Functional Analysis II

Institute of Mathematics, LMU Munich – Winter Term 2011/2012
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HOMEWORK ASSIGNMENT no. 1, issued on Tuesday 18 October 2011
Due: Tuesday 25 October 2011 by 2 pm in the designated "FA2" box on the 1st floor
Info: www.math.lmu.de/~michel/WS11-12_FA2.html

Each exercise sheet is worth a full mark of 40 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.

Exercise 1. Decide which of the following operators is compact and compute their operator norm.

(i) $T : C([0,1]) \to C([0,1])$, $(Tf)(x) = xf(x)$

(ii) $T : C([0,1]) \to C([0,1])$, $(Tf)(x) = f(0) + xf(1)$
(*Hint:* Ascoli-Arzelà.)

(iii) $T : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$ $(1 \leqslant p < \infty)$, $T(x_1, x_2, x_3 \dots) = (x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots)$
(*Hint:* you may use the approximation with finite rank operators, see Problem in class 4(ii).)

Exercise 2. Let $\mathcal{H}$ be a Hilbert spaces. Show that any compact operator $T : \mathcal{H} \to \mathcal{H}$ "attains" its norm, i.e., there exists $x \in \mathcal{H}$ such that $\dfrac{\|Tx\|}{\|x\|} = \|T\|$. (*Hint:* use Banach-Alaoglu and the fact that $\mathcal{H}$ is Hilbert, and not just Banach, to show that T maps bounded sets into compact sets, and exploit the continuity of $y \mapsto \|y\|$.)

Exercise 3. (No surjectivity in infinite dimensions.)

(i) Show that a compact operator on an *infinite* dimensional Banach space is never surjective.

(ii) Consider the compact operator $T : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$, $T(x_1, x_2, x_3, \dots) := (x_1, x_2/2, x_3/3, \dots)$ (where $1 \leqslant p < \infty$). The fact that T is compact is proved in Exercise 1 and is taken for granted here. Find $\mathbf{y} \in \ell^p(\mathbb{N})$ such that $T\mathbf{x} = \mathbf{y}$ has no solution $\mathbf{x}$ in $\ell^p(\mathbb{N})$.

Exercise 4. (Compact projection operators have finite rank.) Let $T : X \to X$ be a compact operator on a Banach space X such that $T^2 = T$. Show that T is a finite rank operator. (*Hint:* use the fact [Problem in class 1.(ii)] that the identity on an infinite-dimensional Banach space cannot be compact.)