Functional Analysis II

Institute of Mathematics, LMU Munich – Winter Term 2011/2012
Prof. T. Ø. Sørensen Ph.D, A. Michelangeli Ph.D

HOMEWORK ASSIGNMENT no. 2, issued on Tuesday 25 October 2011
Due: Wednesday 2 November 2011 by 2 pm in the designated "FA2" box on the 1st floor
Info: `www.math.lmu.de/~michel/WS11-12_FA2.html`

Each exercise sheet is worth a full mark of 40 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.

Exercise 5. Let X and Y be Banach spaces and $T : X \to Y$ be a bounded operator.

(i) Assume that $\|Tx\|_Y \geqslant c\|x\|_X$ $\forall x \in X$, where $c > 0$ is given. Show that under this condition T can be compact only if $\dim X < \infty$.

(ii) Assume that $\dim X = \infty$ and that T is compact. Show that $0 \in Y$ belongs to the closure in norm of the image via T of the unit *sphere* in X. (*Hint:* use (i) to construct an appropriate sequence in the range of T that converges to zero.)

Exercise 6. Recall that an orthogonal projection acting on a Hilbert space $\mathcal{H}$ is an operator $P \in \mathcal{B}(\mathcal{H})$ such that $P = P^* = P^2$. Recall also that in the Hilbert space case the symbol $\oplus$ denotes the orthogonal sum (see the Projection Theorem).

(i) Show that the kernel and the range of an orthogonal projection P are two closed subspaces that decompose $\mathcal{H}$ in the orthogonal decomposition $\mathcal{H} = \mathrm{Ker}P \oplus \mathrm{Ran}P$.

(ii) Conversely, show that if K, R are two closed subspaces of $\mathcal{H}$ such that $\mathcal{H} = K \oplus R$ then there exists $P \in \mathcal{B}(\mathcal{H})$ such that P is the orthogonal projection *onto* R.

Assume in the following that P is an orthogonal projection on $\mathcal{H}$ other than the identity.

(iii) Find the point spectrum $\sigma_{\mathrm{p}}(P)$.

(iv) Find spectrum $\sigma(P)$.

(v) For every $\lambda \notin \sigma(P)$ give the explicit action of the resolvent operator $(\lambda\mathbb{1} - P)^{-1}$.

Exercise 7. Fix an arbitrary $a > 0$. Show that a compact operator on a Banach space has only finitely many linearly independent eigenvectors with eigenvalues having modulus at least a. (*Hint:* assume by contradiction the existence of infinitely many eigenvectors with eigenvalues above a and construct a bounded sequence out of them along which the compactness of T fails. To this aim, use Riesz lemma in this form: if Y is a *finite*-dimensional subspace of a normed space X then $\exists x \in X$, $\|x\| = 1$, such that $d(x, Y) = 1$.)

Exercise 8. Consider the measurable functions f_0 and g_0 such that $f_0(x) = e^{-x^2}$, $g_0(x) = \dfrac{e^{-|x|}}{\sqrt{|x|}}$ and the linear map $f \mapsto Tf$ such that $(Tf)(x) = \Big(\int_{\mathbb{R}} g_0(y)\, f(y)\,\mathrm{d}y\Big)\, f_0(x)$ for a.e. $x \in \mathbb{R}$.

(i) Show that T is a bounded linear operator from the Banach space $X = L^3(\mathbb{R})$ to the Banach space $Y = L^2(\mathbb{R})$ with norm $\|T\| \leqslant \|g_0\|_{3/2}\, \|f_0\|_2$.

(ii) Identify the adjoint operator $T' : Y' \to X'$, that is, say who X' and Y' are and what the explicit action $T' : Y' \to X'$ is.