Institute of Mathematics, LMU Munich – Winter Term 2011/2012
Prof. T. Ø. Sørensen Ph.D, A. Michelangeli Ph.D

HOMEWORK ASSIGNMENT no. 7, issued on Wednesday 30 November 2011
Due: Wednesday 7 December 2011 by 2 pm in the designated "FA2" box on the 1st floor
Info: `www.math.lmu.de/~michel/WS11-12_FA2.html`

Each exercise sheet is worth a full mark of 40 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.

Exercise 25.

(i) Let N be a normal operator on a Hilbert space $\mathcal{H}$. Show that if N has a one-sided inverse, then N is invertible.

(ii) Consider the right shift $R : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, $T(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$. Prove that R cannot be equal to the product of a finite number of normal operators on $\ell^2(\mathbb{N})$.

(*Hint:* go for a contradiction and use (i).)

Exercise 26. Consider the operators $T_1 : \ell^1(\mathbb{N}) \to \ell^1(\mathbb{N})$ and $T_2 : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined on the basis vectors of the form $e_n = (\dots, 0, 0, 1, 0, 0, \dots)$ (that is, all entries are zero but the n-th one that is equal to one) by

$$T_1 e_n := \sum_{m=1}^{\infty} t_{n,m}\, e_m \,, \qquad T_2 e_n := \sum_{m=1}^{\infty} t_{n,m}\, e_m \,,$$

where in both cases

$$t_{n,m} := \begin{cases} \frac{n}{(m-1)m} & \text{if } m > n \\ 0 & \text{if } m \leqslant n \end{cases}$$

(note that the series converge both in ℓ^1 and in ℓ^2), and then are extended by linearity and boundedness to the whole $\ell^1(\mathbb{N})$, respectively the whole $\ell^2(\mathbb{N})$. (Boundedness is in fact proved in (i) here below.)

(i) Prove that both T_1 and T_2 are bounded: $T_1 \in \mathcal{B}(\ell^1(\mathbb{N}))$, $T_2 \in \mathcal{B}(\ell^2(\mathbb{N}))$.

(ii) Prove that $\sigma(T_2) \neq \sigma(T_1)$. (*Hint:* compare $\|T_2\|$ with the spectral radius $r(T_1)$.)

(Therefore: the knowledge of the action of an operator is not enough to determine its spectrum. An operator that acts the same way on different Banach spaces may have different spectra!)

Exercise 27. (Spectral radius of a product and of a sum.) Let X be a Banach space and let $T, S \in \mathcal{B}(X)$. Denote by $r(T)$ the spectral radius of T.

(i) Show that if $[T, S] = \mathbb{O}$ then $r(TS) \leqslant r(T)r(S)$.

(ii) Show that if $[T, S] = \mathbb{O}$ then $r(T + S) \leqslant r(T) + r(S)$.

(iii) Find counterexamples to the conclusions in (i) and (ii) when T and S do not commute.

(iv) Show that $r(T^n) = r(T)^n \ \forall n \in \mathbb{N}$.

Exercise 28. (The norm of a self-adjoint operator.) Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Show that

$$\|A\| \;=\; \sup_{\substack{x \in \mathcal{H} \\ \|x\|=1}} |\langle x, Ax \rangle| \,.$$

(*Hint:* polarisation and parallelogram law.)

Comment: compare the above result with the variational characterisation of the norm of a generic operator $T \in \mathcal{B}(\mathcal{H})$, that is,

$$\|T\| \;=\; \sup_{\substack{x,y \in \mathcal{H} \\ \|x\|=\|y\|=1}} |\langle x, Ty \rangle| \,.$$