1]$.

Important: the full solution here is to produce the explicit "closed" form of the operators U_V and $|V|$. Giving the canonical decomposition of $|V|$ is not enough, show that in fact $|V|$ is an integral operator and U_V is a unitary operator (with an explicit, "easy" form).

Exercise 35. (The discrete Laplacian.)

Consider the operator $\Delta : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined on every $x = (\dots, x_{n-1}, x_n, x_{n+1}, \dots) \in \ell^2$ by

$$(\Delta x)_n := \sum_{\substack{m\in\mathbb{Z}\\ |m-n|=1}} (x_n - x_m) \qquad\qquad (n \in \mathbb{Z})\,.$$

(i) Show that Δ is bounded and self-adjoint.

(ii) Show that $\mathbb{O} \leqslant \Delta \leqslant 4 \cdot \mathbb{1}$.

(iii) Compute $\|\Delta\|$.

(iv) Determine $\sigma(\Delta)$. (*Hint:* Problem 12 (i), Exercise 18 (iii).)

(v) Produce a measure space $(\mathcal{M}, \mu)$, an isomorphism $U : \ell^2(\mathbb{Z}) \to L^2(\mathcal{M}, \mathrm{d}\mu)$, and a function $F : \mathcal{M} \to \mathbb{R}$ such that $U \Delta U^{-1}$ acts on $L^2(\mathcal{M}, \mathrm{d}\mu)$ as the operator of multiplication by F.

Exercise 36. (The spectrum is upper semicontinuous, but not continuous.)

(i) Let X be a Banach space and let $T \in \mathcal{B}(X)$. Show that for every bounded open set $\Omega \subset \mathbb{C}$ such that $\sigma(T) \subset \Omega$ there exists $\varepsilon > 0$ such that if $S \in \mathcal{B}(X)$ with $\|S - T\| \leqslant \varepsilon$ then $\sigma(S) \subset \Omega$.

(ii) Consider the operators $A^{(N)}, A \in \mathcal{B}(\ell^2(\mathbb{Z}))$, $N \in \mathbb{N}$, defined on the canonical orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ of $\ell^2(\mathbb{Z})$ by

$$A^{(N)} e_n := \begin{cases} e_{n+1} & \text{if } n \neq 0 \\ \frac{1}{N}\, e_1 & \text{if } n = 0 \end{cases}, \qquad A\, e_n := \begin{cases} e_{n+1} & \text{if } n \neq 0 \\ 0 & \text{if } n = 0 \end{cases},$$

and then extended by linearity and boundedness. Show that

- $A^{(N)} \xrightarrow[N\to\infty]{\|\ \|} A$,
- $\sigma(A^{(N)}) \subset \{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\}$ for all $N \in \mathbb{N}$,
- $\sigma(A)$ contains points λ with $|\lambda| \neq 1$,

so that the spectrum of A is discontinuously different from the spectra of the $A^{(N)}$'s.

Optional: show that in fact $\sigma(A^{(N)})$ is the unit circle and $\sigma(A)$ is the unit disk of $\mathbb{C}$.