Functional Analysis II

Institute of Mathematics, LMU Munich – Winter Term 2011/2012
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HOMEWORK ASSIGNMENT no. 8, issued on Wednesday 7 December 2011
Due: Wednesday 14 December 2011 by 2 pm in the designated "FA2" box on the 1st floor
Info: `www.math.lmu.de/~michel/WS11-12_FA2.html`

Each exercise sheet is worth a full mark of 40 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.

Exercise 29. (Min-Max.) Let A be a compact, self-adjoint operator on a Hilbert space $\mathcal{H}$. Number its non-zero eigenvalues, possibly repeating them with multiplicity, in such a way that

$$\lambda_{-1} \leqslant \lambda_{-2} \leqslant \cdots < 0 < \cdots \leqslant \lambda_2 \leqslant \lambda_1 \,.$$

Show that

$$\lambda_n \;=\; \inf_{\mathcal{H}_{n-1}} \sup_{\substack{x \perp \mathcal{H}_{n-1} \\ \|x\|=1}} \langle x, Ax\rangle \qquad \text{and} \qquad \lambda_{-n} \;=\; \sup_{\mathcal{H}_{n-1}} \inf_{\substack{x \perp \mathcal{H}_{n-1} \\ \|x\|=1}} \langle x, Ax\rangle \qquad \forall n \in \mathbb{N}\,.$$

Here $\inf_{\mathcal{H}_{n-1}}$ and $\sup_{\mathcal{H}_{n-1}}$ are taken over all possible $(n-1)$-dimensional subspaces $\mathcal{H}_{n-1}$ of $\mathcal{H}$.

Exercise 30. Consider a Banach space X with norm $\|\ \|$ and an operator $T \in \mathcal{B}(X)$ with spectral radius $r(T) < 1$. Show that $\|x\|_0 := \sum_{n=0}^{\infty} \|T^n x\|$ defines for all $x \in X$ a new norm $\|\ \|_0$ on X that is equivalent to the original one.

Exercise 31. (Characterisation of the spectral radius by similarity.) Consider a Hilbert space $\mathcal{H}$ with scalar product $\langle\ ,\ \rangle$.

(i) Show that if $T \in \mathcal{B}(\mathcal{H})$ has spectral radius $r(T) < 1$ then $\langle x, y\rangle_0 := \sum_{n=0}^{\infty} \langle T^n x, T^n y\rangle$ defines for all $x, y \in X$ a new scalar product $\langle\ ,\ \rangle_0$ on X that makes $\mathcal{H}$ complete.

(*Hint:* Exercise 30.)

(ii) Show that if $T \in \mathcal{B}(\mathcal{H})$ has spectral radius $r(T) < 1$ then T is similar to a strict contraction, i.e., $\|STS^{-1}\| < 1$ for some invertible operator $S \in \mathcal{B}(\mathcal{H})$.

(*Hint:* Part (i).)

(iii) Show that $r(T) = r(STS^{-1})$ for any $T \in \mathcal{B}(\mathcal{H})$ and any invertible $S \in \mathcal{B}(\mathcal{H})$.

(iv) Show that if $T \in \mathcal{B}(\mathcal{H})$ then

$$r(T) \;=\; \inf_{\substack{S \in \mathcal{B}(\mathcal{H}) \\ S \,\text{invertible}}} \|STS^{-1}\|\,.$$

(*Hint:* use the operator $\frac{t}{r(T)}T$ for $t \in (0,1)$ and use (ii) and (iii).)

Exercise 32. Here below you have a list of inclusions between Banach spaces (except case (iii), where the space $\mathcal{W}_0^{1,p}$ is in fact a non-complete normed space). In each case decide if the embedding is continuous and if it is compact.

(i) $C^1([0,1]) \subset C([0,1])$

with the standard norm $\|f\|_{C^1} = \|f\|_{\max} + \|f'\|_{\max}$.

(ii) $L^p(\Omega) \subset L^q(\Omega)$

with $\Omega \subset \mathbb{R}^d$, $|\Omega| < \infty$, $1 \leqslant q \leqslant p \leqslant \infty$.

(iii) $\mathcal{W}_0^{1,p}[0,1] \subset C([0,1])$

defining $\mathcal{W}_0^{1,p}[0,1]$, $p \in [1, \infty]$, to be the space of $C^1([0,1])$-functions such that $f(0) = 0$ equipped with the norm $\|f\|_{W_0^{1,p}} := \|f\|_p + \|f'\|_p$.

(iv) $H^{\frac{3}{2}+\varepsilon}(S^1) \subset C(S^1)$

where $\varepsilon > 0$ and S^1 is the unit circle in $\mathbb{R}^2$. Recall the definition of $H^k(S^1)$, $k \geqslant 0$ (see Problems 26, 27, 34 in Functional Analysis, spring term 2011):

- $H^k(S^1) := \{f \in L^2(S^1) \mid \|f\|_{H^k} < \infty\}$
- $\|f\|_{H^k} := \left(\sum_{n\in\mathbb{Z}} (1 + |n|^{2k})\, |c_n|^2\right)^{1/2}$ for all $f = \sum_{n\in\mathbb{Z}} c_n\, e_n \in L^2(S^1)$, where $\{e_n\}_{n\in\mathbb{Z}}$ is the orthonormal basis of $L^2(S^1) \cong L^2[0, 2\pi]$ given by $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$ (the "Fourier" basis).