Functional Analysis II

Institute of Mathematics, LMU Munich – Winter Term 2011/2012
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HOMEWORK ASSIGNMENT no. 3, issued on Wednesday 2 November 2011
Due: Wednesday 9 November 2011 by 2 pm in the designated "FA2" box on the 1st floor
Info: `www.math.lmu.de/~michel/WS11-12_FA2.html`

Each exercise sheet is worth a full mark of 40 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.

Exercise 9. Given $w = (w_1, w_2, w_3, \dots) \in \ell^\infty(\mathbb{N})$ consider the operator $T_w : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, $T_w(x_1, x_2, x_3, \dots) = (w_1x_1, w_2x_2, w_3x_3, \dots)$.

(i) Show that T_w is bounded with norm $\|T_w\| = \|w\|_\infty = \sup_n |w_n|$.

(ii) Find the adjoint T_w^* (i.e., give its explicit action on $\ell^2(\mathbb{N})$).

(iii) Find the subsets of the w's in $\ell^\infty(\mathbb{N})$ for which T_w is normal (i.e., $TT^* = T^*T$) and for which T_w self-adjoint (i.e., $T = T^*$).

(iv) Find the subset of the w's in $\ell^\infty(\mathbb{N})$ for which T_w is compact.

(v) Find the point spectrum $\sigma_{\mathrm{p}}(T_w)$ of T_w.

(vi) Show that the spectrum of T_w is the closure (in $\mathbb{C}$) of its point spectrum, $\sigma(T_w) = \overline{\sigma_{\mathrm{p}}(T_w)}$.

Exercise 10. (The Volterra integral operator on a Hilbert space – I)
Consider the operator $V : L^2[0,1] \to L^2[0,1]$, $(Vf)(x) := \int_0^x f(y)\mathrm{d}y$ for almost all $x \in [0,1]$.

(i) Show that V is bounded.

(*Hint:* computing $\|V\|$ by "guessing" the right inequality and then saturating it is tricky in this case (this will be the goal of Exercise 11), so here just find a cheap bound, for instance $\|V\| \leqslant 1$.)

(ii) Show that V maps the unit ball of $L^2[0,1]$ onto a pre-compact subspace of $L^2[0,1]$, thus concluding that V is compact. (For another proof of the compactness of V see Problem 9.)

(iii) Show that $\sigma_{\mathrm{p}}(V) = \emptyset$.

(iv) Show that $\sigma(V) = \{0\}$ and that $\sigma_{\mathrm{r}}(V) = \emptyset$.

(v) Find the (Hilbert) adjoint V^* of V (i.e., give its explicit action) and show that $V + V^*$ is an orthogonal projection of rank one.

Exercise 11. (The Volterra integral operator on a Hilbert space – II)
Consider the operator $V : L^2[0,1] \to L^2[0,1]$, $(Vf)(x) := \int_0^x f(y)\mathrm{d}y$ for almost all $x \in [0,1]$. In this exercise you may use the results of Exercise 10 without re-proving them.

(i) Show that if $f \in L^2[0,1]$ is an eigenfunction of the operator V^*V with eigenvalue λ, then $\lambda > 0$, f is twice differentiable almost everywhere, and $\lambda f'' + f = 0$ a.e. in $[0,1]$.

(*Hint:* use the explicit expression of V^*V and the fact that $x \mapsto \int_0^x f(y)\,\mathrm{d}y$ is differentiable almost everywhere if f is locally integrable.)

(ii) Find the collection $\{\lambda_n\}_{n=0}^\infty$ of all eigenvalues of V^*V, i.e., $V^*Vf_n = \lambda_n f_n$, $n = 0,1,2,\dots$, and check by inspection that the corresponding family of eigenfunction $\{f_n\}_{n=0}^\infty$ is, up to normalisation, an orthonormal basis of $L^2[0,1]$.

Important: finding the λ_n's and the f_n's is routine computation. Then you are supposed to prove that $\{f_n\}_{n=0}^\infty$ is an orthonormal basis. Orthonormality is routine too. As for the spanning property, this would follow from a very elegant and effective tool, the spectral theorem for compact, self-adjoint operators (aka the Hilbert-Schmidt theorem), but you have not discussed it in class yet and you are not supposed to use it here. Thus, you have to prove the spanning property from scratch. This can be somewhat lengthy or tricky, which should make you appreciate the power of spectral theorem techniques. For instance, you may use that $\{e^{i2\pi nx}\}_{n\in\mathbb{Z}}$ is a well-known orthonormal basis of $L^2[0,1]$.

(iii) Deduce from (ii) that $\|V\| = \frac{2}{\pi}$.

(Note that (ii) provides a concrete, non-trivial realisation of the Riesz-Schauder decomposition for compact operators discussed in class.)

Exercise 12. Let X be a Banach space and consider $T \in \mathcal{B}(X)$. As usual, denote by $\rho(T)$ the resolvent set of T and by $R_\lambda(T) = (\lambda\mathbb{1} - T)^{-1}$ the resolvent of T.

(i) Show that the map $\rho(T) \to \mathcal{B}(X)$, $\lambda \mapsto R_\lambda(T)$ is continuous.

(ii) Show that such a map is also uniformly holomorphic, i.e., it has derivative in $\mathcal{B}(X)$ defined by the limit

$$\frac{\mathrm{d}R_\lambda(T)}{\mathrm{d}\lambda} := \lim_{h\to 0} \frac{R_{\lambda+h}(T) - R_\lambda(T)}{h} = -R_\lambda(T)^2$$

for all $\lambda \in \rho(T)$.

(*Hint:* resolvent identities.)