Functional Analysis II

Institute of Mathematics, LMU Munich – Winter Term 2011/2012
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HOMEWORK ASSIGNMENT no. 4, issued on Wednesday 9 November 2011
Due: Wednesday 16 November 2011 by 2 pm in the designated "FA2" box on the 1st floor
Info: `www.math.lmu.de/~michel/WS11-12_FA2.html`

Each exercise sheet is worth a full mark of 40 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.

Exercise 13. (The Volterra integral operator on a Hilbert space – III)
Consider the operator $V : L^2[0,1] \to L^2[0,1]$, $(Vf)(x) := \int_0^x f(y)\mathrm{d}y$ for almost all $x \in [0,1]$. In this exercise you are asked to compute the resolvent $(\lambda\mathbb{1} - V)^{-1}$ $(\lambda \neq 0)$ in two alternative ways. You may use the results of Exercises 10 and 11 without re-proving them.

• First way:

(i) Let $\lambda \in \mathbb{C}\backslash\{0\}$ and $g \in C^1([0,1])$. Show that the equation $\lambda f - Vf = g$ in $L^2[0,1]$ has a unique solution $f \in L^2[0,1]$ and determine it.

(ii) Release the differentiability assumption on g in (i): take $g \in L^2[0,1]$, $\lambda \neq 0$, and determine the solution f to the problem $\lambda f - Vf = g$ in $L^2[0,1]$, thus obtaining the explicit expression for the action of the resolvent $(\lambda\mathbb{1} - V)^{-1}$. (*Hint:* a density argument.)

• Second way:

(iii) Let $n \in \mathbb{N}$ and $f \in L^2[0,1]$. Show that $(V^n f)(x) = \frac{1}{(n-1)!}\int_0^x (x-y)^{n-1} f(y)\,\mathrm{d}y$ for a.e. x.

(iv) Compute the resolvent $(\lambda\mathbb{1} - V)^{-1}$ $\forall \lambda \in \mathbb{C}\backslash\{0\}$ by means of the resolvent identity $(\lambda\mathbb{1} - V)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1} V^n$. (*Warning:* this identity holds only for $|\lambda| > \|V\|$, make sure your final result holds for all non-zero λ's.)

Exercise 14. (Perturbation of the spectrum with compact operators.)

(i) Let X be a Banach space and let $T, S \in \mathcal{B}(X)$ be two operators such that their difference $T - S$ is compact. Show that their spectra are the same except for eigenvalues, i.e., $\sigma(T)\backslash\sigma_{\mathrm{p}}(T) \subset \sigma(S)$. (*Hint:* Fredholm alternative.)

(ii) Let $\mathcal{H}$ be a Hilbert space and let $U \in \mathcal{B}(\mathcal{H})$ be a unitary operator on $\mathcal{H}$. Show that $\sigma(U) \subset \{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\}$. (*Hint:* Problem 10 in class.)

(iii) The fact proved in (i) does not exclude that the two spectra look considerably different. As an example, produce a bounded operator U and a compact operator K on a Hilbert space $\mathcal{H}$ such that

- $\sigma(U) \subset \{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\}$,
- K is compact,
- $\sigma(U + K) = \{\lambda \in \mathbb{C} \,|\, |\lambda| \leqslant 1\}$.

(*Hint:* part (ii) and Problem 8 in class.)

Exercise 15. (Canonical form of compact operators on a Hilbert space.)
Let $\mathcal{H}$ be a Hilbert space. Let A be a compact, self-adjoint operator on $\mathcal{H}$. Consider the collection of all eigenvalues of A.

(Recall that they form a discrete family, finite or infinite, they are all real (Problem 12(i)), the non-zero ones (if any) have finite degeneracy (Exercise 7), and if they are infinite they accumulate to zero only.)

Pick a (finite) orthonormal basis in each eigenspace with non-zero eigenvalue, and a (possibly infinite, possibly uncountable) orthonormal basis in the kernel of A, if it is non-trivial. Denote by $\{\phi_n\}_{n\in\mathcal{I}}$ the union of all such eigenvectors, i.e., $A\phi_n = \lambda_n\phi_n$ $\forall n \in \mathcal{I}$ and $\langle \phi_n, \phi_m\rangle = \delta_{n,m}$.

(Note that in this notation the λ_n's are repeated with degeneracy and that n runs in an index set $\mathcal{I} \supset \{1, \dots, N\}$, where N is finite or infinite. $\mathcal{I}$ is uncountable if Ker$A$ is not separable. Thus, in this notation $\{\lambda_n\}_{n=\mathcal{I}}$ is a-priori uncountable, but it is still a fact that there are at most countably many *distinct* λ_n's.)

The set $\{\phi_n\}_{n\in\mathcal{I}}$ is by construction and by Problem 12(iii) an orthonormal system of $\mathcal{H}$.

(i) Let $\mathcal{M}$ be the closure in $\mathcal{H}$ of the span of $\{\phi_n\}_{n\in\mathcal{I}}$. Show that $A\mathcal{M} \subset \mathcal{M}$ and $A\mathcal{M}^\perp \subset \mathcal{M}^\perp$.

(ii) Show that the spectrum of the operator $A\big|_{\mathcal{M}^\perp} : \mathcal{M}^\perp \to \mathcal{M}^\perp$ is $\sigma(A\big|_{\mathcal{M}^\perp}) = \{0\}$.

(iii) Deduce from (ii) and from some other fact that $\mathcal{M}^\perp = \{0\}$ and therefore that $\{\phi_n\}_{n\in\mathcal{I}}$ is an orthonormal basis of $\mathcal{H}$.

This proves that *a compact, self-adjoint operator on a Hilbert space admits an orthonormal basis of eigenvectors*. Note that this answers Exercise 11(ii) without the (somewhat tedious) check by inspection that the closure of $\{\phi_n\}_{n\in\mathcal{I}}$ spans the whole $\mathcal{H}$. (In that case Ker$V = \{0\}$.)

Consider now a compact operator C on $\mathcal{H}$.

(iv) Show that the non-zero eigenvalues of C^*C form a family $\{\mu_n\}_{n=1}^N$ of positive real numbers, possibly repeated with degeneracy, where N can be finite or infinite.

(v) Show that there exist two orthonormal system $\{\psi_n\}_{n=1}^N$ and $\{\phi_n\}_{n=1}^N$ in $\mathcal{H}$ and a collection of positive numbers $\{\lambda_n\}_{n=1}^N$, where N can be finite or infinite, such that

$$C \;=\; \sum_{n=1}^{N} \lambda_n \,\langle \psi_n,\,\cdot\,\rangle\, \phi_n$$

(or $C = \sum_{n=1}^N \lambda_n\, |\varphi_n\rangle\langle\psi_n|$ in bra-ket notation) where if $N = \infty$ the series converges in operator norm. (*Hint:* take $\{\psi_n\}_{n=1}^N$ to be the family of normalised eigenvectors corresponding to $\{\mu_n\}_{n=1}^N$ considered in (iv).)

Exercise 16. Consider the operator $T : L^2(S^1) \to L^2(S^1)$, $(Tf)(x) = (h * f)(x)$ for almost all $x \in [0, 2\pi]$ where $h \in L^2(S^1)$ is given. Recall that $(h * f)(x) := \int_0^{2\pi} h(x-y)f(y)\,\mathrm{d}y$ and that in this case the integral makes sense almost everywhere in x thanks to Hölder's inequality.

(i) Show that T is a bounded, compact operator on $L^2(S^1)$.

(ii) Show that T is normal, i.e., $T^*T = TT^*$.

(iii) Find an explicit orthonormal system $\{\phi_n\}_{n=1}^N$ of $\mathcal{H}$ and a collection $\{\lambda_n\}_{n=1}^N$ in $\mathbb{C}$ (where N is finite or infinite depending on h) such that

$$T \;=\; \sum_{n=1}^{N} \lambda_n \,\langle \phi_n,\,\cdot\,\rangle\, \phi_n$$

(or $T = \sum_{n=1}^N \lambda_n |\phi_n\rangle\langle\phi_n|$ in bra-ket notation) where if $N = \infty$ the series converges in operator norm.